

## ON THE NONLINEAR OUTPUT REGULATION PROBLEM – PART 1 – MIMO NONLINEAR SYSTEMS NORMAL FORMS AND A DISCUSSION ON THE NECESSARY CONDITIONS FOR SOLVING THE CONTROL PROBLEM

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*In aceasta lucrare este prezentată problema reglării ieșirii dacă se consideră sisteme neliniare MIMO. În acest context formele normale pentru reprezentarea sistemelor neliniare sunt de mare interes. Două forme normale pentru sisteme neliniare MIMO sunt prezentate. Prima relevă vectorul de grade relative și poate fi aplicată pentru sisteme pătrate inversabile (forma normală clasică), a doua poate fi utilizată și pentru sisteme nepătrate și să dă detalii/informări despre proprietatile de inversabilitate ale sistemului scris în această formă (forma normală recentă). Condițiile necesare pentru rezolvarea problemei reglării ieșirii pentru sisteme scrise în forma clasică și cea recentă sunt prezentate. Din discuția prezentată rezultă că problema pusă se poate rezolva sub presupuneri mai puțin restrictive dacă sistemul este scris în forma normală obținută prin algoritmul structurii zero-urilor la infinit (forma recentă). Această abordare permite de asemenea discutarea problemei de reglare și în cazul sistemelor MIMO nepătrate.*

*In this paper the general problem of output regulation when dealing with nonlinear MIMO systems is presented. In this context, normal forms for nonlinear systems are of great importance. Two normal forms for MIMO nonlinear systems are presented. The first one reveals the relative degrees vector and can be applied for square invertible systems (classic normal form), while the second can be used for non square systems and gives details on the system invertibility properties (recent normal form). The necessary conditions for solving the output regulation problem for the classical normal form and, respectively, the recent normal form are presented. From this discussion it results that the problem can be solved under weaker assumptions if considering the later form (obtained by using the infinite zero structure algorithm). This approach also permits discussing the regulation problem for non square MIMO systems.*

**Keywords:** output regulation, MIMO nonlinear systems, normal forms

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## 1. Introduction

A defining problem in control theory is the design of feedback controllers so as to have certain outputs of a plant to track particular reference trajectories. An appealing idea is dynamic inversion, but this can rarely be carried on in an exact manner through open loop control. In fact, closed-loop control (which achieves an approximate dynamical inversion) is almost always the solution of choice, since, in any realistic scenario, the control goal has to be achieved in spite of a good number of phenomena which would cause unexpected system behavior (for instance: parameter variations, additional undesired inputs).

One particular (deterministic) form of this problem is to consider that the dynamics of the system that generates the references and the disturbances (the exosystem) are known and consequently design a controller that steers to zero certain outputs of the augmented system plant-plus-exosystem, thus achieving what is called the property of *output regulation*. Problems of this kind have been extensively studied in the 1970s for linear MIMO systems; the works of Francis and Wonham for instance, provide an exhaustive presentation of the theory [1, 2]. The results culminated with the Internal Model Principle (IMP), which states that a structurally stable solution (i.e. robust to plant parameter variations) necessarily has to use feedback of the regulated variables and incorporate in the feedback path a (possibly redundant) model of the exosystem.

A nonlinear enhancement of this theory was initiated at the beginning of the 1990s [3, 4]. The seminal paper of Isidori and Byrnes [3], although limited in scope (it only secured local, nonrobust, regulation about an equilibrium point), highlighted fundamental ideas which shaped all subsequent developments in this area of research. For instance, it points out the basic challenges in solving the output regulation problem in a nonlinear setting, namely to create an invariant set on which the desired regulated variable vanishes, and to render this set asymptotically attractive. It also highlights the fundamental link between the problem in question and the notion of “zero dynamics” (a concept introduced and studied earlier by the same authors).

In the past 20 years, the design philosophy introduced in the paper above was extended in several directions. One goal was to move from “local” to “nonlocal” convergence, for which several approaches at increasing level of generality have been proposed [5, 6, 7]. An important advance of [7] was to give a general (nonequilibrium) definition of the problem, through a convenient definition of the notion of “steady state” for nonlinear systems. Another concern was to obtain design methods which are insensitive, or even robust, with respect to model uncertainties (either in the plant or in the exosystem) [8, 9].

A general framework in which the output regulation problem is solved finally emerged. The basic ideas were captured within two fundamental properties, the internal model property and the stabilizability property. Once the first property is achieved, the output regulation problem can be simply solved through high-gain stabilization techniques [10].

A crucial observation was that the problem of achieving the asymptotic IMP is closely related to, and actually can be cast as, the problem of designing a nonlinear observer. By using available observer designs [11, 12, 13], this approach has led to effective design methods that fall in two classes: based on immersion (they imply rather strong assumptions) [14, 15] and newest results dropping the immersion/observability condition [16, 17]. The results referred to above are by no means general; they can be applied to particular classes of systems, under specific hypotheses.

These ideas were pursued mainly for SISO nonlinear systems, leading to some effective designs. The fact that so far there have been limited attempts to solve the problem in the MIMO case (e.g.[18]) is not entirely surprising, and for various reasons. First, seeing how the design of observers is instrumental in the design of controllers that solve the output regulation problem, the design of observers in the multiple-output case is known to pose serious technical difficulties, especially in obtaining the right canonical forms that allow a meaningful (constructive) characterization of the observability properties [19]. Second, MIMO normal forms are not simple extensions of SISO normal forms. For instance, while SISO normal forms lend themselves naturally to the definition of nonlinear equivalents for the linear finite and infinite zero structures and invertibility properties, the extension of these notions to MIMO systems is nowhere near as straightforward. On the other hand, a normal form of some kind represents the only tool to (robustly) handle nonlinear dynamics for control purposes, while normal forms that seem to be adequate for the MIMO nonlinear output regulation problem have been introduced just recently. Last, but not the least, while there is a wealth of stabilization tools for SISO nonlinear systems, there are not so many available for MIMO systems.

In this paper we are going to present the assumptions under which the problem of output regulation can be solved in the case of nonlinear MIMO systems. This problem is strongly linked to the normal forms of nonlinear MIMO systems. The normal forms evolved in close relationship with the control techniques. The paper is structured as follows: in chapter 2 the classic and recent results on normal forms used for MIMO nonlinear control problems are introduced and the zeros dynamics of the system is discussed, in chapter 3 the general problem of output regulation for nonlinear MIMO systems is presented, in chapter 4 the necessary conditions under which the output regulation problem

can be solved are discussed and the assumptions considered in [18] are relaxed and chapter 5 deals with conclusions and further developments.

## 2. Normal forms and zero dynamics for MIMO nonlinear systems

Normal forms for nonlinear MIMO systems are meant to reveal key structural properties of the analyzed/considered system: relative degrees, zero dynamics and invertibility properties. The relative degrees expose the infinite zero structure, while the zero dynamics characterize the finite zero structure of the system. These notions are important (as in the case of linear systems) when control problems are of interest. Considering the following MIMO system:

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x)\end{aligned}\tag{1}$$

with:  $x \in R^n$  the state,  $u \in R^m$  the input,  $y \in R^p$  the output, if the system is input affine it can be represented as:

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}\tag{2}$$

where:  $u = \text{col}(u_1, \dots, u_m)$ ,  $y = \text{col}(y_1, \dots, y_p)$ ,  $g(x) = [g_1(x), \dots, g_m(x)]$ ,  $h(x) = (h_1(x), \dots, h_p(x))$  and  $g(x)$  is a  $n \times m$  matrix and  $u(x)$ ,  $y(x)$ ,  $h(x)$  are vectors and system (2) can be rewritten as (3):

$$\begin{aligned}\dot{x} &= f(x) + \sum_{j=1}^m g_j(x)u_i \\ y_1 &= h_1(x) \\ &\dots \\ y_p &= h_p(x)\end{aligned}\tag{3}$$

In the case of square systems,  $m=p>1$  the system can be written in the normal form introduced by Isidori in [20]. This is based on the existence of some vector relative degrees  $\{r_1, \dots, r_m\}$  at a point  $x_0$ :

1.  $L_{g_j}^k h_i(x) = 0$ ;  $j=1, m$ ;  $i=1, m$ ;  $k < r_i - 1$ ; in a neighborhood of  $x_0$ .

$$2. M = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \dots & L_{g_m} L_f^{r_1-1} h_1(x) \\ \vdots & \ddots & \\ L_{g_1} L_f^{r_m-1} h_m(x) & \dots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix} \text{ nonsingular in } x_0.$$

Each relative degree  $r_i$  is associated to the  $i^{th}$  system output. The sum of the relative degrees is at most  $n$ . By applying the following transform  $\Theta: x \rightarrow (z, e)$ :

$$\Theta(x) = \begin{bmatrix} [h_1(x) \dots L_f^{r_1-1} h_1(x)] \\ \vdots \\ [h_m(x) \dots L_f^{r_m-1} h_m(x)] \end{bmatrix},$$

(considering that  $dL_f^j h_i(x)$ ,  $j=0, r_i-1$ ,  $i=1, m$  are linearly independent) the system in (2) can be written in normal form as:

$$\begin{aligned} \dot{z} &= f_{0i}(z, e) + P(z, e)u \\ e_{i,1} &= e_{i,2} \\ &\dots \\ e_{i,r_i-1} &= e_{i,r_i} \\ e_{i,r_i} &= q(z, e) + \sum_{j=1}^m m_{ij}(z, e)u_j \\ y_i &= e_{i,1} \end{aligned}, \quad i=1, m \quad (4)$$

where  $P_{ij}(e, z) = L_{g_j} z_i \circ \Theta^{-1}(e, z)$ , and the function  $\Theta: x \rightarrow (e, z)$  is a diffeomorphism from  $x$  to  $(e, z)$ .

**Observation 1:** In the case of SISO systems there is always possible to find a set of functions such that  $L_{g_j} z_i(x) \equiv 0, j=1, p; i=1, n - \sum_{i=1}^m r_i$ . For the MIMO case, this is possible only if the distribution spanned by the column vectors  $\{g_1, g_2, \dots, g_m\}$  is involutive in a neighborhood of  $x_0$ .

**Observation 2:** If the matrix  $M(e, z)$  is singular and  $\text{rank } M < m$  is constant (the system can not be written in the normal form with  $m$  relative degrees), a dynamic extension algorithm proposed in [20] can be used in order to extend the system by adding integrators on the input channels such that the system might be written in normal form.

### The zero dynamics of (4)

Considering the system (4), the zero dynamics is given by the following expression:

$$\dot{z} = f_{0i}(z, 0) \quad (5)$$

A generalization of the above algorithm was made in [99]. The normal form of system (2) is:

$$\begin{aligned}\dot{z} &= f_0(x) + g_0(x)u \\ \dot{e}_{i,j} &= e_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x)v_l + \sigma_{i,j}(x)u, j = 1, n_i - 1 \\ \dot{e}_{i,n_i} &= v_i \\ y_i &= e_{i,1}; i = 1, m\end{aligned}\tag{6}$$

with  $n_1 \leq n_2 \leq \dots n_m$

$$v_i = a_i(x) + b_i(x)u; i = 1, m$$

$$a_i(x) = L_g e_{i,n_i}(x)$$

$$b_i(x) = L_f e_{i,n_i}(x)$$

and the matrix  $\{b_1(x), b_2(x), \dots, b_m(x)\}$  is smooth and nonsingular.

Considering  $\delta_{i,j,l}(x) = 0$  the vector  $\{n_1, n_2, \dots, n_m\}$  of system (6) represents exactly the relative degrees vector (which, in the case of linear system gives the infinite zero structure). If  $\delta_{i,j,l}(x) \neq 0$  the vector  $\{n_1, n_2, \dots, n_m\}$  is not linked to the infinite zero structure of a linear system [22].

Under the stronger assumptions that some matrix ranks are constant and the distribution spanned by the column vectors  $\{g_1, g_2, \dots, g_m\}$  is involutive, system (6) takes the form:

$$\begin{aligned}\dot{z} &= f_0(x) \\ \dot{e}_{i,j} &= e_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x)v_l, j = 1, r_i - 1 \\ \dot{e}_{i,r_i} &= v_i \\ y_i &= e_{i,1}; i = 1, m\end{aligned}\tag{7}$$

### The zero dynamics of (6)

Using these generalizations of the normal form - (6) and (7), the zero dynamics is given by:

$$\dot{z} = f_0(x)\tag{8}$$

The assumptions needed for the elaboration of the presented normal forms are rather strong. In a recent publication [22] these assumptions are substantially weakened. Moreover, the infinite zero dynamics algorithm [22], [23] allows the representation of MIMO nonlinear systems that are not necessary square and invertible under the following form:

$$\begin{aligned}
 \dot{z} &= f_e(x) + g_e(x)u_e + \sum_{i=1}^{m_d} \varphi_i(x)v_{d,i} \\
 \dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x)v_{d,l}, \quad j = 1, q_i - 1 \\
 \dot{\xi}_{i,q_i} &= v_{d,i} \\
 y_\Delta &= h_\Delta(z, z_d) \\
 y_{d,i} &= \xi_{i,1}; \quad i = 1, m_d
 \end{aligned} \tag{9}$$

with  $q_1 \leq q_2 \leq \dots \leq q_{m_d}$ ,

$$\xi_i = \{\xi_{i,1}, \xi_{i,2}, \dots, \xi_{i,q_i}\}, \quad i = 1, m_d$$

$$v_{d,i} = a_i(x) + b_i(x)u$$

$$y_d = \text{col}(\xi_{1,1}, \xi_{2,1}, \dots, \xi_{m_d,1})$$

and  $\text{col}\{b_1(x), b_2(x), \dots, b_{m_d}(x)\}$  is nonsingular.

From the infinite zero structure algorithm, the dynamics  $\dot{\xi}_{i,j}$  does not depend on  $v_l$ ,  $l > j$  and it follows that :

$$\delta_{i,j,l}(x) = 0, \quad j < q_i, \quad i = 1, m_d \tag{10}$$

The relations (9), (10) hold under Assumption B[22].

In (9)  $m_d$  represents the largest number for which the system can be transformed in the normal form (this value results by applying the infinite zero structure algorithm). The algorithm also identifies a vector of integer values  $\{q_1, q_2, \dots, q_m\}$  that represents the infinite zero structure.

In addition, if Assumption C [22] holds there exist a coordinate transform that puts the system in the form :

$$\begin{aligned}
 \dot{z} &= f_e(x) + g_e(x)u_e \\
 \dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x)v_{d,l}, \quad j = 1, q_i - 1 \\
 \dot{\xi}_{i,q_i} &= v_{d,i} \\
 y_e &= h_e(x) \\
 y_{d,i} &= \xi_{i,1}; \quad i = 1, m_d
 \end{aligned} \tag{11}$$

and  $\delta_{i,j,l}(x) = 0, \quad j < q_i, \quad i = 1, m_d$ .

Moreover, the form (9) gives an insight on the system invertibility properties. If the term  $u_e$  is absent, the system is left invertible; if the term  $y_e$  is absent, the

system is right invertible; if the terms  $u_e$  and  $y_e$  are both absent, the system is invertible and if the terms  $u_e$  and  $y_e$  exist, the system is degenerate [23].

In the case of square invertible systems with  $m=p=m_d$ , the terms  $u_e$  and  $y_e$  do not exist and the system takes the following normal form [22]:

$$\begin{aligned}\dot{z} &= f_e(z, \xi) \\ \dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(z, \xi) v_l, j = 1, q_i - 1 \\ \dot{\xi}_{i,q_i} &= v_i \\ y_i &= \xi_{i,1}; i = 1, m_d\end{aligned}\tag{12}$$

with  $q_1 \leq q_2 \leq \dots \leq q_m$  and

$$\delta_{i,j,l}(x) = 0, j < q_l, i = 1, m\tag{13}$$

It can be observed that the system in (12) form has a triangular structure [22] between  $\dot{\xi}_{i,j}$  and the inputs. Relation (13) reveals the fact that there is also a triangular dependency of  $\delta_{i,j,l}$  on the state variables [23].

### The zero dynamics of (12)

The system's (12) zero dynamics is given by:

$$\dot{z} = f_e(z, 0)$$

This type of normal form with a structure in which the inputs are entering the system in a triangular fashion and the  $\delta_{i,j,l}$  functions have triangular dependencies on the systems state presents valuable properties (aside weaker assumptions and the fact that it shows the invertibility properties of the system) from the control point of view [23]. Using it to solve the output regulation problem hasn't been yet pursued.

In what follows we are going to give the general context of output regulation problem in the case of MIMO systems and to compare the conditions for the problem solvability in case of using the two normal forms presented above in terms of assumptions restrictivity.

### 3. The general output regulation problem for MIMO nonlinear systems

The problem of output regulation considers that the models for the process to be control and the ecosystem are known. The latter is supposed to contain the reference and/or the perturbations. A regulator solving the problem

in closed loop must assure: the boundedness of the state trajectory and uniform convergence to 0 of the error.

We consider a multivariable system given by the following expression:

$$\begin{aligned} x &= f(w, x, u) \\ y &= k(w, x) \\ e &= h(w, x) \end{aligned} \tag{14}$$

with:  $x \in R^n$  the state,  $u \in R^m$  the control input,  $e \in R^p$  the regulated output,  $y \in R^p$  the measured output,  $w \in R^r$  the exosystem's state.

The exosystem is an autonomous system:

$$w = s(w) \tag{15}$$

The functions  $f(w, x, u)$ ,  $h(w, x)$ ,  $k(w, x)$  and  $s(w)$  are considered to be of class  $C^k$  (sufficiently large) in their arguments. The initial conditions for the system vary on a fixed closed set  $x(0) \in X_0$  and for the exosystem - vary on a invariant compact set  $w(0) \in W$ .

We further consider that the system is of finite dimension, time invariant, and can be put in a normal form:

1. such that it has a well defined relative degrees vector and the zero dynamics is stable (the system is of minimum phase) – Isidori approach in [18] or
2. as described in (9).

The regulator is supposed to be of the form:

$$\begin{aligned} \psi &= \varphi(\psi, y) \\ u &= \gamma(\psi, y) \end{aligned} \tag{16}$$

with:  $\psi \in R^v$  the regulator state and the functions  $\varphi(\psi, y)$  and  $\gamma(\psi, y)$  of class  $C^k$ .

The initial conditions for the regulator can vary on a compact set  $\psi(0) \in \Xi$ .

The system (14), (15) and (16) in closed loop form is:

$$\begin{aligned} w &= s(w) \\ x &= f(w, x, \gamma(\psi, k(w, x))) \\ \psi &= \varphi(\psi, k(w, x)) \\ e &= h(w, x) \end{aligned} \tag{17}$$

Consider that  $X$  is a compact subset of  $X_0$ , the regulator (16) solves the output regulation problem if the positive trajectory on  $W \times X \times \Xi$  is bounded and  $\lim_{t \rightarrow \infty} e(t) = 0$ , uniformly on  $W \times X \times \Xi$  (when the system is in steady state).

The form of the regulator, its initial conditions set and its properties are to be determined. In the context of the output regulation problem as presented in [7], [14] with the notations and lemmas of [7] the trajectories of the system in closed loop are supposed to be bounded.

This leads to the conclusion that the  $\omega$  limit set  $\omega(W \times X \times \Xi)$  is not empty, compact and invariant, and uniformly attracts the trajectories of the system in closed loop and the steady state error is 0 if and only if:

$$\omega(W \times X \times \Xi) \subset \{(w, x, \psi) : h(w, x) = 0\}$$

#### 4. Necessary conditions for solving the output regulation problem MIMO nonlinear systems

##### 4.1. The system can be written in the normal form (4)

In this case the system with the exosystem (15) has the form:

$$\begin{aligned} \dot{z} &= f_0(z, w) + f_1(z, e, w)e & , i = 1, m \\ e &= q(z, e, w) + M(z, w)u \end{aligned} \quad (18)$$

with: initial conditions in the set  $Z \times E \times W$  where  $Z$  is fixed and compact and  $E$  is bounded and the functions  $f_0, f_1, g, s, M$  are smooth enough.

The coupling matrix  $M(z, w)$  is considered invertible (in the SISO systems case the condition is that  $b(w, z, e_1, \dots, e_r) \neq 0$ ). This means that system (18) has a vector of relative degrees:  $\{1, 1, \dots, 1\}$  between the control input  $u$  and the regulated output  $e$ . A system with the relative degrees vector  $\{r_1, \dots, r_m\}$  can always be transformed into the form (18).

Considering that a controller of the form (16) solves the problem of output regulation and applying lemma 2 [7] the steady state locus of the system in closed loop (17) must be a subset of the set for which the error is zero ( $e=0$ ).

If system (8) is in steady state the following conditions are fulfilled:

- the steady state locus of the system in closed loop  $\omega(W \times Z \times E \times \Xi)$  is a subset of:  $R^s \times R^{n-m} \times \{0\} \times R^v$
- the restriction of the system in closed loop to the steady state locus  $\omega(W \times Z \times E \times \Xi)$  is (with the zero dynamics given by the first two relations)

$$w = s(w)$$

$$z = f_0(w, z); \text{ cu } e=0$$

$$\psi = \varphi(\psi, 0)$$

- $\forall (w, z, 0, \dots, 0, \psi) \in \omega(W \times Z \times E \times \Xi)$

$$0 = q(z, 0, w) + M(z, w)\gamma(\psi, 0) \Leftrightarrow 0 = q(z, 0, w) + M(z, w)u$$

It was considered that the positive trajectory of the exosystem (15) on  $W$  is bounded if the trajectories asymptotically approach  $\omega(W)$ . This assumption does not diminish the generality of the problem because it can be considered that  $W = \omega(W)$  which means that the exosystem is in steady state.

### Assumption 1 MIMO

The set  $W \subset R^n$  is compact and invariant under (15).

If the positive orbit of the set  $W \times Z \times E \times \Xi$  under (14), (15) and (16), then the system dynamics in closed loop is the graph of a function defined on the whole of  $W$  parts.

Noting

$$A_{ss} = \{(w, z) \mid (w, z, 0, \dots, 0, \psi) \in \omega(W \times Z \times E \times \Xi), \psi \in R^v\} \quad (19)$$

and considering the function

$$\begin{aligned} u_{ss} : A_{ss} &\rightarrow R^m \\ (w, z) &\rightarrow -M^{-1}(w, z)q(w, z, 0) \end{aligned}$$

by construction the set described by (19) is the codomain of a function defined on the whole of  $W$ , which is invariant under the zero dynamics of the system in the normal form (4) and the exosystem:

$$\begin{aligned} w &= s(w) \\ z &= f_0(w, z) \end{aligned} \quad (20)$$

The function  $u_{ss}$  is the control law that forces the system to evolve on  $A_{ss}$ .

In conclusion, if the controller (16) solves the output regulation problem for the system in normal form with the exosystem, then there is a function defined on the whole of  $W$  which has the codomain  $A_{ss}$  and  $A_{ss}$  is invariant under (20).

Moreover, for each  $(w_0, z_0) \in A_{ss}, \exists \psi_0 \in R^v$  such that the integral curve of (20) is exactly the integral curve of

$$\psi = \varphi(\psi, 0)$$

starting in  $\psi_0$  and satisfying

$$u_{ss}(w(t), z(t)) = \gamma(\psi(t), 0), \forall t \in R.$$

In other words, one can build a controller that reproduces the input for steady state such that the regulated error is 0 (internal model for nonlinear system).

Considering this approach it can be observed that Assumption I MIMO can be considered only if the coupling matrix  $M$  is invertible. The invertibility implies two aspects:

1. the system is square

2. the inverse of  $M$  must be formally computed – which represents a major drawback for real implementation cases. This is probably why no publication follows the article giving this solution in [18].

#### 4.2 Recent normal form

Following the above reasoning, we consider that the system is written in the normal form (12) with the exosystem (15):

$$\begin{aligned}\dot{w} &= s(w) \\ \dot{z} &= f_e(z, w) \\ \dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(z, w, \xi) v_l, j = 1, q_i - 1 \\ \dot{\xi}_{i,q_i} &= v_i \\ y_i &= \xi_{i,1}; i = 1, m_d\end{aligned}$$

Assumption I MIMO takes the following form:

#### Assumption 1 MIMO N

The set  $W \subset R^n$  is compact and invariant under (5).

If the positive orbit of the set  $W \times Z \times E \times \Xi$  under (12), (15) and (16), the system dynamics in closed loop is the graph of a function defined on the whole of  $W$  parts.

If we note:

$$A'_{ss} = \{(w, z) \mid (w, z, 0, \dots, 0, \psi) \in \omega(W \times Z \times E \times \Xi), \psi \in R^v\} \quad (21)$$

and if in relation (13)  $q_l \leq 1$  we can consider the function:

$$\begin{aligned}u'_{ss} : A'_{ss} &\rightarrow R^m \\ (w, z) &\rightarrow \begin{bmatrix} -a_1(w, z, 0) / b_1(w, z, 0) \\ f_p(u'_1, \dots, u'_{p-1}, a_1(w, z, 0), \dots, \\ a_{p-1}(w, z, 0), b_1(w, z, 0), \dots, b_{p-1}(w, z, 0)) \end{bmatrix}.\end{aligned}$$

By construction, the set described by (21) is the codomain of a function defined on the whole of  $W$ , which is invariant under the zero dynamics of the system in the normal form (12) and the exosystem (15):

$$\dot{w} = s(w)$$

$$\dot{z} = f_e(z, 0)$$

The function  $u'_{ss}$  is the control law that forces the system to evolve on  $A'_{ss}$ .

The “triangular” properties of the new normal form (12) lead to a feasible solution for writing the necessary conditions for the output regulation problem in a real case, without formally inverting the coupling matrix.

## 5. Conclusions

In the light of recent advancements on normal forms we consider that a more suitable approach for solving the nonlinear output regulation problem in the case of square MIMO systems is to use the normal form proposed in [22]. In this case the conditions under which the problem of output regulation is solvable are more relaxed from the following points of view: the relative degrees vector is not required; there is no assumption that implies the fact that the coupling  $M$  matrix is to be inverted explicitly, which means that nonsquare MIMO systems could be considered for control too; the assumptions needed for the normal form of [22] are less restrictive than the ones for the normal form of [21].

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