

## FIXED POINT OF $\alpha$ -GERAGHTY CONTRACTION WITH APPLICATIONS

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*Cho, Bae and Karapinar [Fixed point theorems for  $\alpha$ -Geraghty contraction type maps in metric spaces, Fixed Point Theory and Applications 2013, 2013:329] established new fixed point results in complete metric spaces. In this paper, we improve the notion of  $\alpha$ -Geraghty contraction type mappings and establish some common fixed point theorems for a pair of  $\alpha$ -admissible mappings under the improved notion of  $\alpha$ -Geraghty contractive type condition in a complete metric space. An example was constructed to prove the novelty of our results.*

**Keywords:** fixed point; contraction type mapping;  $\alpha$ -Geraghty contraction type mapping; metric space.

**MSC2010:** 46S40; 47H10; 54H25.

### 1. Introduction

The study of fixed point problems in nonlinear analysis emerged as a powerful and very important tool in the last 60 years. Particularly, the techniques of fixed point have been applicable to many diverse fields of sciences such as Economics, Engineering, Chemistry, Biology, Physics and Game Theory. Over the years, fixed point theory has been generalized in multi-directions by several mathematicians.

In 1973, Geraghty [5] studied a generalization of Banach contraction principle. In 2012, Samet et al. [20], introduced a concept of  $\alpha - \psi$ -contractive type mappings and established various fixed point theorems for mappings in complete metric spaces. Afterwards, Karapinar and Samet [12], refined the notion and obtained various fixed point results. Hussain et al. [11], generalized the concept of  $\alpha$ -admissible mappings and proved fixed point theorems. Subsequently, Abdeljawad [1] introduced a pair of  $\alpha$ -admissible mappings satisfying new sufficient contractive conditions different from those in [11, 20], and obtained fixed point and common fixed point theorems. Salimi et al. [19], modified the concept of  $\alpha - \psi$ -contractive mappings and established fixed point results. Recently, Hussain et al. [10] proved some fixed point results for single and set-valued  $\alpha - \eta - \psi$ -contractive mappings in the setting of complete metric space. Mohammadi et al. [17], introduced a new

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notion of  $\alpha - \phi$ -contractive mappings and showed that this is a real generalization for some previous results. Thereafter, many papers have published on  $\alpha - \psi$ -contractive mappings in various spaces. For more detail see [2-3, 6-9, 14, 16, 18, 20] and references therein.

## 2. PRELIMINARIES

In this section, we give some basic definitions, examples and fundamental results which play an essential role in proving our results.

**Definition 2.1.** [20] Let  $S : X \rightarrow X$  and  $\alpha : X \times X \rightarrow [0, \infty)$ . We say that  $S$  is  $\alpha$ -admissible if  $x, y \in X$ ,  $\alpha(x, y) \geq 1 \Rightarrow \alpha(Sx, Sy) \geq 1$ .

**Example 2.1.** [15] Consider  $X = [0, \infty)$ , and define  $S : X \rightarrow X$  and  $\alpha : X \times X \rightarrow [0, \infty)$  by  $Sx = 2x$ , for all  $x, y \in X$  and

$$\alpha(x, y) = \begin{cases} e^{\frac{y}{x}} & \text{if } x \geq y, x \neq 0 \\ 0 & \text{if } x < y. \end{cases}$$

Then  $S$  is  $\alpha$ -admissible.

**Definition 2.2.** [1] Let  $S, T : X \rightarrow X$  and  $\alpha : X \times X \rightarrow [0, +\infty)$ . We say that the pair  $(S, T)$  is  $\alpha$ -admissible if  $x, y \in X$  such that  $\alpha(x, y) \geq 1$ , then we have  $\alpha(Sx, Ty) \geq 1$  and  $\alpha(Tx, Sy) \geq 1$ .

**Example 2.2.** Let  $X = [0, \infty)$ , and define a pair of self mapping  $S, T : X \rightarrow X$  and  $\alpha : X \times X \rightarrow [0, \infty)$  by  $Sx = 2x$ ,  $Tx = x^2$  for all  $x, y \in X$  and

$$\alpha(x, y) = \begin{cases} e^{xy} & \text{if } x, y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then a pair  $(S, T)$  is  $\alpha$ -admissible.

**Definition 2.3.** [13] Let  $S : X \rightarrow X$  and  $\alpha : X \times X \rightarrow [0, +\infty)$ . We say that  $S$  is triangular  $\alpha$ -admissible if  $x, y \in X$ ,  $\alpha(x, z) \geq 1$  and  $\alpha(z, y) \geq 1 \Rightarrow \alpha(x, y) \geq 1$ .

**Example 2.3.** [13] Let  $X = [0, \infty)$ ,  $Sx = x^2 + e^x$  and

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Hence,  $S$  is a triangular  $\alpha$ -admissible mapping.

**Definition 2.4.** [13] Let  $S : X \rightarrow X$  and  $\alpha : X \times X \rightarrow \mathbb{R}$ . We say that  $S$  is a triangular  $\alpha$ -admissible mapping if

- (T1)  $\alpha(x, y) \geq 1$  implies  $\alpha(Sx, Sy) \geq 1$ ,  $x, y \in X$ ,
- (T2)  $\alpha(x, z) \geq 1$ ,  $\alpha(z, y) \geq 1$ , implies  $\alpha(x, y) \geq 1$ ,  $x, y, z \in X$ .

**Example 2.4.** [13] Let  $X = \mathbb{R}$ ,  $Sx = \sqrt[3]{x}$  and  $\alpha(x, y) = e^{x-y}$  then  $S$  is a triangular  $\alpha$ -admissible mapping. Indeed, if  $\alpha(x, y) = e^{x-y} \geq 1$  then  $x \geq y$  which implies  $Sx \geq Sy$ . That is,  $\alpha(Sx, Sy) = e^{Sx-Sy} \geq 1$ . Also, if  $\alpha(x, z) \geq 1$ ,  $\alpha(z, y) \geq 1$  then  $x - z \geq 0$ ,  $z - y \geq 0$ . That is,  $x - y \geq 0$  and so  $\alpha(x, y) = e^{x-y} \geq 1$ .

**Definition 2.5.** [1] Let  $S, T : X \rightarrow X$  and  $\alpha : X \times X \rightarrow \mathbb{R}$ . We say that a pair  $(S, T)$  is triangular  $\alpha$ -admissible if

- (T1)  $\alpha(x, y) \geq 1$ , implies  $\alpha(Sx, Ty) \geq 1$  and  $\alpha(Tx, Sy) \geq 1$ ,  $x, y \in X$ .
- (T2)  $\alpha(x, z) \geq 1$ ,  $\alpha(z, y) \geq 1$ , implies  $\alpha(x, y) \geq 1$ ,  $x, y, z \in X$ .

**Example 2.5.** Let  $X = \mathbb{R}$ , and define a pair of self mapping  $S, T : X \rightarrow X$  and  $\alpha : X \times X \rightarrow \mathbb{R}$  by  $Sx = \sqrt{x}$ ,  $Tx = x^2$  for all  $x, y \in X$  and  $\alpha(x, y) = e^{xy}$ . Then a pair  $(S, T)$  is triangular  $\alpha$ -admissible mappings.

**Definition 2.6.** [19] Let  $S : X \rightarrow X$  and let  $\alpha, \eta : X \times X \rightarrow [0, +\infty)$  be two functions. We say that  $S$  is  $\alpha$ -admissible mapping with respect to  $\eta$  if  $x, y \in X$ ,  $\alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(Sx, Sy) \geq \eta(Sx, Sy)$ . Note that if we take  $\eta(x, y) = 1$ , then this definition reduces to definition in [20]. Also if we take  $\alpha(x, y) = 1$ , then we says that  $S$  is an  $\eta$ -subadmissible mapping.

**Example 2.6.** Let  $X = [0, \infty)$  and  $S : X \rightarrow X$  be defined by  $Sx = \frac{x}{2}$ . Define also  $\alpha, \eta : X \times X \rightarrow [0, +\infty)$  by  $\alpha(x, y) = 3$  and  $\eta(x, y) = 1$  for all  $x, y \in X$ . Then  $S$  is  $\alpha$ -admissible mapping with respect to  $\eta$ .

**Lemma 2.1.** [4] Let  $S : X \rightarrow X$  be a triangular  $\alpha$ -admissible mapping. Assume that there exists  $x_0 \in X$  such that  $\alpha(x_0, Sx_0) \geq 1$ . Define a sequence  $\{x_n\}$  by  $x_{n+1} = Sx_n$ . Then we have  $\alpha(x_n, x_m) \geq 1$  for all  $m, n \in \mathbb{N} \cup \{0\}$  with  $n < m$ .

**Lemma 2.2.** Let  $S, T : X \rightarrow X$  be a pair of triangular  $\alpha$ -admissible. Assume that there exists  $x_0 \in X$  such that  $\alpha(x_0, Sx_0) \geq 1$ . Define sequence  $x_{2i+1} = Sx_{2i}$ , and  $x_{2i+2} = Tx_{2i+1}$ , where  $i = 0, 1, 2, \dots$ . Then we have  $\alpha(x_n, x_m) \geq 1$  for all  $m, n \in \mathbb{N} \cup \{0\}$  with  $n < m$ .

We denote by  $\Omega$  the family of all functions  $\beta : [0, +\infty) \rightarrow [0, 1)$  such that, for any bounded sequence  $\{t_n\}$  of positive reals,  $\beta(t_n) \rightarrow 1$  implies  $t_n \rightarrow 0$ .

**Theorem 2.1.** [5] Let  $(X, d)$  be a metric space. Let  $S : X \rightarrow X$  be a self mapping. Suppose that there exists  $\beta \in \Omega$  such that for all  $x, y \in X$ ,

$$d(Sx, Sy) \leq \beta(d(x, y))d(x, y).$$

then  $S$  has a fixed unique point  $p \in X$  and  $\{S^n x\}$  converges to  $p$  for each  $x \in X$ .

### 3. MAIN RESULTS

In this section, we prove some fixed point theorems satisfying  $\alpha$ -Geraghty contraction type mappings in complete metric space.

Let  $(X, d)$  be a metric space and  $\alpha : X \times X \rightarrow \mathbb{R}$  be a function. Two mappings  $S, T : X \rightarrow X$  is called a pair of generalized  $\alpha$ -Geraghty contraction type mappings if there exists  $\beta \in \Omega$  such that for all  $x, y \in X$ ,

$$\alpha(x, y)d(Sx, Ty) \leq \beta(M(x, y))M(x, y) \quad (1)$$

where

$$M(x, y) = \max \left\{ d(x, y), d(x, Sx), d(y, Ty), \frac{d(y, Sx) + d(x, Ty)}{2} \right\}.$$

If  $S = T$  then  $T$  is called generalized  $\alpha$ -Geraghty contraction type mapping if there exists  $\beta \in \Omega$  such that for all  $x, y \in X$ ,

$$\alpha(x, y)d(Sx, Ty) \leq \beta(N(x, y))N(x, y)$$

where

$$N(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}.$$

**Theorem 3.1.** *Let  $(X, d)$  be a complete metric space,  $\alpha : X \times X \rightarrow \mathbb{R}$  be a function. Let  $S, T : X \rightarrow X$  be two mappings. Suppose that the following holds:*

- (i)  $(S, T)$  is a pair of generalized  $\alpha$ -Geraghty contraction type mapping;
- (ii)  $(S, T)$  is triangular  $\alpha$ -admissible;
- (iii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Sx_0) \geq 1$ ;
- (iv)  $S$  and  $T$  are continuous;

Then  $(S, T)$  have common fixed point.

*Proof.* Let  $x_1$  in  $X$  be such that  $x_1 = Sx_0$  and  $x_2 = Tx_1$ . Continuing this process, we construct a sequence  $x_n$  of points in  $X$  such that,

$$x_{2i+1} = Sx_{2i}, \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where } i = 0, 1, 2, \dots$$

By assumption  $\alpha(x_0, x_1) \geq 1$  and a pair  $(S, T)$  is  $\alpha$ -admissible, By Lemma 14, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Then, we have

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &= d(Sx_{2i}, Tx_{2i+1}) \leq \alpha(x_{2i}, x_{2i+1})d(Sx_{2i}, Tx_{2i+1}) \\ &\leq \beta(M(x_{2i}, x_{2i+1}))M(x_{2i}, x_{2i+1}), \end{aligned}$$

for all  $i \in \mathbb{N} \cup \{0\}$ . Now

$$\begin{aligned} M(x_{2i}, x_{2i+1}) &= \max \left\{ d(x_{2i}, x_{2i+1}), d(x_{2i}, Sx_{2i}), d(x_{2i+1}, Tx_{2i+1}), \frac{d(x_{2i}, Tx_{2i+1}) + d(x_{2i+1}, Sx_{2i})}{2} \right\} \\ &= \max \left\{ d(x_{2i}, x_{2i+1}), d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2}), \frac{d(x_{2i}, x_{2i+2})}{2} \right\} \\ &\leq \max \left\{ d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2}), \frac{d(x_{2i}, x_{2i+1}) + d(x_{2i+1}, x_{2i+2})}{2} \right\} \\ &= \max \{d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2})\}. \end{aligned}$$

Thus

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &\leq \beta(M(x_{2i}, x_{2i+1}))M(x_{2i}, x_{2i+1}); \\ &\leq \beta(d(x_{2i}, x_{2i+1}))d(x_{2i}, x_{2i+1}) < d(x_{2i}, x_{2i+1}). \end{aligned}$$

That is

$$d(x_{2i+1}, x_{2i+2}) < d(x_{2i}, x_{2i+1}). \tag{2}$$

This, implies that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

So, sequence  $\{d(x_n, x_{n+1})\}$  is nonnegative and nonincreasing. Now, we prove that  $d(x_n, x_{n+1}) \rightarrow 0$ . It is clear that  $\{d(x_n, x_{n+1})\}$  is a decreasing sequence. Therefore, there exists some positive number  $r$  such that  $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = r$ . From (2), we have

$$\frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \leq \beta(d(x_n, x_{n+1})) \leq 1.$$

Now by taking limit  $n \rightarrow \infty$ , we have

$$1 \leq \beta(d(x_n, x_{n+1})) \leq 1,$$

that is

$$\lim_{n \rightarrow \infty} \beta(d(x_n, x_{n+1})) = 1.$$

By the property of  $\beta$ , we have

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (3)$$

Now, we show that sequence  $\{x_n\}$  is a Cauchy. Suppose on contrary that  $\{x_n\}$  is not a Cauchy sequence. Then there exists  $\epsilon > 0$  and sequences  $\{x_{m_k}\}$  and  $\{x_{n_k}\}$  such that, for all positive integers  $k$ , we have  $m_k > n_k > k$ ,

$$d(x_{m_k}, x_{n_k}) \geq \epsilon$$

and

$$d(x_{m_k}, x_{n_{k-1}}) < \epsilon.$$

By the triangle inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{m_k}, x_{n_k}); \\ &\leq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}); \\ &< \epsilon + d(x_{n_{k-1}}, x_{n_k}). \end{aligned}$$

That is

$$\epsilon < \epsilon + d(x_{n_{k-1}}, x_{n_k}). \quad (4)$$

for all  $k \in \mathbb{N}$ . In the view of (4), (3), we have

$$\lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_k}) = \epsilon. \quad (5)$$

Again using triangle inequality, we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k})$$

and

$$d(x_{m_{k+1}}, x_{n_{k+1}}) \leq d(x_{m_{k+1}}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}}).$$

Taking limit as  $k \rightarrow +\infty$  and using (3) and (5), we obtain

$$\lim_{k \rightarrow +\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon.$$

By Lemma 14,  $\alpha(x_{n_k}, x_{m_{k+1}}) \geq 1$ , we have

$$\begin{aligned} d(x_{n_{k+1}}, x_{m_{k+2}}) &= d(Sx_{n_k}, Tx_{m_{k+1}}) \leq \alpha(x_{n_k}, x_{m_{k+1}})d(Sx_{n_k}, Tx_{m_{k+1}}) \\ &\leq \beta(M(x_{n_k}, x_{m_{k+1}}))M(x_{n_k}, x_{m_{k+1}}). \end{aligned}$$

Finally, we conclude that

$$\frac{d(x_{n_k+1}, x_{m_k+2})}{M(x_{n_k}, x_{m_k+1})} \leq \beta(M(x_{n_k}, x_{m_k+1})).$$

Keeping (3) in mind and letting  $k \rightarrow +\infty$  in the above inequality, we obtain

$$\lim_{k \rightarrow \infty} \beta(d(x_{n_k}, x_{m_k+1})) = 1.$$

So,  $\lim_{k \rightarrow \infty} d(x_{n_k}, x_{m_k+1}) = 0 < \epsilon$ , which is a contradiction. Using similar technique for other cases, it can be easily seen that  $\{x_n\}$  is a Cauchy sequence. Since  $X$  is complete so there exists  $p \in X$  such that  $x_n \rightarrow p$  implies that  $x_{2i+1} \rightarrow p$  and  $x_{2i+2} \rightarrow p$ . As  $S$  and  $T$  are continuous, so we get  $Tx_{2i+1} \rightarrow Tp$  and  $Sx_{2i+2} \rightarrow Sp$ . Thus  $p = Sp$  similarly,  $p = Tp$ , we have  $Sp = Tp = p$ . Then  $(S, T)$  have common fixed point.  $\square$

In the following Theorem, we dropped the continuity.

**Theorem 3.2.** *Let  $(X, d)$  be a complete metric space,  $\alpha : X \times X \rightarrow \mathbb{R}$  be a function. Let  $S, T : X \rightarrow X$  be two mappings. Suppose that the following holds:*

- (i)  $(S, T)$  is a pair of generalized  $\alpha$ -Geraghty contraction type mapping;
- (ii)  $(S, T)$  is triangular  $\alpha$ -admissible;
- (iii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Sx_0) \geq 1$ ;
- (iv) if  $\{x_n\}$  is a sequence in  $X$  such that  $\alpha(x_n, x_{n+1}) \geq 1$  for all  $n \in \mathbb{N} \cup \{0\}$  and  $x_n \rightarrow p \in X$  as  $n \rightarrow +\infty$ , then there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $\alpha(x_{n_k}, p) \geq 1$  for all  $k$ .

Then  $(S, T)$  have common fixed point.

*Proof.* Follows the similar lines of the Theorem 16. Define a sequence  $x_{2i+1} = Sx_{2i}$ , and  $x_{2i+2} = Tx_{2i+1}$ , where  $i = 0, 1, 2, \dots$  converges to  $p \in X$ . By the hypotheses of (iv) there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $\alpha(x_{2n_k}, p) \geq 1$  for all  $k$ . Now by using (3.1) for all  $k$ , we have

$$\begin{aligned} d(x_{2n_k+1}, Tp) &= d(Sx_{2n_k}, Tp) \leq \alpha(x_{2n_k}, p)d(Sx_{2n_k}, Tp) \\ &\leq \beta(M(x_{2n_k}, p))M(x_{2n_k}, p). \end{aligned}$$

On the other hand, we obtain

$$M(x_{2n_k}, p) = \max \left\{ d(x_{2n_k}, p), d(x_{2n_k}, Sx_{2n_k}), d(p, Tp), \frac{d(x_{2n_k}, Tp) + d(p, Sx_{2n_k})}{2} \right\}.$$

Letting  $k \rightarrow \infty$  then we have

$$\lim_{k \rightarrow \infty} M(x_{2n_k}, p) = d(p, Tp). \quad (6)$$

Suppose that  $d(p, Tp) > 0$ . From (6), for an enough large  $k$ , we have  $M(x_{2n_k}, p) > 0$ , which implies that

$$\beta(M(x_{2n_k}, p)) < M(x_{2n_k}, p).$$

Then, we have

$$d(x_{2n_k}, Tp) < M(x_{2n_k}, p) \quad (7)$$

Letting  $k \rightarrow \infty$  inequality (7), we obtain that  $d(p, Tp) < d(p, Tp)$ , which is a contradiction. Thus, we find that  $d(p, Tp) = 0$ , implies  $p = Tp$ . Similarly  $p = Sp$ . Thus  $p = Tp = Sp$ .  $\square$

If  $M(x, y) = \max \left\{ d(x, y), d(x, Sx), d(y, Sy), \frac{d(y, Sx) + d(x, Sy)}{2} \right\}$  and  $S = T$  in Theorem 16 and Theorem 17, we have the following corollaries.

**Corollary 3.1.** *Let  $(X, d)$  be a complete metric space and let  $S$  is  $\alpha$ -admissible mappings such that the following holds:*

- (i)  $S$  is a generalized  $\alpha$ -Geraghty contraction type mapping;
- (ii)  $S$  is triangular  $\alpha$ -admissible;
- (iii) there exists  $x_0 \in X$  such that  $\alpha(x_0, T_0) \geq 1$ ;
- (iv)  $S$  is continuous;

Then  $S$  has a fixed point  $p \in X$ , and  $S$  is a Picard operator, that is,  $\{S^n x_0\}$  converges to  $p$ .

**Corollary 3.2.** *Let  $(X, d)$  be a complete metric space and let  $S$  is  $\alpha$ -admissible mappings such that the following holds:*

- (i)  $S$  is a generalized  $\alpha$ -Geraghty contraction type mapping;
- (ii)  $S$  is triangular  $\alpha$ -admissible;
- (iii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Sx_0) \geq 1$ ;
- (iv) if  $\{x_n\}$  is a sequence in  $X$  such that  $\alpha(x_n, x_{n+1}) \geq 1$  for all  $n \in \mathbb{N} \cup \{0\}$  and  $x_n \rightarrow p \in X$  as  $n \rightarrow +\infty$ , then there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $\alpha(x_{n_k}, p) \geq 1$  for all  $k$ .

Then  $S$  has a fixed point  $p \in X$ , and  $S$  is a Picard operator, that is,  $\{S^n x_0\}$  converges to  $p$ .

If  $M(x, y) = \max \{d(x, y), d(x, Sx), d(y, Sy)\}$  and  $S = T$  in Theorem 16, Theorem 17, we obtain the following corollaries.

**Corollary 3.3.** [4] *Let  $(X, d)$  be a complete metric space,  $\alpha : X \times X \rightarrow \mathbb{R}$  be a function. Let  $S : X \rightarrow X$  be a mapping then suppose that the following holds:*

- (i)  $S$  is a generalized  $\alpha$ -Geraghty contraction type mapping;
- (ii)  $S$  is triangular  $\alpha$ -admissible;
- (iii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Sx_0) \geq 1$ ;
- (iv)  $S$  is continuous;

Then  $S$  has a fixed point  $p \in X$ , and  $S$  is a Picard operator, that is,  $\{S^n x_0\}$  converges to  $p$ .

**Corollary 3.4.** [4] *Let  $(X, d)$  be a complete metric space,  $\alpha : X \times X \rightarrow \mathbb{R}$  be a function. Let  $S : X \rightarrow X$  be a mapping then suppose that the following holds:*

- (i)  $S$  is a generalized  $\alpha$ -Geraghty contraction type mapping;
- (ii)  $S$  is triangular  $\alpha$ -admissible;
- (iii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Sx_0) \geq 1$ ;
- (iv) if  $\{x_n\}$  is a sequence in  $X$  such that  $\alpha(x_n, x_{n+1}) \geq 1$  for all  $n \in \mathbb{N} \cup \{0\}$  and  $x_n \rightarrow p \in X$  as  $n \rightarrow +\infty$ , then there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $\alpha(x_{n_k}, p) \geq 1$  for all  $k$ .

Then  $S$  has a fixed point  $p \in X$ , and  $S$  is a Picard operator, that is,  $\{S^n x_0\}$  converges to  $p$ .

Let  $(X, d)$  be a metric space, and  $\alpha, \eta : X \times X \rightarrow \mathbb{R}$  be two functions. Two mappings  $S, T : X \rightarrow X$  is called a pair of generalized  $\alpha$ - $\eta$ -Geraghty contraction type mappings if there exists  $\beta \in \Omega$  such that for all  $x, y \in X$ ,

$$\alpha(x, y) \geq \eta(x, y) \Rightarrow d(Sx, Ty) \leq \beta(M(x, y)) M(x, y) \quad (3.8)$$

where

$$M(x, y) = \max \left\{ d(x, y), d(x, Sx), d(y, Ty), \frac{d(y, Sx) + d(x, Ty)}{2} \right\}.$$

**Theorem 3.3.** *Let  $(X, d)$  be a complete metric space. Let  $S$  is  $\alpha$ -admissible mappings with respect to  $\eta$  such that the following holds:*

- (i)  $(S, T)$  is a pair of generalized  $\alpha$ - $\eta$ -Geraghty contraction type mapping;
- (ii)  $(S, T)$  is triangular  $\alpha$ -admissible;
- (iii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$ ;
- (iv)  $S$  and  $T$  are continuous;

Then  $(S, T)$  have common fixed point.

*Proof.* Let  $x_1$  in  $X$  be such that  $x_1 = Sx_0$  and  $x_2 = Tx_1$ . Continuing this process, we construct a sequence  $x_n$  of points in  $X$  such that,

$$x_{2i+1} = Sx_{2i}, \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where } i = 0, 1, 2, \dots$$

By assumption  $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$  and a pair  $(S, T)$  is  $\alpha$ -admissible with respect to  $\eta$ , we have,  $\alpha(Sx_0, Tx_1) \geq \eta(Sx_0, Tx_1)$  from which we deduce that  $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$  which also implies that  $\alpha(Tx_1, Sx_2) \geq \eta(Tx_1, Sx_2)$ . Continuing in this way we obtain  $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$  for all  $n \in \mathbb{N} \cup \{0\}$ .

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &= d(Sx_{2i}, Tx_{2i+1}) \leq \alpha(x_{2i}, x_{2i+1})d(Sx_{2i}, Tx_{2i+1}) \\ &\leq \beta(M(x_{2i}, x_{2i+1})) M(x_{2i}, x_{2i+1}), \end{aligned}$$

for all  $i \in \mathbb{N} \cup \{0\}$ . Now

$$\begin{aligned} M(x_{2i}, x_{2i+1}) &= \max \left\{ d(x_{2i}, x_{2i+1}), d(x_{2i}, Sx_{2i}), d(x_{2i+1}, Tx_{2i+1}), \frac{d(x_{2i}, Tx_{2i+1}) + d(x_{2i+1}, Sx_{2i})}{2} \right\} \\ &= \max \left\{ d(x_{2i}, x_{2i+1}), d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2}), \frac{d(x_{2i}, x_{2i+2})}{2} \right\} \\ &\leq \max \left\{ d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2}), \frac{d(x_{2i}, x_{2i+1}) + d(x_{2i+1}, x_{2i+2})}{2} \right\} \\ &= \max \{d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2})\}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &\leq \beta(M(x_{2i}, x_{2i+1})) M(x_{2i}, x_{2i+1}) \\ &\leq \beta(d(x_{2i}, x_{2i+1})) d(x_{2i}, x_{2i+1}) < d(x_{2i}, x_{2i+1}). \end{aligned}$$

This, implies that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Follows the similar lines of the Theorem 16. Hence  $p$  is common fixed point of  $S$  and  $T$ .  $\square$

**Theorem 3.4.** *Let  $(X, d)$  be a complete metric space and let  $(S, T)$  are  $\alpha$ -admissible mappings with respect to  $\eta$  such that the following holds:*

- (i)  $(S, T)$  is a pair of generalized  $\alpha$ -Geraghty contraction type mapping;
- (ii)  $(S, T)$  is triangular  $\alpha$ -admissible;
- (iii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$ ;
- (iv) if  $\{x_n\}$  is a sequence in  $X$  such that  $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$  for all  $n \in \mathbb{N} \cup \{0\}$  and  $x_n \rightarrow p \in X$  as  $n \rightarrow +\infty$ , then there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $\alpha(x_{n_k}, p) \geq \eta(x_{n_k}, p)$  for all  $k$ .

Then  $S$  and  $T$  has common fixed point.

*Proof.* Follows the similar lines of the Theorem 17.  $\square$

If  $M(x, y) = \max \left\{ d(x, y), d(x, Sx), d(y, Sy), \frac{d(y, Sx) + d(x, Sy)}{2} \right\}$  and  $S = T$  in the Theorem 22, Theorem 23, we get the following corollaries.

**Corollary 3.5.** *Let  $(X, d)$  be a complete metric space and let  $S$  is  $\alpha$ -admissible mappings with respect to  $\eta$  such that the following holds:*

- (i)  $S$  is a generalized  $\alpha$ -Geraghty contraction type mapping;
- (ii)  $S$  is triangular  $\alpha$ -admissible;
- (iii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$ ;
- (iv)  $S$  is continuous;

Then  $S$  has a fixed point  $p \in X$ , and  $S$  is a Picard operator, that is,  $\{S^n x_0\}$  converges to  $p$ .

**Corollary 3.6.** *Let  $(X, d)$  be a complete metric space and let  $S$  is  $\alpha$ -admissible mappings with respect to  $\eta$  such that the following holds:*

- (i)  $S$  is a generalized  $\alpha$ -Geraghty contraction type mapping;
- (ii)  $S$  is triangular  $\alpha$ -admissible;
- (iii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$ ;
- (iv) if  $\{x_n\}$  is a sequence in  $X$  such that  $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$  for all  $n \in \mathbb{N} \cup \{0\}$  and  $x_n \rightarrow p \in X$  as  $n \rightarrow +\infty$ , then there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $\alpha(x_{n_k}, p) \geq \eta(x_{n_k}, p)$  for all  $k$ .

Then  $S$  has a fixed point  $p \in X$ , and  $S$  is a Picard operator, that is,  $\{S^n x_0\}$  converges to  $p$ .

**Example 3.1.** *Let  $X = \{i, j, k\}$  with metric*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \frac{5}{7} & \text{if } x, y \in X - \{j\} \\ 1 & \text{if } x, y \in X - \{k\} \\ \frac{4}{7} & \text{if } x, y \in X - \{i\}. \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in X, \\ 0 & \text{otherwise} \end{cases}.$$

Define mapping  $T : X \rightarrow X$  as follows:

$$T(x) = \begin{cases} i & \text{if } x \neq j \\ k & \text{if } x = j. \end{cases}$$

and  $\beta : [0, +\infty) \rightarrow [0, 1)$ . Then

$$\alpha(x, y)d(Tx, Ty) \not\leq \beta(M(x, y))M(x, y).$$

Indeed, let  $x = j$  and  $y = k$  then

$$\begin{aligned} M(j, k) &= \max \left\{ d(j, k), d(j, T(j)), d(k, T(k)), \frac{d(j, T(k)) + d(k, T(j))}{2} \right\} \\ &= \max \left\{ \frac{4}{7}, \frac{4}{7}, \frac{5}{7}, \frac{1}{2} \right\} = \frac{5}{7}. \end{aligned}$$

Theorem 2.1[4], is not valid to get fixed point of  $T$ . Since

$$\alpha(j, k)d(T(j), T(k)) \not\leq \beta(M(j, k))M(j, k).$$

Now, we prove that Theorem 16 can be applied to common fixed point of  $S$  and  $T$ . Now, consider  $S : X \rightarrow X$  be a mapping such that  $Sx = i$  for each  $x \in X$ .

where

$$\begin{aligned} M(j, k) &= \max \left\{ d(j, k), d(j, S(j)), d(k, T(k)), \frac{d(j, T(k)) + d(k, S(j))}{2} \right\} \\ &= \max \left\{ \frac{4}{7}, 1, \frac{5}{7}, \frac{12}{14} \right\} = 1. \end{aligned}$$

and

$$d(Sj, Tk) = d(i, i) = 0.$$

$$\alpha(x, y)d(Sx, Ty) \leq \beta(M(x, y)) (M(x, y)).$$

Hence all the hypothesis of the Theorem 16 is satisfied, So  $S$  and  $T$  have common fixed point.

**Remark 3.1.** More detailed, applications and examples see in [4] and references therein. Our results are more general than those in [4, 10, 19] and improve several results existing in literature.

#### 4. Conclusions

This paper presents some common fixed point theorems for a pair of  $\alpha$ -admissible mappings under the improved notion of  $\alpha$ -Geraghty contractive type condition. The presented theorems extend, generalize and improve many new and classical results in fixed point theory, in particular the very famous Banach contraction principle. The present version of these results make significant and useful contribution in the existing literature.

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