

## SOME REMARKS ON INVERSE LAPLACE TRANSFORMS INVOLVING CONJUGATE BRANCH POINTS WITH APPLICATIONS

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*In this paper, we state a theorem for the inverse Laplace transform of functions involving conjugate branch points on imaginary axis. We get two equivalent integral representations for this inversion in terms of the Fourier sine and cosine transforms. Also, as applications of this theorem we obtain the solutions of dual integral equations with kernels of the Struve and Bessel functions and show an integral representation for the Gregory-Nörlund numbers. Moreover, new representations of the powers of the Airy functions are given in terms of the fractional integrals of order  $\frac{1}{2}$ .*

**Keywords:** Laplace transform, Bromwich integral, Dual integral equation, Volterra function, Airy function.

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### 1. Introduction

From past to present, the classical Laplace transform

$$\mathcal{L}\{f(x); s\} = F(s) = \int_0^{\infty} e^{-sx} f(x) dx, \quad (1)$$

and its inversion formula (complex inversion formula or Bromwich integral)

$$\mathcal{L}^{-1}\{F(s); x\} = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx} F(s) ds, \quad \Re(s) > c, \quad (2)$$

have many applications in the applied sciences. One of the most important application of this transform is its impact on partial differential equations which have been attracted much attention for many years, see [10, 11, 17, 23]. The main aspect of this application is finding suitable contours for the Bromwich integral with respect to the singular points of  $F(s)$ .

For example, Apelblat found new classes for the inverse Laplace transform of logarithmic functions [2], Puri and Kythe showed new classes for the inverse Laplace transform of exponential functions with two or three branch points on the real axis [19]. Also, Bobylev and Cercignani modified the well-known Titchmarsh theorem on the Hankel contour for the inverse Laplace transform and applied it for solving

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the Boltzmann equation [8].

Now, in this paper we intend to choose the functions  $F(s)$  involving expression  $\sqrt{s^2 + a^2}$  with two different branch points  $s = \pm ai$  (or functions involving expression  $\ln(s^2 + a^2)$ ) and write the corresponding Bromwich integrals in terms of the Fourier sine and cosine transforms. For this purpose, by considering two different contours for the Bromwich integral and using the residue theorem, we state a theorem for these classes of functions in Section 2. In Section 3, we intend to introduce classes of dual integral equations with kernels of the Struve and Bessel functions and solve this type of integral equations. In Section 4, as another application in the case  $\ln(s^2 + a^2)$ ,  $a \rightarrow 0$ , we introduce the Gregory-Nörlund numbers and using the associated Laguerre polynomials, we find an integral representation for these numbers in terms of the Volterra functions. In Section 5, we obtain the semi-integrals of powers of the Airy functions  $\text{Ai}^n(x)$ ,  $n = 1, 2, 3, 4$ , using the main theorem. Finally the main conclusions are set.

## 2. Main theorem

**Theorem 2.1.** Let  $F(s)$  be an analytic function for  $\Re(s) > c$ , also it has two conjugate branch points  $\pm ai$  and  $F(re^{-i\pi}) = F(re^{i\pi})$ , where  $a > 0$  and  $r > 0$ . Furthermore,  $F(s)$  satisfy the conditions

$$\begin{aligned} F(s) &= O(1), & |s| \rightarrow \infty, \\ F(s) &= O\left(\frac{1}{|s|}\right), & |s| \rightarrow 0, \end{aligned}$$

for any sector  $|\arg(s)| < \pi - \eta$ , where  $0 < \eta < \pi$ . Then the inverse Laplace transform  $f(t)$  can be written as two integral representations

$$f(t) = \mathcal{L}^{-1}\{F(s); t\} = -\frac{2}{\pi} \int_a^\infty \sin(rt) \Im(F(re^{i\frac{\pi}{2}})) dr, \quad (1)$$

$$f(t) = \mathcal{L}^{-1}\{F(s); t\} = \frac{2}{\pi} \int_0^a \cos(rt) \Re(F(re^{i\frac{\pi}{2}})) dr. \quad (2)$$

*Proof.* We consider Figure 1 as the modifications of the Bromwich integral and suppose that  $F(s) \rightarrow 0$  uniformly on circular arcs with radius  $R$ , when  $R \rightarrow \infty$ . In this sense, by applying the Cauchy theorem we can show on the closed contour  $\Gamma_j$ ,  $j = 1, 2$

$$\frac{1}{2\pi i} \int_{\Gamma_j} e^{st} F(s) ds = 0, \quad j = 1, 2. \quad (3)$$

On the contour  $\Gamma_1$ , if we set suitable values for lines  $C_j$  ( $s = re^{i\frac{\pi}{2}}$  for  $C_j, j = 1, 2$  and  $s = re^{-i\frac{\pi}{2}}$  for  $C_j, j = 3, 4$ ) and infinitesimal circles, and use the well-known

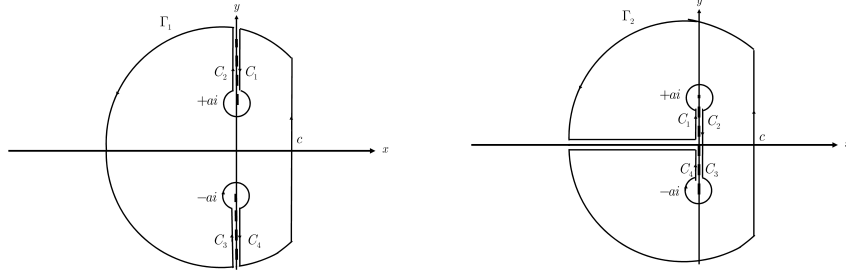


FIGURE 1. Deformations of the Bromwich integral for functions involving two branch points  $\pm ai$

Jordan lemma for vanishing the integral on arcs we get the relation (1) as follows

$$\begin{aligned}
 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds &= -\frac{1}{2\pi i} \left( \int_{C_1} + \int_{C_3} \right) - \frac{1}{2\pi i} \left( \int_{C_2} + \int_{C_4} \right) \\
 &= \frac{1}{2\pi i} \int_a^\infty e^{irt} F(re^{i\frac{\pi}{2}}) (idr) + \frac{1}{2\pi i} \int_a^\infty e^{-irt} F(re^{i\frac{\pi}{2}}) (-idr) \\
 &\quad + \frac{1}{2\pi i} \int_\infty^a e^{irt} F(re^{-i\frac{\pi}{2}}) (idr) + \frac{1}{2\pi i} \int_\infty^a e^{-irt} F(re^{-i\frac{\pi}{2}}) (-idr) \\
 &= -\frac{1}{2\pi i} \int_a^\infty 2 \sin(rt) [F(re^{i\frac{\pi}{2}}) - F(re^{-i\frac{\pi}{2}})] (idr) \\
 &= -\frac{2}{\pi} \int_a^\infty \sin(rt) \Im(F(re^{i\frac{\pi}{2}})) dr.
 \end{aligned}$$

In the same procedure to derive the relation (2), by writing the integral on the contour  $\Gamma_2$  we can obtain

$$\begin{aligned}
 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds &= -\frac{1}{2\pi i} \left( \int_{C_1} + \int_{C_3} \right) - \frac{1}{2\pi i} \left( \int_{C_2} + \int_{C_4} \right) \\
 &= \frac{1}{2\pi i} \int_0^a e^{irt} F(re^{i\frac{\pi}{2}}) (idr) + \frac{1}{2\pi i} \int_0^a e^{-irt} F(re^{i\frac{\pi}{2}}) (-idr) \\
 &\quad + \frac{1}{2\pi i} \int_a^0 e^{irt} F(re^{-i\frac{\pi}{2}}) (idr) + \frac{1}{2\pi i} \int_a^0 e^{-irt} F(re^{-i\frac{\pi}{2}}) (-idr) \\
 &= \frac{1}{2\pi i} \int_0^a 2 \cos(rt) [F(re^{i\frac{\pi}{2}}) + F(re^{-i\frac{\pi}{2}})] (idr) \\
 &= \frac{2}{\pi} \int_0^a \cos(rt) \Re(F(re^{i\frac{\pi}{2}})) dr.
 \end{aligned}$$

□

**Corollary 2.1.** In view of the Efros theorem for the following relation [4]

$$\mathcal{L}^{-1} \left\{ \frac{1}{s} F\left(\frac{1}{s}\right); x \right\} = \int_0^\infty J_0(2\sqrt{ux}) f(u) du, \quad (4)$$

and applying the identities [13]

$$\int_0^\infty J_0(2\sqrt{xt}) \sin(tr) dt = \frac{1}{r} \cos\left(\frac{x}{r}\right), \quad (5)$$

$$\int_0^\infty J_0(2\sqrt{xt}) \cos(tr) dt = \frac{1}{r} \sin\left(\frac{x}{r}\right), \quad (6)$$

we get new integral representation for function  $\mathcal{L}^{-1}\{\frac{1}{s}F(\frac{1}{s}); x\}$ , as follows

$$f(x) = \mathcal{L}^{-1}\left\{\frac{1}{s}F\left(\frac{1}{s}\right); x\right\} = -\frac{2}{\pi} \int_0^a \cos(rx) \left(\Im(F(\xi e^{i\frac{\pi}{2}}))\right)_{\xi=\frac{1}{r}} \frac{dr}{r}, \quad (7)$$

$$f(x) = \mathcal{L}^{-1}\left\{\frac{1}{s}F\left(\frac{1}{s}\right); x\right\} = \frac{2}{\pi} \int_a^\infty \sin(rx) \left(\Re(F(\xi e^{i\frac{\pi}{2}}))\right)_{\xi=\frac{1}{r}} \frac{dr}{r}. \quad (8)$$

**Corollary 2.2.** In view of the Hilbert transform

$$\mathcal{H}\{f(x); y\} = \frac{1}{\pi} P.V. \int_{-\infty}^\infty \frac{f(x)}{y-x} dx, \quad (9)$$

of trigonometric functions, i.e.,  $\mathcal{H}\{\sin(ax); y\} = -\cos(ay)$  and  $\mathcal{H}\{\cos(ax); y\} = \sin(ay)$ , and using relations (1) and (2), we get new integral representations as follows

$$\mathcal{H}\{f(t); y\} = \frac{2}{\pi} \int_a^\infty \cos(ry) \Im(F(re^{i\frac{\pi}{2}})) dr, \quad (10)$$

$$\mathcal{H}\{f(t); y\} = \frac{2}{\pi} \int_0^a \sin(ry) \Re(F(re^{i\frac{\pi}{2}})) dr. \quad (11)$$

**Remark 2.1.** The relations (1) and (2) can be interpreted as the Fourier sine and cosine transforms of functions  $-\frac{2}{\pi} \Im(F(re^{i\frac{\pi}{2}}))H(r-a)$  and  $\frac{2}{\pi} \Re(F(re^{i\frac{\pi}{2}}))H(a-r)$  respectively, where  $H$  is the Heaviside unit step function. In this sense, we get the following integral representations as the inverse Fourier sine and cosine transforms

$$-\Im(F(re^{i\frac{\pi}{2}}))H(r-a) = \int_0^\infty \sin(rt) f(t) dt, \quad (12)$$

$$\Re(F(re^{i\frac{\pi}{2}}))H(a-r) = \int_0^\infty \cos(rt) f(t) dt. \quad (13)$$

**Example 2.1.** Using the well-known relation  $\mathcal{L}\{J_0(at); s\} = \frac{1}{\sqrt{s^2+a^2}}$  for the Bessel function of first kind and zero order, we have the following integral representations

$$J_0(at) = \frac{2}{\pi} \int_a^\infty \sin(rt) \frac{1}{\sqrt{r^2-a^2}} dr, \quad (14)$$

$$J_0(at) = \frac{2}{\pi} \int_0^a \cos(rt) \frac{1}{\sqrt{a^2-r^2}} dr. \quad (15)$$

Also, using the relations (10) and (11), we have the following integral representations for the Bessel function of second kind and the Struve function of zero order

$$Y_0(ay) = -\frac{2}{\pi} \int_a^\infty \cos(ry) \frac{1}{\sqrt{r^2-a^2}} dr, \quad (16)$$

$$\mathbf{H}_0(ay) = \frac{2}{\pi} \int_0^a \sin(ry) \frac{1}{\sqrt{a^2-r^2}} dr. \quad (17)$$

**Remark 2.2.** If we have other singularities inside the Bromwich contours (poles and essential singularities) or branch points, then, the sum of residues of the function  $F(s)e^{sx}$  at these singularities is added to the relations (1) and (2) in Theorem 2.1. For example, we state the following examples.

**Example 2.2.** Using the fact that  $\mathcal{L}\{Y_0(at); s\} = -\frac{2}{\pi} \frac{\ln(\frac{s+\sqrt{s^2+a^2}}{a})}{\sqrt{s^2+a^2}}$ , for the Bessel function of second kind and zero order we have the following integral representation [13]

$$Y_0(at) = \frac{4}{\pi^2} \int_0^a \sin(rt) \frac{\sin^{-1}(\frac{r}{a})}{\sqrt{a^2-r^2}} dr - \frac{4}{\pi^2} \int_a^\infty \sin(rt) \frac{\ln(\frac{r+\sqrt{r^2-a^2}}{a})}{\sqrt{r^2-a^2}} dr. \quad (18)$$

Also, for the function  $F(s) = \frac{1}{\ln(s^2+a^2)}$  with the poles  $\pm\sqrt{1-a^2}$ ,  $a > 1$ , we have the following representation [2]

$$\mathcal{L}^{-1}\left\{\frac{1}{\ln(s^2+a^2)}, t\right\} = 2 \int_a^\infty \frac{\sin(rt)}{\pi^2 + \ln^2(r^2-a^2)} dr + \frac{\sin(t\sqrt{a^2-1})}{\sqrt{a^2-1}}, \quad (19)$$

and for  $\mathcal{L}^{-1}\left\{\frac{e^{-\frac{xs}{\sqrt{s^2+a^2}}}}{s}, t\right\}$ , we have

$$\mathcal{L}^{-1}\left\{\frac{e^{-\frac{xs}{\sqrt{s^2+a^2}}}}{s}, t\right\} = 1 - \frac{2}{\pi} \int_0^a \cos(rt) \sin\left(\frac{xr}{\sqrt{a^2-r^2}}\right) \frac{dr}{r}, \quad x > 0. \quad (20)$$

**Example 2.3.** Using the well-known relation

$$\mathcal{L}\{\mathbf{H}_0(at); s\} = \frac{2}{\pi} \frac{\ln(\frac{a+\sqrt{s^2+a^2}}{s})}{\sqrt{s^2+a^2}} = -\frac{1}{s} \mathcal{L}\{Y_0(at); \frac{1}{s}\}, \quad (21)$$

for the Struve function of zero order [13]

$$\mathbf{H}_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+1}}{\Gamma^2(n + \frac{3}{2})}, \quad (22)$$

and applying the relations (7) and (8), we get a new integral representation for the Struve function

$$\mathbf{H}_0(at) = -\frac{4}{\pi^2} \int_a^\infty \cos(rt) \frac{\sin^{-1}(\frac{a}{r})}{\sqrt{r^2-a^2}} dr + \frac{4}{\pi^2} \int_0^a \cos(rt) \frac{\ln(\frac{a+\sqrt{a^2-r^2}}{r})}{\sqrt{a^2-r^2}} dr. \quad (23)$$

### 3. Application to dual integral equations

The dual integral equations are pairs of integral equations arising in solution of mixed boundary problems, for example in potential theory and crack problems in elasticity [12]. In this section, as an application of the obtained results in previous section, we introduce dual integral equations with kernels of the Struve and Bessel functions  $\mathbf{H}_0$  and  $Y_0$ . For this purpose, we consider the  $Y$ -transform of zero order and its inversion formula with respect to the Bessel function of second kind and Struve function

$$g(y) = \int_0^\infty \sqrt{xy} Y_0(xy) f(x) dx, \quad y > 0, \quad (24)$$

$$f(x) = \int_0^\infty \sqrt{xy} \mathbf{H}_0(xy) g(y) dy. \quad (25)$$

**Theorem 3.1.** The solution of the dual integral equations

$$\begin{cases} \int_0^\infty t^{-1} f(t) \mathbf{H}_0(xt) dt = A(x), & 0 < x < 1, \\ \int_0^\infty f(t) \mathbf{H}_0(xt) dt = 0, & 1 < x < \infty, \end{cases} \quad (26)$$

is given by

$$f(t) = t \int_0^1 h_1(r) \sin(rt) dr + t \int_1^\infty h_2(r) \sin(rt) dr, \quad (27)$$

where  $h_1$  and  $h_2$  satisfy the following Abel type integral equation

$$A(x) = \int_x^1 \frac{h_1(r)}{\sqrt{r^2 - x^2}} dr + \int_x^\infty \frac{h_2(r)}{\sqrt{r^2 - x^2}} dr, \quad 0 < x < 1. \quad (28)$$

*Proof.* First we consider the second equation of (26), we use the inverse  $\mathbf{H}$ -transform to get the following relation

$$t^{-\frac{1}{2}} f(t) = \int_0^1 g(x) \sqrt{xt} Y_0(xt) dx. \quad (29)$$

Now, if we apply the integral representation (18) for the Bessel function of second kind, we obtain

$$f(t) = t \int_0^1 h_1(r) \sin(rt) dr + t \int_1^\infty h_2(r) \sin(rt) dr, \quad (30)$$

where  $h_1(r)$  and  $h_2(r)$  are auxiliary functions given by

$$h_1(r) = \frac{4}{\pi^2} \int_r^1 \sqrt{x} g(x) \frac{\sin^{-1}(\frac{r}{x})}{\sqrt{x^2 - r^2}} dx - \frac{4}{\pi^2} \int_0^r \sqrt{x} g(x) \frac{\ln(\frac{r + \sqrt{r^2 - x^2}}{x})}{\sqrt{r^2 - x^2}} dx, \quad (31)$$

$$h_2(r) = -\frac{4}{\pi^2} \int_0^1 \sqrt{x} g(x) \frac{\ln(\frac{r + \sqrt{r^2 - x^2}}{x})}{\sqrt{r^2 - x^2}} dx. \quad (32)$$

At this point, we set  $f(t)$  in the first equation of (26) to get an Abel type integral equation for the unknown functions  $h_1$  and  $h_2$

$$\begin{aligned} A(x) &= \int_0^\infty \mathbf{H}_0(xt) \left[ \int_0^1 h_1(r) \sin(rt) dr + \int_1^\infty h_2(r) \sin(rt) dr \right] dt \\ &= \int_0^1 h_1(r) \int_0^\infty \sin(rt) \mathbf{H}_0(xt) dt dr + \int_1^\infty h_2(r) \int_0^\infty \sin(rt) \mathbf{H}_0(xt) dt dr \\ &= \int_x^1 \frac{h_1(r)}{\sqrt{r^2 - x^2}} dr + \int_x^\infty \frac{h_2(r)}{\sqrt{r^2 - x^2}} dr, \quad 0 < x < 1, \end{aligned} \quad (33)$$

where we used the following fact for simplification [13]

$$\int_0^\infty \sin(rt) \mathbf{H}_0(xt) dt = \begin{cases} \frac{1}{\sqrt{x^2 - r^2}}, & r < x, \\ 0, & r > x. \end{cases} \quad (34)$$

□

**Theorem 3.2.** The solution of the dual integral equations

$$\begin{cases} \int_0^\infty t^{-1} f(t) Y_0(xt) dt = A(x), & 0 < x < 1, \\ \int_0^\infty f(t) Y_0(xt) dt = 0, & 1 < x < \infty, \end{cases} \quad (35)$$

is given by

$$f(t) = t \int_0^1 h_1(r) \cos(rt) dr + t \int_1^\infty h_2(r) \cos(rt) dr, \quad (36)$$

where  $h_1$  and  $h_2$  satisfy the following Abel type integral equation

$$A(x) = - \int_0^x \frac{h_1(r)}{\sqrt{x^2 - r^2}} dr - \int_1^x \frac{h_2(r)}{\sqrt{x^2 - r^2}} dr, \quad 0 < x < 1. \quad (37)$$

*Proof.* First we consider the second equation of (35), we use the inverse  $Y$ -transform to get the following relation

$$t^{-\frac{1}{2}} f(t) = \int_0^1 g(x) \sqrt{xt} \mathbf{H}_0(xt) dx. \quad (38)$$

Now, if we apply the integral representation (23) for the Struve function, we obtain

$$f(t) = t \int_0^1 h_1(r) \cos(rt) dr + t \int_1^\infty h_2(r) \cos(rt) dr, \quad (39)$$

where  $h_1$  and  $h_2$  are auxiliary functions given by

$$h_1(r) = -\frac{4}{\pi^2} \int_0^r \sqrt{x} g(x) \frac{\sin^{-1}(\frac{x}{r})}{\sqrt{r^2 - x^2}} dx + \frac{4}{\pi^2} \int_r^1 \sqrt{x} g(x) \frac{\ln(\frac{x + \sqrt{r^2 - x^2}}{r})}{\sqrt{r^2 - x^2}} dx. \quad (40)$$

$$h_2(r) = -\frac{4}{\pi^2} \int_0^1 \sqrt{x} g(x) \frac{\sin^{-1}(\frac{x}{r})}{\sqrt{r^2 - x^2}} dx. \quad (41)$$

At this point, we set  $f(t)$  in the first equation of (35) to get an Abel integral equation for the unknown function  $h$

$$\begin{aligned} A(x) &= \int_0^\infty Y_0(xt) \left[ \int_0^1 h_1(r) \cos(rt) dr + \int_1^\infty h_2(r) \cos(rt) dr \right] dt \\ &= \int_0^1 h_1(r) \int_0^\infty \cos(rt) Y_0(xt) dt dr + \int_1^\infty h_2(r) \int_0^\infty \cos(rt) Y_0(xt) dt dr \\ &= - \int_0^x \frac{h_1(r)}{\sqrt{x^2 - r^2}} dr - \int_1^x \frac{h_2(r)}{\sqrt{x^2 - r^2}} dr, \quad 0 < x < 1, \end{aligned} \quad (42)$$

where we used the following fact for simplification [13]

$$\int_0^\infty \cos(rt) Y_0(xt) dt = \begin{cases} -\frac{1}{\sqrt{r^2 - x^2}}, & r > x, \\ 0, & r < x. \end{cases} \quad (43)$$

□

#### 4. Application to Gregory-Nörlund numbers

In this section, as an application of the inverse Laplace transforms of the logarithmic functions when  $a \rightarrow 0$ , we introduce the Gregory-Nörlund numbers and present an integral representation for it. First, we recall the definition of the Gregory numbers. The Gregory numbers  $i_n$  are defined by the following formula [6, 7, 9]

$$\frac{x}{\ln(1-x)} = -1 + \sum_{n=1}^{\infty} i_n x^n, \quad -1 < x < 1, \quad (44)$$

and was introduced for the first time by Gregory in 1670. As an interesting representation of the Gregory numbers, Apelblat showed these numbers with respect to the improper integrals of the Volterra function  $\nu(t)$  [2]

$$\nu(t) = \int_0^\infty \frac{t^u}{\Gamma(u+1)} du. \quad (45)$$

He used the complex inversion formula of the Laplace transform for the Volterra function  $\nu(t)$  and derived the following integral representation for the Gregory numbers by the Laguerre polynomials  $L_n(t)$

$$\int_0^\infty \frac{e^{-xt}}{x[\pi^2 + \ln^2(x)]} dx = \sum_{n=0}^\infty i_{n+1} L_n(t). \quad (46)$$

In other point of view to the Gregory numbers, we consider the generating function of the Bernoulli numbers [1]

$$\frac{x}{e^x - 1} = \sum_{n=0}^\infty B_n \frac{x^n}{n!}. \quad (47)$$

It is obvious that the relation between the Gregory numbers and Bernoulli numbers for  $-1 < x < 1$  is given by

$$\frac{1}{\sum_{n=0}^\infty B_n \frac{\ln^n(1-x)}{n!}} = 1 - \sum_{n=1}^\infty i_n x^n. \quad (48)$$

Now, in this section in view of the generating function of Nörlund's generalized Bernoulli numbers [16, 24]

$$\left( \frac{x}{e^x - 1} \right)^\alpha = \sum_{n=0}^\infty B_n^{(\alpha)} \frac{x^n}{n!}, \quad \alpha > 0, \quad (49)$$

we intend to find the Nörlund's generalized Gregory numbers  $i_n^{(\alpha)}$  (we name it as the *Gregory-Nörlund numbers*). For this purpose, we start with the reciprocal of the Nörlund's generalized Bernoulli numbers as the generating function of Gregory-Nörlund numbers

$$\left( \frac{x}{\ln(1-x)} \right)^\alpha = (-1)^\alpha + \sum_{n=1}^\infty i_n^{(\alpha)} x^n, \quad -1 < x < 1, \quad (50)$$

and in next stage, we try to find an integral representation for the Gregory-Nörlund numbers in terms of the Volterra function of two parameters  $\mu(t, \alpha, \beta)$ . This integral representation and the associated Laguerre polynomials enable us to show a formula for obtaining these numbers. At this point, we start with the relation (50) as an assumption and we set  $x = 1 - \frac{1}{s}$ . We see that the left hand side of the relation is reduced to

$$F(s) = \frac{(s-1)^{\alpha-1}}{s^{2\alpha} \ln^\alpha(s)} = \frac{(-1)^\alpha}{(s-1)s^\alpha} \left( \frac{1 - \frac{1}{s}}{\ln(1 - (1 - \frac{1}{s}))} \right)^\alpha, \quad (51)$$

whose the inverse Laplace transform can be obtained by the Volterra function of two parameters  $\mu(t, \alpha, \beta)$  for  $\alpha \in \mathbb{N}$  [3, 5]

$$\mu(t, \alpha, \beta) = \int_0^\infty \frac{t^{u+\alpha} u^\beta}{\Gamma(\beta+1)\Gamma(u+\alpha+1)} du, \quad \Re(\alpha) > -1, \Re(\beta) > -1. \quad (52)$$



Now, by substituting the relation (50) in  $F(s)$ , we get

$$F(s) = \frac{(-1)^\alpha}{(s-1)s^\alpha} \left( \frac{1 - \frac{1}{s}}{\ln(1 - (1 - \frac{1}{s}))} \right)^\alpha \quad (53)$$

$$= \frac{(-1)^\alpha}{(s-1)s^\alpha} \left( (-1)^\alpha + \sum_{n=1}^{\infty} i_n^{(\alpha)} (1 - \frac{1}{s})^n \right) \quad (54)$$

$$= \frac{1}{(s-1)s^\alpha} + \frac{(-1)^\alpha}{s^{\alpha+1}} \sum_{n=0}^{\infty} i_{n+1}^{(\alpha)} (1 - \frac{1}{s})^n, \quad (55)$$

and the inverse Laplace transform of  $\frac{1}{s^{\alpha+1}}(1 - \frac{1}{s})^n$  is given by the associated Laguerre polynomials as [18]

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha+1}}(1 - \frac{1}{s})^n; x\right\} = \frac{n!}{\Gamma(n + \alpha + 1)} x^\alpha L_n^\alpha(x), \quad \Re(\alpha) > -1. \quad (56)$$

Therefore, the following identity holds for the associated Laguerre polynomials and Gregory-Nörlund numbers

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{(s-1)^{\alpha-1}}{s^{2\alpha} \ln^\alpha(s)}; x\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)s^\alpha}; x\right\} + (-1)^\alpha \sum_{n=0}^{\infty} n! \frac{x^\alpha i_{n+1}^{(\alpha)}}{\Gamma(n + \alpha + 1)} L_n^\alpha(x), \\ &= \frac{1}{\Gamma(\alpha)} e^x \gamma(\alpha, x) + (-1)^\alpha \sum_{n=0}^{\infty} n! \frac{i_{n+1}^{(\alpha)} x^\alpha}{\Gamma(n + \alpha + 1)} L_n^\alpha(x), \end{aligned} \quad (57)$$

where  $\gamma(\alpha, x)$  is the lower incomplete gamma function. Now, in order to obtain an integral representation for the Gregory-Nörlund numbers, we use the orthogonality relation of the associated Laguerre polynomials

$$\int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) L_m^\alpha(x) dx = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{m,n}, \quad (58)$$

to get the following integral representation for  $i_{n+1}^{(\alpha)}, n = 0, 1, \dots$ ,

$$(-1)^\alpha i_{n+1}^{(\alpha)} = \int_0^\infty e^{-x} L_n^\alpha(x) \left( \mathcal{L}^{-1}\left\{\frac{(s-1)^{\alpha-1}}{s^{2\alpha} \ln^\alpha(s)}; x\right\} \right) dx - \frac{1}{\Gamma(\alpha)} \int_0^\infty L_n^\alpha(x) \gamma(\alpha, x) dx. \quad (59)$$

The above representation can be simplified into the following relation for  $\alpha \in \mathbb{N}$

$$\begin{aligned} (-1)^\alpha i_{n+1}^{(\alpha)} &= \int_0^\infty e^{-x} L_n^\alpha(x) \left( \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n-1}{k} \mu(x, n-1, 2n-k-1) \right) dx \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^\infty L_n^\alpha(x) \gamma(\alpha, x) dx, \end{aligned} \quad (60)$$

where we used the following facts for the inverse laplace transforms of Volterra functions

$$\mathcal{L}^{-1}\left\{\frac{1}{s^\beta \ln^\alpha(s)}; x\right\} = \mu(x, \alpha-1, \beta-1), \quad (61)$$

$$\mathcal{L}^{-1}\left\{\frac{(s-1)^{n-1}}{s^{2n} \ln^n(s)}; x\right\} = \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n-1}{k} \mu(x, n-1, 2n-k-1). \quad (62)$$

$i_{n+1}^{(\alpha)}$	$i_1^{(\alpha)}$	$i_2^{(\alpha)}$	$i_3^{(\alpha)}$	$i_4^{(\alpha)}$	$i_5^{(\alpha)}$	$i_6^{(\alpha)}$	$i_7^{(\alpha)}$	$i_8^{(\alpha)}$	$i_9^{(\alpha)}$	$i_{10}^{(\alpha)}$
$\alpha = 2$	-1	$\frac{1}{12}$	0	$-\frac{1}{240}$	$-\frac{1}{240}$	$-\frac{221}{60480}$	$-\frac{19}{6048}$	$-\frac{9829}{3628800}$	$-\frac{407}{172800}$	$-\frac{330157}{159667200}$
$\alpha = 3$	$\frac{3}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{240}$	$-\frac{1}{480}$	$-\frac{1}{945}$	$-\frac{11}{20160}$	$-\frac{172800}{199}$	$-\frac{161280}{79}$	$-\frac{15966720}{8213}$
$\alpha = 4$	-2	$\frac{7}{6}$	$-\frac{1}{6}$	$-\frac{1}{720}$	0	$\frac{3024}{1}$	$\frac{3024}{1}$	$\frac{725760}{19}$	$\frac{362880}{1}$	$\frac{47900160}{53}$
$\alpha = 5$	$\frac{5}{2}$	$-\frac{25}{12}$	$\frac{5}{8}$	$-\frac{1}{24}$	0	$\frac{6048}{1}$	$\frac{12096}{1}$	$\frac{725760}{19}$	$-\frac{483840}{1}$	$-\frac{3548160}{1}$

TABLE 1. The first ten Gregory-Nörlund numbers for  $\alpha = 2, 3, 4, 5$ .

At the end, the first ten Gregory-Nörlund numbers have been shown in Table 1 for  $\alpha = 2, 3, 4, 5$ .

### 5. Semi-integrals of powers of Airy functions

In this section, we intend to show new integral representations for the powers of Airy functions, i.e.,  $\text{Ai}^n(x)$ ,  $n = 1, 2, 3, 4$ . These integral representations are written in terms of the semi-integrals and semi-derivatives (the Riemann-Liouville fractional integral and derivative of order  $\frac{1}{2}$ ) of associated functions. For this purpose, we start with the definitions of the fractional integral and derivative, and powers of Airy functions.

**Definition 5.1.** For  $n - 1 < \alpha < n$ ,  $n \in \mathbb{N}$  and  $f \in L_1(a, b)$ , the Riemann-Liouville fractional integrals and derivatives are defined as [14, 17]

$$(I_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) ds, \quad (63)$$

$$(D_{a+}^{\alpha} f)(x) = D^n (I_{a+}^{n-\alpha} f)(x), \quad (64)$$

where  $D = \frac{d}{dx}$ . In similar way, for  $n - 1 < \alpha < n$ ,  $n \in \mathbb{N}$  and  $f \in L_1(\mathbb{R})$ , the Weyl fractional integrals and derivatives are defined as

$$(I_+^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-s)^{\alpha-1} f(s) ds, \quad (65)$$

$$(W_+^{\alpha} f)(x) = D^n (I_+^{n-\alpha} f)(x). \quad (66)$$

**Definition 5.2.** The following integral representations hold for the powers of Airy functions [20, 21, 22]

$$\text{Ai}(x) = \frac{1}{\pi} \int_0^{\infty} \cos\left(xt + \frac{t^3}{3}\right) dt, \quad (67)$$

$$\text{Ai}^2(x) = \frac{1}{2\pi^{\frac{3}{2}}} \int_0^{\infty} t^{-\frac{1}{2}} \cos\left(xt + \frac{t^3}{12} + \frac{\pi}{4}\right) dt, \quad (68)$$

$$\text{Ai}^3(x) = \frac{1}{(18\pi^3)^{\frac{1}{2}}} \int_0^{\infty} t^{-\frac{1}{2}} J_{-\frac{1}{6}}\left(\frac{4}{27}t^3\right) [\cos(\frac{5}{27}t^3 + xt) - \sin(\frac{5}{27}t^3 + xt)] dt, \quad (69)$$

$$\begin{aligned} \text{Ai}^4(x) &= -\frac{3}{16\pi^2} \int_0^{\infty} J_0\left(\frac{1}{32}t^3\right) \sin\left(\frac{5}{96}t^3 + xt\right) dt \\ &\quad - \frac{3}{16\pi^2} \int_0^{\infty} Y_0\left(\frac{1}{32}t^3\right) \cos\left(\frac{5}{96}t^3 + xt\right) dt, \end{aligned} \quad (70)$$

**Theorem 5.1.** The following semi-integrals and semi-derivatives hold for the powers of Airy functions

$$f_n(\sqrt{x}) = \frac{\sqrt{\pi}}{2} I_0^{\frac{1}{2}} \text{Ai}^n(\sqrt{x}), \quad \text{Ai}^n(\sqrt{x}) = \frac{2\sqrt{x}}{\sqrt{\pi}} D_0^{\frac{1}{2}} f_n(\sqrt{x}), \quad n = 1, 2, 3, 4, \quad (71)$$

where

$$f_1(x) = \frac{1}{\pi} \int_0^\infty \left( J_0(\xi x) \cos\left(\frac{\xi^3}{3}\right) - \mathbf{H}_0(\xi x) \sin\left(\frac{\xi^3}{3}\right) \right) d\xi, \quad (72)$$

$$f_2(x) = \frac{1}{4\sqrt{\pi}} \int_0^\infty \xi^{-\frac{1}{2}} \left( J_0(\xi x) \left( \cos\left(\frac{\xi^3}{12}\right) - \sin\left(\frac{\xi^3}{12}\right) \right) - \mathbf{H}_0(\xi x) \left( \cos\left(\frac{\xi^3}{12}\right) + \sin\left(\frac{\xi^3}{12}\right) \right) \right) d\xi, \quad (73)$$

$$\begin{aligned} f_3(x) &= (6\sqrt{2}\pi)^{-1} \int_0^\infty \xi^{\frac{1}{2}} \left( J_0(\xi x) J_{-\frac{1}{6}}\left(\frac{4}{27}\xi^3\right) \cos\left(\frac{5\xi^3}{27}\right) - \mathbf{H}_0(\xi x) J_{-\frac{1}{6}}\left(\frac{4}{27}\xi^3\right) \sin\left(\frac{5\xi^3}{27}\right) \right) d\xi, \\ &- (6\sqrt{2}\pi)^{-1} \int_0^\infty \xi^{\frac{1}{2}} \left( J_0(\xi x) J_{\frac{1}{6}}\left(\frac{4}{27}\xi^3\right) \sin\left(\frac{5\xi^3}{27}\right) + \mathbf{H}_0(\xi x) J_{\frac{1}{6}}\left(\frac{4}{27}\xi^3\right) \cos\left(\frac{5\xi^3}{27}\right) \right) d\xi, \end{aligned} \quad (74)$$

$$\begin{aligned} f_4(x) &= -\frac{3}{32\pi} \int_0^\infty \left( J_0(\xi x) J_0\left(\frac{1}{32}\xi^3\right) \sin\left(\frac{5\xi^3}{96}\right) + \mathbf{H}_0(\xi x) J_0\left(\frac{1}{32}\xi^3\right) \cos\left(\frac{5\xi^3}{96}\right) \right) d\xi, \\ &- \frac{1}{32\pi} \int_0^\infty \left( J_0(\xi x) Y_0\left(\frac{1}{32}\xi^3\right) \cos\left(\frac{5\xi^3}{96}\right) - \mathbf{H}_0(\xi x) Y_0\left(\frac{1}{32}\xi^3\right) \sin\left(\frac{5\xi^3}{96}\right) \right) d\xi. \end{aligned} \quad (75)$$

*Proof.* We use the relations (15) and (17) to construct the Airy function  $\text{Ai}(x)$  in terms of its addition formula in integral (67). After a little algebra, we get

$$\int_0^\infty \left( J_0(\xi x) \cos\left(\frac{\xi^3}{3}\right) - \mathbf{H}_0(\xi x) \sin\left(\frac{\xi^3}{3}\right) \right) d\xi = \frac{2}{\pi} \int_0^x \frac{1}{\sqrt{x^2 - \xi^2}} \text{Ai}(\xi) d\xi, \quad (76)$$

which by applying the suitable change of variables, we obtain the result (71) for  $n = 1$ . In the same procedures, we can show other representations for  $\text{Ai}^n(x)$ ,  $n = 2, 3, 4$ .  $\square$

**Remark 5.1.** If we start by the relations (14) and (16) to construct the powers of Airy functions, then we get new representations in terms of the Weyl fractional integral (65) and fractional derivative (66), see [15].

## 6. Concluding remarks

This paper provides new integral representations for the inverse Laplace transform of multivalued functions involving two conjugate branch points on imaginary axis. These representations were given in terms of the Fourier sine and cosine transforms. In this sense, we obtained new integral representation for some special functions and we get an Abel type integral equation for certain dual integral equations. Also, we presented an integral representation for the Gregory-Nörlund numbers and showed new identities for the powers of the Airy functions with respect to the Riemann-Liouville fractional integrals of order  $\frac{1}{2}$ . This representation for the Airy function can be extended to order  $\alpha$ ,  $0 < \alpha < 1$ , by obtaining new integral representations for the Bessel functions in future works.

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