

## BLOWING UP AND INTEGRABILITY OF HOMOGENEOUS PLANAR VECTOR FIELDS

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*In this paper, we consider the integrability of the strongly degenerate planar polynomial systems, i.e., the systems with no linear term. Using a special blowing up via the Newton polyhedra, we construct a magnification of the singular point. The effectiveness of this blowing up is that to study a quasi-homogeneous system, a single blowing up is enough, instead of iterative blowing ups. We show that for the homogeneous systems and a generic family of quasi-homogeneous systems, this magnification is a desingularization and we give a criterion for the integrability of such systems. We give a rational first integral for those systems which have a first integral.*

**Keywords:** blow up, Newton polyhedra, inverse integral factor, integrability problem, first integral.

### 1. Introduction

Throughout this paper, the sets of integer, real and complex numbers are denoted by  $\mathbb{Z}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  respectively. Also the ring of all polynomials in the variables  $x$  and  $y$  with coefficients in  $\mathbb{C}$ , is denoted by  $C[x, y]$  and  $C(x, y)$  is its quotient field, that is, the field of rational functions.

One of the main problems in the theory of differential equations, is the integrability problem. The existence of a first integral, that is, a function which remains constant along the trajectories of the system, completely determines the phase portrait of the system.

We consider the polynomial vector field

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (1)$$

where  $P(x, y), Q(x, y) \in C[x, y]$  and  $\dot{x}$  denotes the derivative of  $x$  with respect to an independent real or complex variable  $t$ .

We recall that a function  $H \in C[x, y]$  (or a curve  $H = 0$ ) is an invariant function (or curve) for the system (1) on an open set  $U \subseteq \mathbb{C} \times \mathbb{C}$ , if  $H$  is not identically zero and if there exists  $K \in C[x, y]$  such that:

$$P(x, y) \frac{\partial H}{\partial x} + Q(x, y) \frac{\partial H}{\partial y} = KH.$$

The function  $K$  is called the cofactor of the invariant function  $H$ .

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Let  $U \subseteq \mathbb{C} \times \mathbb{C}$  be an open set. A continuous and not locally constant function  $H: U \rightarrow \mathbb{C}$ , is called a first integral of the system (1), if  $H$  is constant on each trajectory of the system contained in  $U$ . Note that if  $H$  is of class at least  $C^1$  (the class of differentiable functions with continuous derivative) in  $U$ , then  $H$  is a first integral if it is not locally constant and satisfies the following partial differential equation

$$P(x, y) \frac{\partial H}{\partial x} + Q(x, y) \frac{\partial H}{\partial y} \equiv 0,$$

in  $U$ . In other words  $H$  is an invariant function with zero cofactor.

We recall that the integrability problem is the problem of finding such a first integral and the functional class where it belongs. It is said that the system has a polynomial first integral if there exists a first integral  $H(x, y) \in C[x, y]$ . Analogously, the system has a rational first integral if there exists a first integral  $H(x, y) \in C(x, y)$ .

Let  $W$  be an open set and consider the function  $V: W \subseteq \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ . We recall that  $V$  is an inverse integrating factor of system (1), if  $V$  is of class  $C^1$ , not locally zero and satisfies the following linear partial differential equation

$$P(x, y) \frac{\partial V}{\partial x} + Q(x, y) \frac{\partial V}{\partial y} = \left( \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} \right) V(x, y),$$

in  $W$ .

Once we know the inverse integrating factor defined in  $W$ , we can compute the first integral in  $U = W \setminus \{V = 0\}$  by the following line integral

$$H(x, y) = \int_{(x_0, y_0)}^{(x, y)} \frac{P(x, y)dy - Q(x, y)dx}{V(x, y)},$$

where  $(x_0, y_0) \in U$  is an arbitrary point.

The integrability of the system (1) for special cases have been studied by various authors see e.g.[4, 7, 9, 12, 13, 14, 15] and references therein. All of them study the eigenvalues of the Jacobian matrix of the system at the singular point, whenever the system either has at least one linear term or is homogeneous (quasi-homogeneous) of degree 2 or 3. Recall that a scalar polynomial  $p: \mathbb{R}^2 \rightarrow \mathbb{R}$  is quasi-homogeneous of type  $(t_1, t_2)$  and degree  $k$  if  $p(\lambda^{t_1} x, \lambda^{t_2} y) = \lambda^k p(x, y)$  for all  $\lambda \in \mathbb{R}$ . The vector field  $F = (P, Q)$  is called quasi-homogeneous of type  $(t_1, t_2)$  and degree  $k$  if  $P, Q$  are quasi-homogeneous polynomials of type  $(t_1, t_2)$  and degree  $t_1 + k, t_2 + k$  respectively. If both the eigenvalues of the Jacobian matrix of (1), at a singular point have nonzero real part, the singular point is called hyperbolic and if just one of the eigenvalues is zero the singular point is called semi-hyperbolic. If both eigenvalues of the Jacobian matrix at a singular point equal zero the singular

point is degenerate. In this case if the Jacobian matrix is not identically zero, the system is called nilpotent. The integrability of a class of nilpotent systems is studied in [6]. If the Jacobian matrix is identically zero, i.e., the system has no linear part, the singular point is called strongly degenerate. One of the ways to study a system with strongly degenerate singular point is the blowing up method. In this method one explodes a singular point to a line or closed curve (divisor). Studying those singular points of the new system which lie on the divisor, can help the study the original singular point. If some of the singular points on the divisor are strongly degenerate, then the blowing up process is repeated.

For the analytic nondegenerate planar systems, the Poincaré Theorem states that the system has a center if and only if there exist analytic first integrals for the system. But, there are analytic nilpotent systems which have a center and do not admit analytic first integrals [16]. The theoretical characterization of centers and analytic first integrals for analytic nilpotent systems can be found in [17]. For the strongly degenerate systems only naive analyses have been carried out, see for instance [2]. The ideas presented in this paper should contribute to a deeper understanding of this case.

In this paper we use a blowing up method that relates to Newton polyhedra of the system. Using this special blowing up, one can avoid the iterative polar or other blowing ups, to study the homogeneous or quasi-homogeneous systems, see Example 2.1 and Example 2.2. In other words all the new singular points which lie on the divisor are hyperbolic or semi-hyperbolic and none of them are degenerate. Using this blowing up method we give an integrability criterion for (quasi-) homogeneous polynomial planar systems. Unlike the previous works our integrability criterion can be applied for the systems with arbitrary degree.

The paper is organized as follows. In Section 2, we give the preliminary definitions and construct a magnification of the origin for an arbitrary planar polynomial system. We prove that this magnification is actually a desingularization of the origin for homogeneous vector fields and a generic family of quasi-homogeneous vector field. In Section 3, using the desingularization built in Section 2, we give an integrability criterion for homogeneous vector fields. We give a rational first integral for those systems which have a first integral.

## 2 Blow up via the Newton polyhedra

In this section we recall the blowing up method using the Newton polyhedra. Indeed it is a *magnification* of the origin, associated to the *skeleton of simple fan extending* of the vectors which are normal to *Newton polyhedra*. This magnification was introduced by Brunella and Miari [8] and it often prevents the iterative polar or quasi-polar blowing up processes, see Example 2.1. Throughout the following sections we suppose that the origin is an isolated singular point of the

vector fields we deal with. So it is assumed that the degree of homogeneous systems is greater than one and the homogeneous linear systems are not the subject of our study.

**Definition 2.1** [8] A *magnification* of  $0 \in \mathbb{R}^2$  is a pair  $(M, \pi)$  such that:

1. the set  $M$  is a  $C^\infty$  2-dimensional manifold;  $\pi: M \rightarrow \mathbb{R}^2$  is a  $C^\infty$  (infinitely many differentiable), surjective and proper map.
2. Let  $Z = \pi^{-1}(0)$  be the divisor of the magnification. Then  $Z$  is union of one-dimensional manifolds in general position on  $M$ . Furthermore  $\pi|_{M \setminus Z}$  is a diffeomorphism from  $M \setminus Z$  to  $\mathbb{R}^2 \setminus \{0\}$ .

A *desingularization* of a vector field  $X$  is a magnification  $(M, \pi)$  such that:

3. if  $\tilde{X}$  is the unique vector field on  $M$  defined by means of the following commutative diagram

$$\begin{array}{ccc}
 TM & \xrightarrow{\pi_*} & TR^2 \\
 \uparrow \tilde{X} & & \uparrow X \\
 M & \xrightarrow{\pi} & R^2
 \end{array}$$

then for any  $p \in Z$  there exist a neighborhood  $U$  of  $p$  and a function  $f$ , different from zero outside  $Z$ , such that  $\tilde{X}|_U = fX$  and if  $\tilde{X}(p) = 0$  then  $p$  is a hyperbolic or semi-hyperbolic singularity of the field  $\tilde{X}$ .

**Proposition 2.1** [8] Suppose that  $(M, \pi)$  is a magnification of  $0 \in \mathbb{R}^2$ . Let  $\tilde{X}, \tilde{Y}$  be the corresponding vector fields of the vector fields  $X, Y$  defined by means of the previous diagram. If  $\tilde{X}, \tilde{Y}$  are topologically equivalent in a neighborhood of  $Z$ , then  $X$  and  $Y$  are topologically equivalent in a neighborhood of  $0 \in \mathbb{R}^2$ .

Let  $F = (P, Q)$  be a planar vector field, where

$$P(x, y) = \sum_{i=0}^n \sum_{j=1}^n a_{ij} x^i y^{j-1}, \quad Q(x, y) = \sum_{i=1}^n \sum_{j=0}^n b_{ij} x^{i-1} y^j.$$

The support of  $F$  is defined as the set  $S = \{(i, j) : (a_{ij}, b_{ij}) \neq (0, 0)\}$ .

Consider the set  $\bigcup_{(i,j) \in S} \{(i, j) + \mathbb{R}_+^2\}$ ,

where  $\mathbb{R}_+^2$  is the positive quadrant. The boundary of the convex hull of this set is made of two open rays and a polygon. The polygon together with the rays that do not lie on a coordinate axis, if there existed, is called the Newton diagram of the vector field  $F$  and we denote it by  $\gamma$ . The component parts of the Newton diagram are called the edges and their endpoints are the vertices of the Newton diagram. If a vertex of the Newton diagram does not lie on a coordinate axis, then it is said to be inner, otherwise, it is an exterior vertex.

The principal part of a vector field  $X$ , is made of those terms in the formal series of  $X$  such that the corresponding points in the support of  $X$  are vertices of the Newton diagram. It was shown in [8] that there is a topological equivalence between the planar vector fields and their principal part for an open and dense family of vector fields.

Let  $\{(a_1, b_1), \dots, (a_k, b_k)\}$  be a collection of  $k$  vectors  $(a_j, b_j) \in \mathbb{R}_+^2 \subset \mathbb{R}^2$  with mutually prime integer components, each normal to an edge  $\gamma_j$  of the Newton diagram  $\gamma$ .

**Definition 2.2** [8] A skeleton of simple fan extending the collection  $\{(a_1, b_1), \dots, (a_k, b_k)\}$ ,

is a finite collection  $\{(\alpha_j, \beta_j)\}$  of vectors  $e_j = (\alpha_j, \beta_j) \in \mathbb{R}_+^2, j = 0, \dots, n, n \in \mathbb{N}$ , each with mutually prime integer components such that

1.  $e_0 = (\alpha_0, \beta_0) = (1, 0); e_n = (\alpha_n, \beta_n) = (0, 1),$
2.  $\{(a_1, b_1), \dots, (a_k, b_k)\} \subseteq \{(\alpha_j, \beta_j)\}_{j=0}^n,$
3.  $\det \begin{pmatrix} \alpha_{j-1} & \alpha_j \\ \beta_{j-1} & \beta_j \end{pmatrix} = 1, j = 1, \dots, n.$

**Remark 2.1** Condition (3) of the definition implies, in particular, that any pair of consecutive vectors  $B_j = \{e_{j-1}, e_j\}$  is a basis of  $\mathbb{Z}^2$ ; the inverse of the matrix

$$\begin{pmatrix} \alpha_{j-1} & \alpha_j \\ \beta_{j-1} & \beta_j \end{pmatrix},$$

still has integer elements and furthermore the matrix defining the transformation from the basis  $B_j$  to the basis  $B_l$ ,  $j, l = 1, \dots, n, j \neq l$ , has integer elements and its determinant equals one.

Now we associate a manifold  $M$  to the skeleton  $\{e_j\}_{j=0}^n$  by means of the transition maps between two local charts on  $M$ .

To every pair  $B_j = \{e_{j-1}, e_j\}$  a chart  $(\Phi_j, U_j)$  on  $M$ ,  $\Phi_j : U_j \rightarrow \mathbb{D}^2, p \mapsto (x_j, y_j)$  is associated. The transition map from  $(\Phi_j, U_j)$  to  $(\Phi_l, U_l)$ ,  $j \neq l$ , is given by

$$h_{jl}(x_j, y_j) = (x_j^{a_{jl}} y_j^{c_{jl}}, x_j^{b_{jl}} y_j^{d_{jl}}), \quad (2)$$

where  $(x_j, y_j) = \Phi_j(p)$  for  $p \in U_j$  and  $a_{jl}, b_{jl}, c_{jl}, d_{jl}$  are defined by

$$\begin{pmatrix} e_{j-1} \\ e_j \end{pmatrix} = \begin{pmatrix} a_{jl} & b_{jl} \\ c_{jl} & d_{jl} \end{pmatrix} \begin{pmatrix} e_{l-1} \\ e_l \end{pmatrix}. \quad (3)$$

The set  $M$  obtained by glueing the  $n$  copies of  $\mathbb{R}^2$  by means of the  $n(n-1)$  maps  $h_{jl}$  is an analytic manifold [8]. In order to construct the map  $\pi: M \rightarrow \mathbb{R}^2$ , one expresses  $B_j = \{e_{j-1}, e_j\}$  in terms of the basis  $\{e_0, e_n\}$ ; the local representation of  $\pi$  in the chart  $(\Phi_j, U_j)$  is defined by

$$h_j(x_j, y_j) = (x_j^{\alpha_{j-1}} y_j^{\alpha_j}, x_j^{\beta_{j-1}} y_j^{\beta_j}). \quad (4)$$

Note that  $h_j^{-1}$  is still a monomial map with integer exponents

$$h_j^{-1}(x, y) = (x^{\beta_j} y^{-\alpha_j}, x^{-\beta_{j-1}} y^{\alpha_{j-1}}). \quad (5)$$

The maps  $h_j: U_j \rightarrow \mathbb{R}^2, j = 1, \dots, n$  are compatible and define an analytic map  $\pi: M \rightarrow \mathbb{R}^2$ , which is surjective and proper. The divisor  $Z = \pi^{-1}(0)$  is the union of circles  $S^1$  intersecting transversally. Furthermore,  $\pi: M \setminus Z \rightarrow \mathbb{R}^2 \setminus \{0\}$  is a diffeomorphism. The pair  $(M, \pi)$  is therefore a magnification of  $0 \in \mathbb{R}^2$ , [8].

We remark that  $Z$  is expressed locally by :

- $\{y_1 = 0\}$  in the chart  $(\Phi_1, U_1)$ ;
- $\{x_j = 0\} \cup \{y_j = 0\}$  in the chart  $(\Phi_j, U_j)$ ,  $j = 2, \dots, n-1$ ;
- $\{x_n = 0\}$  in the chart  $(\Phi_n, U_n)$ .

In the following example we show how this blowing up is effective to shorten the blowing up process.

**Example 2.1** Consider the system

$$\begin{aligned} \dot{x} &= y + ax^2 + \mathcal{O}(|x, y|^3), \\ \dot{y} &= bx^2 + \mathcal{O}(|x, y|^3). \end{aligned} \quad (6)$$

Using two iterated polar blow ups, some change of variables and normal form theory, it was shown in [11, pp. 362-364] that the origin is a cusp of the system (6). The Newton diagram of the system (6) has two vertices  $(0,2), (3,0)$  and the associated extending skeleton of simple fan is

$$\{e_0 = (1,0), e_1 = (1,1), e_2 = (2,3), e_3 = (1,2), e_4 = (0,1)\}.$$

The principal part of the system (6) is

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= bx^2. \end{aligned} \quad (7)$$

It is a quasi-homogeneous system of type (2,3) and degree 1. Using the blow up associated with the Newton polyhedra  $(x, y) \rightarrow (x^2 y, x^3 y^2)$  on the chart  $(U_3, \Phi_3)$  (follow by canceling a common factor  $x$ ), the system can be written as

$$\begin{aligned}\dot{x} &= \frac{xy}{2} + \frac{a}{2}x^2, \\ \dot{y} &= b - \frac{3}{2}y^2 - \frac{3a}{2}xy.\end{aligned}\tag{8}$$

The system (8) has two singular points  $(0, \pm\sqrt{2b/3})$ , both of them are saddles. The phase portrait of the system after blowing up and going back through blowing up (shrinking the divisor to a point) is shown in Fig.1.

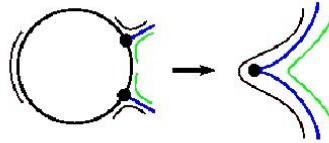


Fig. 1: Cusp at the origin

**Example 2.2** Let  $X = (P_n, Q_n)$  be a homogeneous vector field, where  $P_n(x, y)$ ,  $Q_n(x, y)$  are homogeneous polynomials of degree  $n$ . The Newton diagram of  $X$  has a single side and the vector  $(1, 1)$  is normal to it. So  $\{e_0 = (1, 0)$ ,  $e_1 = (1, 1)$ ,  $e_2 = (0, 1)\}$  is a skeleton of simple fan extending of the set  $\{(1, 1)\}$ . Consider two charts  $(\Phi_1, U_1)$ ,  $(\Phi_2, U_2)$  with the transition map

$$h_{12}(x_1, y_1) = \left( x_1 y_1, \frac{1}{x_1} \right).$$

This map joins together two copies of  $\mathbb{R}^2$  by glueing the half-plane  $x_1 > 0$  with the half-plane  $y_2 > 0$ , and  $x_1 < 0$  with  $y_2 < 0$  but reversing the orientation. The manifold  $M$  so obtained is a Moebius band. On the charts  $(\Phi_i, U_i)$ ,  $i = 1, 2$ , the local representations of  $\pi$  are:

$$\begin{aligned}h_1(x_1, y_1) &= (x_1 y_1, y_1), \\ h_2(x_2, y_2) &= (x_2, x_2 y_2),\end{aligned}$$

and the divisor  $Z$  is expressed by  $\{y_1 = 0\}$  in the chart  $(\Phi_1, U_1)$ ,  $\{x_2 = 0\}$  in the chart  $(\Phi_2, U_2)$ .

**Remark 2.2** The blow up described in Example 2.2 on the chart  $(\Phi_2, U_2)$  is indeed the directional blow up in the  $x$  direction. The system after blowing up on this chart becomes

$$\begin{aligned}\dot{x}_2 &= P_n(x_2, x_2 y_2) = x_2^n P_n(1, y_2), \\ \dot{y}_2 &= \frac{Q_n(x_2, x_2 y_2) - y_2 P_n(x_2, x_2 y_2)}{x_2} = x_2^{n-1} (Q_n(1, y_2) - y_2 P_n(1, y_2)).\end{aligned}\quad (9)$$

Using the change of the variable  $\frac{d\tau}{dt} = x_2^{n-1}$  and denoting the derivative with respect to  $\tau$  with dot, the system (9) becomes

$$\begin{aligned}\dot{x}_2 &= x_2 P_n(1, y_2), \\ \dot{y}_2 &= Q_n(1, y_2) - y_2 P_n(1, y_2).\end{aligned}\quad (10)$$

Similarly on the chart  $(\Phi_1, U_1)$  the magnification via the Newton polyhedra is actually the directional blow up in the  $y$  direction. The system after the blowing up and canceling  $y_1^{n-1}$  becomes

$$\begin{aligned}\dot{x}_1 &= P_n(x_1, 1) - x_1 Q_n(x_1, 1), \\ \dot{y}_1 &= y_1 P_n(x_1, 1).\end{aligned}\quad (11)$$

From now on we suppose that  $P_n, Q_n$  are co-prime in the sense that each element of the set  $\{(x, y) : P_n(x, y) = Q_n(x, y) = 0\}$  is isolated.

**Theorem 2.1** The magnification  $(M, \pi)$  built in Example 2.2, is a desingularization of the origin.

*Proof.* Let  $(M, \pi)$  are such that described in Example 2.2. On the chart  $(\Phi_2, U_2)$  the divisor  $Z$  is expressed by  $\{x_2 = 0\}$  and the local representation of  $\pi$  on this chart is  $h_2(x_2, y_2) = (x_2, x_2 y_2)$ . Under this transformation after dividing by  $x_2^{n-1}$  the system becomes

$$\begin{aligned}\dot{x}_2 &= x_2 P_n(1, y_2), \\ \dot{y}_2 &= Q_n(1, y_2) - y_2 P_n(1, y_2).\end{aligned}\quad (12)$$

Define the function  $V(x_2, y_2) := y_2 P_n(x_2, y_2) - x_2 Q_n(x_2, y_2)$ . Note that  $V$  is a polynomial of degree  $n+1$ . We can factorize  $V$  as

$$V(x_2, y_2) = c(y_2 - \alpha_1 x_2) \dots (y_2 - \alpha_{n+1} x_2), \quad (13)$$

where  $\alpha_1, \dots, \alpha_{n+1} \in \mathbb{C}$ ,  $c \in \mathbb{C}$ .

In the equation (13) note that  $c$  equals the coefficient of  $y^n$  in  $P_n(x, y)$ , since  $P_n, Q_n$  are co-prime we have  $c \neq 0$  and without loss of generality one can suppose that  $c = 1$ . On the exceptional divisor  $x_2 = 0$ , the singularities of the system (12) are the roots of the function

$$V(1, y_2) = y_2 P_n(1, y_2) - Q_n(1, y_2).$$

Thus  $(0, \alpha_1), \dots, (0, \alpha_{n+1})$  are the singular points of the system.

The Jacobian matrix of the system (12) in a neighborhood of the singular point  $(0, \alpha_i)$ ,  $1 \leq i \leq n+1$ , is

$$\begin{pmatrix} P_n(1, y_2) & 0 \\ 0 & \frac{\partial}{\partial y_2}(Q_n(1, y_2) - y_2 P_n(1, y_2)) \end{pmatrix} \Big|_{y_2=\alpha_i}.$$

Denote  $P_n(1, \alpha_i)$  by  $\lambda_i$  and  $\frac{\partial}{\partial y_2}(Q_n(1, y_2) - y_2 P_n(1, y_2))|_{y_2=\alpha_i}$  by  $\mu_i$ . We shall show that  $\lambda_i \neq 0$ , and hence if  $\mu_i \neq 0$  then the singular point is hyperbolic and the singular point is semi-hyperbolic if  $\mu_i = 0$ . If on the contrary  $\lambda_i = 0$  then since  $Q_n(1, \alpha_i) - \alpha_i P_n(1, \alpha_i) = 0$ , we should have  $Q_n(1, \alpha_i) = 0$ . Hence  $P_n, Q_n$  are not co-prime, in contradiction to the hypothesis. Thus  $\lambda_i \neq 0$ . Furthermore let  $\alpha_i$  be a simple root of the function  $Q_n(1, y_2) - y_2 P_n(1, y_2)$ . Then we have

$$\mu_i = \frac{\partial}{\partial y_2}(-V(1, y_2)) \Big|_{y_2=\alpha_i} = - \prod_{j=1, j \neq i}^{n+1} (\alpha_i - \alpha_j),$$

since  $\alpha_i$  is a simple root then  $\mu_i \neq 0$  and in this case the singular point is hyperbolic. A similar argument shows that on the chart  $(\Phi_1, U_1)$  all the singular points are hyperbolic or semi-hyperbolic. Thus  $(M, \pi)$  is a desingularization of the the origin.

### 3 Integrability of homogeneous systems

In this section we use the desingularization built in Example 2.2, in order to solve the integrability problem for the homogeneous planar vector fields. Consider the homogeneous planar polynomial system

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y), \quad (14)$$

where  $P_n, Q_n$  are co-prime homogeneous polynomials of degree  $n$ . Associate to the system (14) the vector field  $F = (P_n, Q_n)$ .

**Lemma 3.1** Using the same notation as in the proof of Theorem 1, we have

$$\sum_{i=1}^{n+1} \frac{\lambda_i}{\mu_i} = -1.$$

*Proof.* Consider the function

$$\frac{P_n(1, y)}{Q_n(1, y) - y P_n(1, y)}.$$

The degree of the numerator of this fraction is smaller than the degree of its denominator. Using the decomposition in simple fractions of a rational function, by some direct calculation one can easily show that

$$\frac{P_n(1, y)}{Q_n(1, y) - yP_n(1, y)} = \sum_{i=1}^{n+1} \frac{\frac{\lambda_i}{\mu_i}}{(y - \alpha_i)}. \quad (15)$$

The multiplier of  $y^n$  in the numerator of the left side of the equation (15) equals  $-1$ , but in the numerator of the right hand side the multiplier is  $\sum_{i=1}^{n+1} \frac{\lambda_i}{\mu_i}$ , as desired.

**Theorem 3.1** If all the roots of  $V(x, y)$  are simple, then the homogeneous vector field has a rational first integral.

*Proof.* Since all the roots of  $V$  are simple, using the similar notation as the proof of Theorem 1 we have  $\lambda_i \neq 0, \mu_i \neq 0$ . Let  $H(x, y) = \prod_{i=1}^{n+1} (y - \alpha_i x)^{\frac{\lambda_i}{\mu_i}}$ , we show that  $H$  is a first integral for the system (14). On the chart  $(\Phi_2, U_2)$  the system (14) becomes

$$\begin{aligned} \dot{x}_2 &= x_2 P_n(1, y_2), \\ \dot{y}_2 &= Q_n(1, y_2) - y_2 P_n(1, y_2). \end{aligned} \quad (16)$$

Thus  $H(x, y)$  is a first integral for the system (14) if and only if  $H(x_2, x_2 y_2)$  is a first integral for the system (16). We have

$$\begin{aligned} H(x_2, x_2 y_2) &= \prod_{i=1}^{n+1} (x_2 y_2 - \alpha_i x_2)^{\left(\frac{\lambda_i}{\mu_i}\right)} \\ &= x_2^{\left(\sum_{i=1}^{n+1} \frac{\lambda_i}{\mu_i}\right)} \prod_{i=1}^{n+1} (y_2 - \alpha_i)^{\left(\frac{\lambda_i}{\mu_i}\right)} \\ &= \frac{1}{x_2} \prod_{i=1}^{n+1} (y_2 - \alpha_i)^{\left(\frac{\lambda_i}{\mu_i}\right)}. \end{aligned}$$

By some direct calculation one has

$$\begin{aligned} \frac{\partial H(x_2, x_2 y_2)}{\partial x_2} x_2 P_n(1, y_2) + \frac{\partial H(x_2, x_2 y_2)}{\partial y_2} (Q_n(1, y_2) - y_2 P_n(1, y_2)) &= \\ H(x_2, x_2 y_2) \left( -P_n(1, y_2) + \sum_{i=1}^{n+1} \frac{\lambda_i}{\mu_i} \left( \frac{Q_n(1, y_2) - y_2 P_n(1, y_2)}{y_2 - \alpha_i} \right) \right) &\equiv 0. \end{aligned} \quad (17)$$

To show the last equivalence note that

$$\frac{Q_n(1, y_2) - y_2 P_n(1, y_2)}{y_2 - \alpha_i} = - \prod_{j=1, j \neq i}^{n+1} (y_2 - \alpha_j),$$

since  $\lambda_i = P_n(1, \alpha_i)$  and  $\mu_i = \prod_{j=1, j \neq i}^{n+1} (\alpha_i - \alpha_j)$  we have

$$\sum_{i=1}^{n+1} \frac{\lambda_i}{\mu_i} \left( \frac{Q_n(1, y_2) - y_2 P_n(1, y_2)}{y_2 - \alpha_i} \right) = \sum_{i=1}^{n+1} P_n(1, \alpha_i) \prod_{j=1, j \neq i}^{n+1} \frac{y_2 - \alpha_i}{\alpha_i - \alpha_j}.$$

The last one is the expression of the Lagrange polynomial which interpolates the  $n+1$  points  $(\alpha_i, P_n(1, \alpha_i)), i = 1, \dots, n+1$ , thus it equals  $P_n(1, y_2)$  and the equivalence in (17) obtained. By a similar argument one can easily see that on the chart  $(\Phi_1, U_1)$ ,  $H(x, y)$  is a first integral for the system (14).

**Remark 3.1** The magnification constructed in Section 2 is a desingularization for a quasi-homogeneous systems too and our method can be used to solve the integrability problem for quasi-homogeneous systems. The quasi-homogeneous systems has recently considered for their relation with the principal part of a system in generalizations of Hartman-Grobman theorem, for instance see [1, 3, 5, 6].

**Remark 3.2** It was shown in [10] that the function  $V(x, y) = xQ_n(x, y) - yP_n(x, y)$  is an inverse integrating factor for the system (14).

One can prove (for example using a generalization of the Euler's formula) that a quasi-homogeneous vector field  $(P, Q)$  of type  $(t_1, t_2)$  has an inverse integrating factor in the form  $V = t_1 x Q - t_2 y P$ .

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