

RELATION-THEORETIC CONTRACTION PRINCIPLE IN SYMMETRIC SPACES

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In this paper, we prove two contraction mapping theorems in symmetric spaces under a binary relation which generalize a multitude of fixed point theorems of the existing literature. Examples have been furnished to substantiate the utility of our newly proved results.

Keywords: Fixed points, symmetric spaces, binary relations, property (W_3) , regular spaces.

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1. Introduction

In 1922, Polish mathematician Banach [1] formulated one of the most natural and useful theorems ever proved in analysis often referred as the Banach contraction principle which asserts that every contraction map on a complete metric space admits a unique fixed point. This classical result continues to inspire researchers of metric fixed point theory and due to its simplicity and applicability, this principle has been generalized and improved in various ways.

One way of improving the celebrated Banach contraction principle is to enlarge the class of underlying spaces and generalizing the principle on those larger classes of spaces. This has led to the extension of fixed point theory to several variants of metric spaces, e.g., rectangular metric spaces [10], generalized metric spaces [11–13], partial metric spaces [14–16], b-metric spaces [17–19], partial b-metric spaces [20], symmetric spaces [3], quasi metric spaces [21], quasi-partial metric spaces [22] and many more.

In 1976, Cicchese [3] initiated the study of fixed points for contraction mappings in symmetric spaces. Indeed the idea of such spaces is due to Wilson [2]. By now, there exists a considerable literature on fixed point theory in symmetric spaces. For the work of this kind one can be referred to [26–28]. In several noted articles written in subsequent years, numerous fixed point results in this setting were established which include Jachymski et al. [29], Hicks and Rhoades [30], Aamri and El Moutawakil [31], Aamri et al. [32], Aliouche [33] and Imdad et al. [34]. In 2014, Bessenyei and Páles [8] proved a novel fixed point theorem for regular symmetric spaces which inspires our results of the present paper.

On the other hand, in 2004, Ran and Reurings [5] obtained a very useful generalization of the Banach contraction principle in a partially ordered complete metric space under a relatively weaker contraction condition which is required to hold merely on the elements which are comparable with respect to the associated partial ordering. In doing so, they were essentially motivated by Turinici [4]. The result of Ran and Reurings was further improved

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by Nieto and Rodríguez-López in 2005 [6] and in 2007 [7]. Subsequently, in 2015, Alam and Imdad [23] presented a natural extension of the Banach contraction principle in a complete metric space equipped with an arbitrary binary relation (not necessarily a partial ordering).

In this paper, our aim is to extend the relation-theoretic analogue of the Banach contraction principle due to Alam and Imdad [23] to the class of symmetric spaces possessing the property (W_3) . We also deduce the corresponding result for regular symmetric spaces.

2. Symmetric Spaces

In this section, we collect some relevant background materials (especially about symmetric spaces) to make our presentation possibly self-contained. Throughout the paper $\mathbb{N}, \mathbb{N}_0, \mathbb{Q}, \mathbb{R}$, and \mathbb{R}^+ denote the set of natural numbers, whole numbers, rational numbers, real numbers and nonnegative real numbers respectively.

Firstly, we summarize some basic ideas on symmetric spaces essentially due to Wilson [2].

Definition 2.1. Let E be a nonempty set and ρ a mapping from $E \times E$ to \mathbb{R}^+ satisfying the following axioms:

- (i) $\rho(r, s) = 0$ if and only if $r = s$,
- (ii) $\rho(r, s) = \rho(s, r)$ for each $r, s \in E$.

Then ρ is a symmetric on E and the pair (E, ρ) is called a symmetric space.

In such spaces, the notions of convergent and Cauchy sequences are defined in the usual way. A sequence $\{r_n\} \subset E$ is said to be convergent to $r \in E$ if $\lim_{n \rightarrow \infty} \rho(r_n, r) = 0$. Also, a sequence is Cauchy if for each $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that $\rho(r_n, r_m) < \epsilon \forall n, m \geq N$. The space E is said to be complete if every Cauchy sequence in E converges. For an open ball with center at p and radius ϵ , we use the notation $B(p, \epsilon)$. The diameter of $B(p, \epsilon)$ is the supremum of distances taken over the pairs of points of the ball. The topology of such spaces is the topology induced by the open balls.

In this context, due to the absence of triangle inequality, the following problems arise:

- (a) there is nothing to assure the uniqueness of limits of convergent sequences and consequently these spaces are not Hausdorff in general,
- (b) a convergent sequence is not necessarily a Cauchy sequence,
- (c) the mapping $\rho : E \times E \rightarrow \mathbb{R}$ is not continuous in general.

To avoid the aforementioned difficulties, we need some additional axioms to establish fixed point theorems in such spaces. The following axioms have been quite instrumental in the literature.

- (W_3) : For $\{r_n\}, r$ and s in E ;

$$\rho(r_n, r) \rightarrow 0 \text{ and } \rho(r_n, s) \rightarrow 0 \implies r = s.$$

- (W_4) : For $\{r_n\}, \{s_n\}$ and r in E ;

$$\rho(r_n, r) \rightarrow 0 \text{ and } \rho(r_n, s_n) \rightarrow 0 \implies \rho(s_n, r) \rightarrow 0.$$

- (HE) : For $\{r_n\}, \{s_n\}$ and r in E ;

$$\rho(r_n, r) \rightarrow 0 \text{ and } \rho(s_n, r) \rightarrow 0 \implies \rho(r_n, s_n) \rightarrow 0.$$

- $(1C)$: For $\{r_n\}, r$ and s in E ;

$$\rho(r_n, r) \rightarrow 0 \implies \rho(r_n, s) \rightarrow \rho(r, s).$$

If (E, ρ) satisfies the property $(1C)$ then the symmetric ρ is called 1-continuous.

- (CC) : For $\{r_n\}, \{s_n\}$ and r and s in E ;

$$\rho(r_n, r) \rightarrow 0 \text{ and } \rho(s_n, s) \rightarrow 0 \implies \rho(r_n, s_n) \rightarrow \rho(r, s).$$

If (E, ρ) satisfies the property (CC) then the symmetric ρ is called continuous.

We observe that

$$(CC) \implies (1C), (W_4) \implies (W_3) \text{ and } (1C) \implies (W_3),$$

but the converse of the above implications are not true in general. Moreover, (CC) implies all the other four conditions, namely $(W_3), (W_4), (HE)$ and $(1C)$.

Assuming these additional axioms optimally, several fixed point results have been established in the setting of symmetric spaces. Cicchese [3] firstly established a variant of the Banach contraction principle on bounded symmetric spaces having the property (W_3) . A slightly modified version of this classical result on symmetric space (without boundedness) is available in literature, which runs as follows:

Theorem 2.1. *Let (E, ρ) be a symmetric space which is complete and enjoys the property (W_3) . Let f be a self mapping on E and there exists some $\alpha \in (0, 1)$ such that*

$$\rho(fr, fs) \leq \alpha \rho(r, s)$$

$\forall r, s \in E$. If there exists $r_0 \in E$ such that

$$\delta(\rho, f, r_0) = \sup_{i, j \in \mathbb{N}} \{\rho(f^i r_0, f^j r_0)\} < \infty,$$

then f possesses a unique fixed point $\bar{r} \in E$. Moreover, the sequence $\{f^n r_0\}$ converges to \bar{r} .

Recently, Bessenyei and Páles [8] recognized a new class of symmetric spaces and termed such spaces as regular symmetric spaces.

Definition 2.2. [8] *Let (E, ρ) be a symmetric space. A function $\varphi : \overline{\mathbb{R}}_+^2 \rightarrow \overline{\mathbb{R}}_+$ is called a triangle function with respect to the symmetric ρ if*

- (i) φ is symmetric,
- (ii) φ is monotonically increasing in both the arguments,
- (iii) $\varphi(0, 0) = 0$,
- (iv) $\rho(r, s) \leq \varphi(\rho(r, t), \rho(s, t))$ for all $r, s, t \in E$.

Proposition 2.1. [8] *Every symmetric space (E, ρ) admits a unique triangle function Φ_ρ such that $\Phi_\rho \leq \varphi$, where φ is any other triangle function with respect to ρ .*

Such a unique triangle function Φ_ρ is called the basic triangle function.

Definition 2.3. [8] *A symmetric space (E, ρ) is said to be a regular space if the basic triangle function with respect to the symmetric ρ is continuous at the origin $(0, 0)$.*

The following lemma ensures the adequacy of the topology of regular symmetric spaces.

Lemma 2.1. [8] *The topology of a regular symmetric space is always Hausdorff. A convergent sequence in a regular symmetric space possesses a unique limit and it has the Cauchy property. Moreover, a symmetric space (E, ρ) is regular if and only if*

$$\lim_{\epsilon \rightarrow 0} \sup_{p \in E} \text{diam } B(p, \epsilon) = 0.$$

We begin with the following crucial observation.

Proposition 2.2. *Every regular symmetric space possesses the property (W_3) .*

Proof. Let (E, ρ) is a regular symmetric space. Let $\Phi_\rho : \overline{\mathbb{R}}_+^2 \rightarrow \overline{\mathbb{R}}_+$ be the basic triangle function. From definition of regular spaces we know that Φ_ρ is continuous at the origin $(0,0)$.

Let $\{r_n\}$ be a sequence in E such that $\rho(r_n, r) \rightarrow 0$ and $\rho(r_n, s) \rightarrow 0$. We want to show that $r = s$. Let ϵ be any arbitrarily chosen positive number. As $\Phi_\rho(0,0) = 0$, there is a neighbourhood V of the origin such that $\Phi_\rho(u, v) < \epsilon \forall (u, v) \in V$. In other words, $\exists \delta > 0$ such that, $\Phi_\rho(u, v) < \epsilon \forall u, v : 0 \leq u, v \leq \delta$. Using property (iv) of triangle functions, we have

$$\rho(r, s) \leq \Phi_\rho(\rho(r, r_n), \rho(r_n, s)).$$

Now, as $\rho(r_n, r) \rightarrow 0$ and $\rho(r_n, s) \rightarrow 0$, there exist $N, M \in \mathbb{N}$ such that

$$\rho(r_n, r) < \delta \forall n \geq N \text{ and } \rho(r_n, s) < \delta \forall n \geq M.$$

Therefore, for $n \geq \max\{N, M\}$, we have

$$\rho(r, s) \leq \Phi_\rho(\delta, \delta) < \epsilon.$$

ϵ being arbitrary, we conclude $r = s$. Thus we see that (E, ρ) has the property (W_3) . \square

Recently, Bessenyei and Páles [8] proved a fixed point theorem under nonlinear contraction on regular symmetric spaces. Its variant under linear contraction assumes the following form.

Theorem 2.2. *Let (E, ρ) is a regular symmetric space which is complete and f a self-mapping on E . Suppose that there exists some $\alpha \in (0, 1)$ such that the following holds:*

$$\rho(fr, fs) \leq \alpha(\rho(r, s)) \forall r, s \in E.$$

Then f has a unique fixed point.

3. Relation-theoretic Notions

Before we proceed to the main results, we recall some relevant notions regarding binary relations which will be needed in our main results.

Definition 3.1. [39] *A binary relation \mathcal{R} on a nonempty set E is a subset of $E \times E$. For $r, s \in E$ when $(r, s) \in \mathcal{R}$, we say r is related to s . Sometimes, we write $r\mathcal{R}s$ instead of $(r, s) \in \mathcal{R}$. If $(r, s) \notin \mathcal{R}$, we say r is not related to s . A binary relation \mathcal{R} is said to be connected if either $(r, s) \in \mathcal{R}$ or $(s, r) \in \mathcal{R}$ for all $r, s \in E$.*

Definition 3.2. [40] *Let E be a nonempty set and \mathcal{R} be a binary relation on E . For $r, s \in E$, we say that r and s are \mathcal{R} -comparative if either $(r, s) \in \mathcal{R}$ or $(s, r) \in \mathcal{R}$. When r and s are \mathcal{R} -comparative, we denote it by $[r, s] \in \mathcal{R}$.*

The following result is immediate directly from symmetry of ρ .

Proposition 3.1. [23] *Let f be a self mapping on a symmetric space (E, ρ) endowed with a binary relation \mathcal{R} . Then the following conditions are equivalent:*

- (i) $\rho(fr, fs) \leq \alpha(\rho(r, s)) \forall (r, s) \in \mathcal{R}$,
- (ii) $\rho(fr, fs) \leq \alpha(\rho(r, s)) \forall [r, s] \in \mathcal{R}$.

Definition 3.3. [36] *Let E be a nonempty set endowed with a binary relation \mathcal{R} . A sequence $\{r_n\} \subset E$ is said to be \mathcal{R} -preserving if $(r_n, r_{n+1}) \in \mathcal{R} \forall n \in \mathbb{N}$.*

Definition 3.4. [23] *Let E be a nonempty set endowed with a binary relation \mathcal{R} and f a self-mapping on E . The relation \mathcal{R} on E is said to be*

- *f -closed if (for any $r, s \in E$)*

$$(r, s) \in \mathcal{R} \Rightarrow (fr, fs) \in \mathcal{R}.$$

- f -transitive if (for any $r, s, t \in E$)

$$(fr, fs), (fs, ft) \in \mathcal{R} \Rightarrow (fr, ft) \in \mathcal{R}.$$

Definition 3.5. [39] Let E be a nonempty set endowed with a binary relation \mathcal{R} and $U \subset E$. Then the restriction of \mathcal{R} to U is the set $\mathcal{R} \cap U^2$ and is denoted by $\mathcal{R}|_U$.

Definition 3.6. [24] Let E be a nonempty set endowed with a binary relation \mathcal{R} . The relation \mathcal{R} is said to be locally transitive if for any \mathcal{R} -preserving sequence $\{r_n\} \subset E$ the binary relation $\mathcal{R}|_U$ is transitive, where $U = \{r_n | n \in \mathbb{N}_0\}$.

Definition 3.7. [24] Let E be a nonempty set endowed with a binary relation \mathcal{R} and f a self-mapping on E . The relation \mathcal{R} is said to be locally f -transitive if for any \mathcal{R} -preserving sequence $\{r_n\} \subset f(E)$ the binary relation $\mathcal{R}|_U$ is transitive, where $U = \{r_n | n \in \mathbb{N}_0\}$.

Definition 3.8. [25, 40] Let E be a nonempty set endowed with a binary relation \mathcal{R} . A subset U of E is said to be \mathcal{R} -connected if for each pair $r, s \in U$, there is a finite sequence $\{r_1, r_2, \dots, r_k\} \subset E$ satisfying the following conditions:

- (i) $r_0 = r$ and $r_k = s$,
- (ii) $(r_i, r_{i+1}) \in \mathcal{R}$ for each i ($0 \leq i \leq k-1$).

The sequence $\{r_1, r_2, \dots, r_k\}$ is called a path in \mathcal{R} from r to s .

Definition 3.9. [24] Let E be a nonempty set endowed with a binary relation \mathcal{R} . A subset U of E is said to be \mathcal{R}^s -connected if for each pair $r, s \in U$, there is a finite sequence $\{r_1, r_2, \dots, r_k\} \subset E$ that satisfies the following conditions:

- (i) $r_0 = r$ and $r_k = s$,
- (ii) $[r_i, r_{i+1}] \in \mathcal{R}$ for each i ($0 \leq i \leq k-1$).

Now, we define the analogues of the notions of ρ -self-closedness, \mathcal{R} -continuity and \mathcal{R} -completeness due to [24, 25] in the framework of symmetric spaces.

Definition 3.10. A binary relation \mathcal{R} on a symmetric space (E, ρ) is said to be ρ -self-closed if for every \mathcal{R} -preserving sequence $\{r_n\}$ in E converging to r , there exists a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ with $[r_{n_k}, r] \in \mathcal{R}$.

Definition 3.11. Let (E, ρ) be a symmetric space endowed with a binary relation \mathcal{R} . A mapping $f : E \rightarrow E$ is \mathcal{R} -continuous at $r \in E$ if for any \mathcal{R} -preserving sequence $\{r_n\}$ (in E) converging to r , we have $fr_n \rightarrow fr$. Moreover, f is called \mathcal{R} -continuous if it is so at each point of E .

Definition 3.12. Let (E, ρ) be a symmetric space endowed with a binary relation \mathcal{R} . We say that (E, ρ) is \mathcal{R} -complete if every \mathcal{R} -preserving Cauchy sequence in E converges.

The following notations will be used in this paper:

- (i) $\text{Fix}(f) := \{r \in E : f(r) = r\}$,
- (ii) $E(f, \mathcal{R}) := \{r \in E : (r, fr) \in \mathcal{R}\}$.

4. Main Results

Our main result remains a relation theoretic variant of Theorem 2.1.

Theorem 4.1. Let (E, ρ) be a symmetric space which enjoys the property (W_3) and \mathcal{R} a binary relation on E . Let f be a self-mapping on E and the following conditions hold:

- (a) (E, ρ) is \mathcal{R} -complete,
- (b) \mathcal{R} is locally f -transitive and f -closed,
- (c) either f is \mathcal{R} -continuous or \mathcal{R} is ρ -self-closed,
- (d) there is $r_0 \in E(f, \mathcal{R})$ such that $\delta(\rho, f, r_0) = \sup_{i,j \in \mathbb{N}} \rho(f^i r_0, f^j r_0) < \infty$,

(e) there exists some $\alpha \in [0, 1)$ such that

$$\rho(fr, fs) \leq \alpha(\rho(r, s)) \quad \forall r, s \in E \text{ with } (r, s) \in \mathcal{R}.$$

Then f possesses a fixed point in E . In addition, if

(f) $f(E)$ is connected.

Then the fixed point of f in E is unique.

Proof. In view of (d), there is some $r_0 \in E$ such that

$$\delta(\rho, f, r_0) = \sup_{i, j \in \mathbb{N}} \rho(f^i r_0, f^j r_0) < \infty.$$

Set $M := \delta(\rho, f, r_0)$. Then, $0 \leq M < \infty$. As \mathcal{R} is f -closed, for the given $r_0 \in E(f, \mathcal{R})$ we obtain (using induction)

$$(fr_0, f^2 r_0), (f^2 r_0, f^3 r_0), \dots, (f^n r_0, f^{n+1} r_0), \dots \in \mathcal{R};$$

which shows that the sequence $\{f^n r_0\}$ is \mathcal{R} -preserving. As \mathcal{R} is locally f -transitive, we have $(f^m r_0, f^n r_0) \in \mathcal{R} \quad \forall n > m$. Using the contraction condition (e), we get

$$\rho(f^{n+i} r_0, f^{n+j} r_0) \leq \alpha \rho(f^{n+i-1} r_0, f^{n+j-1} r_0).$$

Therefore,

$$\delta(\rho, f, f^n r_0) \leq \alpha \delta(\rho, f, f^{n-1} r_0).$$

Similarly, we obtain

$$\begin{aligned} \delta(\rho, f, f^{n-1} r_0) &\leq \alpha \delta(\rho, f, f^{n-2} r_0) \\ \delta(\rho, f, f^{n-2} r_0) &\leq \alpha \delta(\rho, f, f^{n-3} r_0) \\ &\vdots \\ \delta(\rho, f, f^2 r_0) &\leq \alpha \delta(\rho, f, fr_0) \\ \delta(\rho, f, fr_0) &\leq \alpha \delta(\rho, f, r_0), \end{aligned}$$

so that

$$\delta(\rho, f, f^n r_0) \leq \alpha^n \delta(\rho, f, r_0) = \alpha^n M \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now, $\rho(f^{n+1} r_0, f^{n+m} r_0) \leq \delta(\rho, f, f^n r_0)$ and $\delta(\rho, f, f^n r_0) \rightarrow 0$. Thus we conclude that the sequence $\{f^n r_0\}$ is Cauchy. Also, the sequence $\{f^n r_0\}$ is \mathcal{R} -preserving. Hence, \mathcal{R} -completeness of (E, ρ) guarantees the existence of some $r \in E$ such that $f^n r_0 \rightarrow r$.

If f is \mathcal{R} -continuous, then $f(f^n r_0) \rightarrow fr$, i.e., $f^{n+1} r_0 \rightarrow fr$. We observe that $f^n r_0 \rightarrow r$ and $f^n r_0 \rightarrow fr$. As (E, ρ) possesses the property (W_3) , we conclude $fr = r$. Hence $\{f^n r_0\}$ converges to a fixed point of f .

On the other hand, if \mathcal{R} is ρ -self-closed, then there is a subsequence $\{f^{n_k} r_0\}$ of $\{f^n r_0\}$ such that $(f^{n_k} r_0, r) \in \mathcal{R} \quad \forall k \in \mathbb{N}$. Hence,

$$\rho(f^{n_k+1} r_0, fr) = \rho(f(f^{n_k} r_0), fr) \leq \alpha \rho(f^{n_k} r_0, r).$$

As $\rho(f^{n_k} r_0, r) \rightarrow 0$, we obtain $\rho(f^{n_k+1} r_0, fr) \rightarrow 0$. Owing to property (W_3) of E , we obtain $fr = r$. Hence $\{f^n r_0\}$ converges to a fixed point of f .

For uniqueness part; let r, s be two fixed points of f such that $r \neq s$. We see that $r, s \in f(E)$ as $r = f(r)$ and $s = f(s)$. Now, $f(E)$ being connected $[r, s] \in \mathcal{R}$. Therefore,

$$\rho(r, s) = \rho(fr, fs) \leq \alpha \rho(r, s) < \rho(r, s);$$

which is a contradiction. Hence the fixed point of f is unique. This accomplishes the proof. \square

Corollary 4.1. *Under the universal relation Theorem 4.1 reduces to Theorem 2.1. Observe that when $\mathcal{R} = E^2$ the hypotheses of Theorem 4.1 hold trivially.*

Proposition 4.1. *Let \mathcal{R} a binary relation on a regular symmetric space (E, ρ) and f a self-mapping on E . Let \mathcal{R} be f -closed and locally f -transitive. If there exists $\alpha \in (0, 1)$ such that*

$$\rho(fr, fs) \leq \alpha(\rho(r, s)) \quad \forall (r, s) \in \mathcal{R},$$

then for each $r_0 \in E(f, \mathcal{R})$

$$\delta(\rho, f, r_0) = \sup_{i, j \in \mathbb{N}} \{\rho(f^i r_0, f^j r_0)\} < \infty,$$

provided \mathcal{R} is locally f -transitive and f -closed.

Proof. Consider $r_0 \in E(f, \mathcal{R})$. From the definition of $E(f, \mathcal{R})$, we have $(r_0, fr_0) \in \mathcal{R}$. If $f(r_0) = r_0$, then we are done; as

$$\delta(\rho, f, r_0) = \sup_{i, j \in \mathbb{N}} \rho(f^i r_0, f^j r_0) = \sup_{i, j \in \mathbb{N}} \rho(r_0, r_0) = 0 < \infty.$$

Suppose that $fr_0 \neq r_0$. Since $(r_0, fr_0) \in \mathcal{R}$ and \mathcal{R} is f -closed, we get by induction on n that

$$(f^n r_0, f^{n+1} r_0) \in \mathcal{R} \quad \forall n \in \mathbb{N}.$$

Construct the sequence $\{r_n\}$ of Picard iterates with initial point r_0 , i.e.,

$$r_n = f^n(r_0), \text{ so that } (r_n, r_{n+1}) \in \mathcal{R} \quad \forall n \in \mathbb{N}_0.$$

Therefore, the sequence $\{r_n\}$ is \mathcal{R} -preserving. \mathcal{R} being locally f -transitive $(r_n, r_m) \in \mathcal{R} \quad \forall m > n$. Observe that the sequence $\rho(r_n, r_{n+k})$ tends to zero for all fixed $k \in \mathbb{N}$,

$$\begin{aligned} \rho(r_n, r_{n+k}) &= \rho(fr_{n-1}, fr_{n+k-1}) \\ &\leq \alpha(\rho(r_{n-1}, r_{n+k-1})) \\ &\leq \alpha^2(\rho(r_{n-2}, r_{n+k-2})) \\ &\vdots \\ &\leq \alpha^n \rho(r_0, r_k) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Now, we prove that $\{r_n\}$ is Cauchy. Let $\epsilon > 0$ be any positive number. As (E, ρ) is regular, the basic triangle function Φ_ρ is continuous at the origin $(0, 0)$. Therefore, there exists a neighbourhood U of the origin such that $\Phi_\rho(u, v) < \epsilon \quad \forall (u, v) \in U$. In other words, $\exists \delta > 0$ such that, $\Phi_\rho(u, v) < \epsilon \quad \forall u, v : 0 \leq u, v \leq \delta$. We take $\delta < \epsilon$. We can find $N \in \mathbb{N}$ such that $\alpha^N \epsilon < \delta$. Set $T = f^N$. We can see that

$$\rho(Tr, Ts) = \rho(f^N r, f^N s) \leq \alpha^N \rho(r, s) \quad \text{when } (r, s) \in \mathcal{R}.$$

Define $m_k : \rho(r_n, f^k Tr_n) < \delta \quad \forall n \geq m_k$ and set $m = \max\{m_0, m_1, \dots, m_N\}$.

If $V = \{r_m, r_{m+1}, r_{m+2}, \dots, r_{m+k}, \dots\}$ then for any $s \in B(r_m, \epsilon) \cap V, s \neq r_m$

$$\begin{aligned} \rho(f^k Tr_m, f^k Ts) &= \rho(T f^k r_m, T f^k s) \leq \alpha^N \rho(f^k r_m, f^k s) \quad \text{as } (f^k r_m, f^k s) \in \mathcal{R} \\ &\leq \alpha^N \alpha^k \rho(r_m, s) < \alpha^N \rho(r_m, s) < \alpha^N \epsilon < \delta, \end{aligned}$$

yielding thereby

$$\begin{aligned} \rho(f^k Ts, r_m) &\leq \Phi_\rho(\rho(f^k Ts, f^k Tr_m), \rho(f^k Tr_m, r_m)) \\ &\leq \Phi_\rho(\delta, \delta) \quad \forall k = 0, 1, 2, \dots, N \end{aligned}$$

which implies that

$$\rho(f^k Ts, r_m) < \epsilon, \quad \forall k = 0, 1, 2, \dots, N.$$

Also, for $s = r_m$, $\rho(f^k T r_m, r_m) < \delta < \epsilon, \forall k = 0, 1, 2, \dots, N$.

Thus we see that $f^k T$ maps $V \cap B(r_m, \epsilon)$ into itself for all $k = 0, 1, 2, \dots, N$. In particular, each iterate of T maps $V \cap B(r_m, \epsilon)$ into itself (as $T = f^N$). Now, if $n > m$ be an arbitrarily given natural number, i.e., $n = Nk + M$ where $k \in \mathbb{N}_0$ and $0 \leq M < N$, then

$$f^n T = f^{Nk+M} T = f^M T^{k+1}.$$

Henceforth,

$$\begin{aligned} f^n T(V \cap B(r_m, \epsilon)) &= f^M T^{k+1}(V \cap B(r_m, \epsilon)) \\ &= f^M T(T^k(V \cap B(r_m, \epsilon))) \\ &\subset f^M T(V \cap B(r_m, \epsilon)) \\ &\subset V \cap B(r_m, \epsilon); \text{ as } 0 \leq M < N. \end{aligned}$$

Therefore, $f^n T(r_m) \in B(r_m, \epsilon) \forall n > m$, i.e., $r_{m+N+k} \in B(r_m, \epsilon) \forall k \in \mathbb{N}$.

As (E, ρ) is regular, $\text{diam}(r_m, \epsilon) \rightarrow 0$ when $\epsilon \rightarrow 0$, which means the sequence $\{r_n\}$ is a Cauchy sequence. Therefore, for each $r_0 \in E(f, \mathcal{R})$

$$\delta(\rho, f, r_0) = \sup_{i,j \in \mathbb{N}} \rho(f^i r_0, f^j r_0) = \sup_{i,j \in \mathbb{N}} \rho(r_i, r_j) < \infty,$$

as $\rho(r_i, r_j) \rightarrow 0$ when $i, j \rightarrow \infty$. This accomplishes the proof. \square

In view of Propositions 2.2 and 4.1, Theorem 4.1 yields the following consequence.

Corollary 4.2. *Let (E, ρ) be a regular symmetric space endowed with a binary relation \mathcal{R} . Let f be a self-mapping on E and the following conditions hold:*

- (a) $E(f, \mathcal{R})$ is nonempty,
- (b) (E, ρ) is \mathcal{R} -complete,
- (c) \mathcal{R} is locally f -transitive and f -closed,
- (d) either f is \mathcal{R} is ρ -self-closed or \mathcal{R} -continuous,
- (e) there exists $\alpha \in (0, 1)$ satisfying the following:

$$\rho(fr, fs) \leq \alpha(\rho(r, s)) \quad \forall (r, s) \in \mathcal{R}.$$

Then T possesses a fixed point. Moreover, if

- (f) $f(E)$ is connected,
- then the fixed point of f is unique.*

Proof. As (E, ρ) is a regular space, using Proposition 2.2, we infer that it has the property (W_3) . Also, in view of condition (a), there is some $r_0 \in E(f, \mathcal{R})$. From proposition 4.1, we have $\delta(\rho, f, r_0) < \infty$. Hence we observe that all the hypotheses of Theorem 4.1 hold. Therefore, f possesses a unique fixed point in E . \square

Theorem 4.2. *In the hypotheses of Corollary 4.2, if we replace assumption (f) by the following weaker condition:*

- (f') $f(E)$ is \mathcal{R}^s -connected;
- then the fixed point of f is unique.*

Proof. Existence of fixed point is guaranteed from the assumptions (a)-(e) of Corollary 4.2. To show uniqueness let r, s be two fixed points of f such that $r \neq s$. We see that $r, s \in f(E)$ as $r = f(r)$ and $s = f(s)$. As $f(E)$ being \mathcal{R}^s -connected, there exist $r_0, r_1, r_2, \dots, r_k \in E$ satisfying the following conditions:

- (i) $r_0 = r, r_k = s$;
- (ii) $[r_i, r_{i+1}] \in \mathcal{R}$ for each i ($0 \leq i \leq k-1$).

Now, due to condition (ii), we have $\rho(fr_i, fr_{i+1}) \leq \alpha(\rho(r_i, r_{i+1}))$. Using induction, we obtain $\rho(f^n r_i, f^n r_{i+1}) \leq \alpha^n \rho(r_i, r_{i+1})$. For $\epsilon > 0$, $\exists \delta > 0$ such that

$$\Phi_\rho(u, v) < \epsilon \quad \forall u, v : 0 \leq u, v < \delta.$$

Let $\delta_1 = \delta$ and define δ_i ($2 \leq i \leq k-1$) : $\Phi_\rho(u, v) < \delta_{i-1} \quad \forall u, v : 0 \leq u, v < \delta_i$ and set $\gamma = \min\{\delta_1, \delta_2, \dots, \delta_{k-1}\}$.

Also, set $M' = \max\{N_1, N_2, \dots, N_{k-1}\}$ where,

$$N_i : \rho(f^n r_i, f^n r_{i+1}) \leq \alpha^n \rho(r_i, r_{i+1}) < \gamma \quad \forall n \geq N_i.$$

Hence, for $n \geq M'$, we have,

$$\begin{aligned} \rho(f^n r_{k-1}, f^n s) &= \rho(f^n r_{k-1}, f^n r_k) < \gamma \leq \delta_{k-1} \\ \rho(f^n r_{k-2}, f^n s) &\leq \Phi_\rho(\rho(f^n r_{k-2}, f^n r_{k-1}), \rho(f^n r_{k-1}, f^n s)) \\ &\leq \Phi_\rho(\gamma, \delta_{k-1}) \leq \Phi_\rho(\delta_{k-1}, \delta_{k-1}) < \delta_{k-2} \\ \rho(f^n r_{k-3}, f^n s) &\leq \Phi_\rho(\rho(f^n r_{k-3}, f^n r_{k-2}), \rho(f^n r_{k-2}, f^n s)) \\ &\leq \Phi_\rho(\gamma, \delta_{k-2}) \leq \Phi_\rho(\delta_{k-2}, \delta_{k-2}) < \delta_{k-3} \\ &\vdots \\ \rho(f^n r_1, f^n s) &\leq \Phi_\rho(\rho(f^n r_1, f^n r_2), \rho(f^n r_2, f^n s)) \\ &\leq \Phi_\rho(\gamma, \delta_2) \leq \Phi_\rho(\delta_2, \delta_2) < \delta_1 \\ \rho(f^n r, f^n s) &\leq \Phi_\rho(\rho(f^n a, f^n r_1), \rho(f^n r_1, f^n s)) \\ &\leq \Phi_\rho(\gamma, \delta_1) \leq \Phi_\rho(\delta_1, \delta_1) < \epsilon. \end{aligned}$$

This is true for any $\epsilon > 0$. Therefore, $\rho(f^n r, f^n s) = \rho(r, s) = 0$, i.e., $r = s$. Hence, fixed point of f is unique. \square

Corollary 4.3. *Under the universal relation Corollary 4.2 reduces to Theorem 2.2.*

Clearly, under the universal relation $\mathcal{R} = E^2$ the hypotheses of Corollary 4.2 hold trivially.

Finally, we present two examples in support of our newly proved theorems.

We adopt the example below to exhibit that Theorem 4.1 is genuinely different as compared to Corollary 4.2 as well as Theorem 4.2.

Example 4.1. *Let $E = [0, 1)$. Define $\rho : E \times E \rightarrow \mathbb{R}^+$ by*

$$\rho(r, s) = \begin{cases} r + s & \text{if } r \neq s, \\ 1 & \text{if } r = s \neq 0, \\ 0 & \text{if } r = s = 0. \end{cases}$$

Here, it easy to check that (E, ρ) is a symmetric space having the property (W_3) . Consider the binary relation \mathcal{R} on E as given below:

$$\mathcal{R} = \left\{ \left(\frac{1}{m}, \frac{1}{n} \right) \mid m, n \in \mathbb{N}, 5 \leq m < n \right\}.$$

Also, define $f : E \rightarrow E$ such that

$$f(r) = \begin{cases} \frac{r}{2} & \text{if } 0 \leq r \leq \frac{1}{5}, \\ \frac{1}{8}(9r - 1) & \text{if } \frac{1}{5} < r < 1. \end{cases}$$

Then, for all $(r, s) \in \mathcal{R}$, we have

$$\rho(fr, fs) = d\left(\frac{r}{2}, \frac{s}{2}\right) = \frac{r}{2} + \frac{s}{2} \leq \frac{1}{2}\rho(r, s).$$

It follows that f is a contraction for the elements related by \mathcal{R} with Lipschitz constant $\alpha = \frac{1}{2}$. It can be easily seen that rest of the hypotheses of Theorem 4.1 are also satisfied and hence f possesses a fixed point. We can see that 0 is a fixed point of f .

Observe that this map f in the above example is not a contraction on E for any $\alpha \in (0, 1)$ but remains a relational contraction.

Here, it is worth mentioning that the above considered symmetric space (E, ρ) is not a regular one as the associated basic triangle function Φ_ρ is not continuous at the origin $(0, 0)$. The proof is simple and runs as follows:

The basic triangle function $\Phi_\rho : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is defined as (see [8])

$$\Phi_\rho(u, v) = \sup\{\rho(r, s) \mid \exists t \in E : \rho(t, r) \leq u, \rho(t, s) \leq v\}.$$

Consider $u = v = \frac{1}{n}$ for some $n(\geq 2) \in \mathbb{N}$, then

$$\Phi_\rho\left(\frac{1}{n}, \frac{1}{n}\right) = \sup\{\rho(r, s) \mid \exists t \in E : \rho(t, r) \leq \frac{1}{n}, \rho(t, s) \leq \frac{1}{n}\}.$$

Then for $r = s = \frac{1}{3n}$, there exists $t = \frac{1}{2n} \in E = [0, 1)$ such that

$$\rho(t, r) = \rho(t, s) = \frac{1}{3n} + \frac{1}{2n} < \frac{1}{n},$$

so that

$$\Phi_\rho\left(\frac{1}{n}, \frac{1}{n}\right) \geq \rho\left(\frac{1}{3n}, \frac{1}{3n}\right) = 1.$$

This is true for all $n \geq 2$. Therefore, $\lim_{n \rightarrow \infty} \Phi_\rho\left(\frac{1}{n}, \frac{1}{n}\right) \neq 0 = \Phi_\rho(0, 0)$. Thus we conclude that

$$\lim_{u, v \rightarrow 0} \Phi_\rho(u, v) \neq \Phi_\rho(0, 0).$$

Thus it is established that Φ_ρ is not continuous at the origin $(0, 0)$.

Now, we give another example to substantiate the utility of Theorem 4.2.

Example 4.2. Consider $E = \mathbb{R}$ and define a symmetric ρ on E by $\rho(r, s) = (r - s)^2$, then (E, ρ) is a regular symmetric space which is complete. Take a binary relation \mathcal{R} on E as follows:

$$\mathcal{R} = \{(r, s) \in \mathbb{R}^2 : r \geq s \geq 0, r \in \mathbb{Q}\}.$$

Define a mapping $f : E \rightarrow E$ by

$$f(r) = \begin{cases} 2r, & \text{if } r < 0, \\ \frac{r}{3}, & \text{if } r \geq 0. \end{cases}$$

Consider $(r, s) \in \mathcal{R}$, then

$$\rho(fr, fs) = d\left(\frac{r}{3}, \frac{s}{3}\right) = \left(\frac{r}{3} - \frac{s}{3}\right)^2 \leq \frac{1}{9}(r - s)^2 = \frac{1}{9}\rho(r, s)$$

i.e., f is a contraction (with Lipschitz constant $\alpha = \frac{1}{9} < 1$) for those elements which are related. It can be easily shown that all the hypotheses of Corollary 4.2 hold and hence f possesses a unique fixed point.

We note that the map f considered in the above example is not a contraction on the whole space (E, ρ) for any $\alpha \in (0, 1)$ but remains a relational contraction for $\alpha = \frac{1}{9}$.

An observation (on the existing results) we have made is presented below. Under the universal relation $\mathcal{R} = E^2$ Proposition 4.1 reduces to the following result.

Proposition 4.2. Let (E, ρ) be a regular symmetric space and f is a contraction mapping on E . Then for each $r_0 \in E$,

$$\delta(\rho, f, r_0) = \sup_{i, j \in \mathbb{N}} \rho(f^i r_0, f^j r_0) < \infty.$$

Henceforth, by using Propositions 2.2 and 4.2, it can be concluded that the Theorem 2.2, which is in fact a variant (under linear contraction) of the fixed point theorem of Bessenyei and Páles [8], becomes a corollary of Theorem 2.1.

5. Conclusions

The existing literature already contains a multitude of fixed point theorems in symmetric spaces; especially using EA property, common EA property and common limit range property (see [26] and references cited therein). Recently, the authors in [8] proved a fixed point theorem in regular symmetric spaces without using the observations of Wilson [2] in the form of some special properties; namely, (W_3) , (W_4) , (HE) , $(1C)$, (CC) etc. Our results are the first of their kind owing to the use of relational contractions to obtain fixed point theorems in symmetric spaces. Also, our newly proved results exhibit that the two above mentioned approaches are independent of each other.

Our results deduce many well known fixed point theorems if the underlined binary relation is universal. The ideas used in our paper may be utilized in many relatively more general situations (e.g. [37, 38]) by slightly varying the involved conditions. Thus there is every possibility for similar results in near future.

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