

## A COMMON FIXED POINT OF MULTI-VALUED MAPS IN B-METRIC SPACE

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*In this work we are interested to prove a general fixed point theorem for a pair of multi-valued mappings satisfying a new type of implicit relation in b-metric spaces. The results in this paper generalize the results obtained in [18], [19], [20], [22], [23] and to obtain other particular results.*

**Keywords:** Metric space, b-metric space, fixed point, implicit relation, multivalued maps.

**MSC2010:** 54H25, 47H10.

### 1. Introduction and Preliminary

The study of fixed point theory in metric spaces has several applications in mathematics, especially in solving differential equations. In [4] Bakhtin introduced a new class of generalized metric space called b-metric space which has been studied by many authors. For example, see [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 21]. This new notion of spaces has been used to obtain several results in fixed point theory. For example : in [18] the authors have generalized Banach fixed point theorem in b-metric space, see Theorem 1.3 and in [19], [20], [22], [23]. The authors have used an other form of contraction to obtain the fixed point theorem of uni-valued mappings in b-metric spaces.

In our work, using the Hausdorff-Pompeiu metric, we are dealing with the common fixed point of two multi-valued mappings verifying a certain relation in b-metric space. As a consequence of our work we obtain some results known in the case of uni-valued mappings that we will point out in the following paragraph.

**Definition 1.1** ([9]). *Let  $X$  be a nonempty set and  $s \geq 1$  be a given real number. A function  $d : X \times X \longrightarrow \mathbb{R}^+$  is said to be a b-metric on  $X$  if the following conditions hold:*

- (i)  $d(x, y) = 0$  if and only if  $x = y$ ;
- (ii)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- (iii)  $d(x, y) \leq s[d(x, z) + d(z, y)]$  for all  $x, y, z \in X$ .

*Note that every metric space is a b-metric space with  $s = 1$ . But the converse need not be true as is shown in the following example.*

**Example 1.1** ([22]). *Let  $X = \{-1, 0, 1\}$ . Define  $d : X \times X \longrightarrow \mathbb{R}^+$  by :  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ,  $d(x, x) = 0$ ,  $d(-1, 0) = 3$  and  $d(-1, 1) = d(0, 1) = 1$ .*

*Then  $(X, d)$  is a b-metric space with  $s = \frac{3}{2}$  but it is not a metric space since the triangle inequality is not satisfied. Indeed, we have*

$$d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0).$$

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**Definition 1.2** ([7]). Let  $(X, d)$  be a  $b$ -metric space,  $x \in X$  and  $(x_n)$  be a sequence in  $X$ . Then

(i)  $(x_n)$  converges to  $x$  if and only if  $\lim_{n \rightarrow \infty} d(x, x_n) = 0$ . We denote this by  $x_n \rightarrow x$  ( $n \rightarrow \infty$ ) or  $\lim_{n \rightarrow \infty} x_n = x$ .

(ii)  $(x_n)$  is Cauchy if and only if  $\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$ .

(iii)  $(X, d)$  is complete if and only if every Cauchy sequence in  $X$  is convergent.

(iv) A subset  $A \subset X$  is said to be closed if for every sequence  $x_n \in A$  such that  $x_n \rightarrow x$  we have  $x \in A$ .

(v) A subset  $A \subset X$  is said to be bounded is  $\sup_{x, y \in A} d(x, y) < +\infty$ .

(vi) A subset  $A \subset X$  is said to be compact if every sequence  $x_n \in A$  has a convergent subsequence.

Let  $(X, d)$  be a  $b$ -metric space, we denote  $CB(X)$  the set of nonempty closed bounded subsets of  $X$  provided with the Hausdorff-Pompeiu metric  $H$  defined by

$$H(A, B) = \max \left( \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right),$$

where  $A, B \in CB(X)$  and  $d(x, A) = \inf_{y \in A} d(x, y)$ . We define also  $\delta(A, B)$  by

$$\delta(A, B) = \sup \{d(a, b), \quad a \in A \quad b \in B\},$$

it follows immediately from the definition of  $\delta$  that

$$\delta(A, B) = 0 \iff A = B = \{.\} \text{ and } \delta(\{.\}, B) = H(\{.\}, B) \quad \text{and}$$

$$d(a, b) \leq \delta(A, B) \quad \forall a \in A \quad \forall b \in B.$$

In the following,  $C(X)$  means the set of nonempty compact subsets of  $X$ .

We begin by quoting the theorems which we generalize in this work.

**Theorem 1.1** (see theorem 4 in [22]). Let  $(X, d)$  be a complete  $b$ -metric space with constant  $s \geq 1$  and let  $T : X \rightarrow X$  be such that

$$d(T(x), T(y)) \leq \alpha d(x, y) + \beta d(x, T(x)) + \gamma d(y, T(y))$$

for every  $x, y \in X$ , where  $\alpha, \beta, \gamma \geq 0$  with  $\alpha + \beta + \gamma < \frac{1}{s}$ . Then  $T$  has a unique fixed point in  $X$ .

**Theorem 1.2** (see theorem 3.2 in [20]). Let  $(X, d)$  be a complete  $b$ -metric space with constant  $s \geq 1$  and let  $T : X \rightarrow X$  be such that

$$d(T(x), T(y)) \leq ad(x, T(x)) + bd(y, T(y)) + cd(x, y)$$

for every  $x, y \in X$ , where  $a, b, c \geq 0$  with  $a + s(b + c) < 1$ . Then  $T$  has a unique fixed point in  $X$ .

**Theorem 1.3** (see theorem 1 in [18]). Let  $(X, d)$  be a complete  $b$ -metric space with constant  $s \geq 1$  and let  $T : X \rightarrow X$  be such that

$$d(T(x), T(y)) \leq kd(x, y)$$

for every  $x, y \in X$ , where  $ks < 1$ . Then  $T$  has a unique fixed point in  $X$ .

**Theorem 1.4** (see theorem 3.1.2 in [19]). Let  $(X, d)$  be a complete  $b$ -metric space with constant  $s \geq 1$  and let  $T : X \rightarrow X$  be such that

$$d(T(x), T(y)) \leq \alpha d(x, y) + \beta d(x, T(x)) + \gamma d(y, T(y)) + \mu[d(x, T(y)) + d(y, T(x))]$$

for every  $x, y \in X$ , where  $\alpha, \beta, \gamma, \mu \geq 0$ , with  $s(\alpha + \beta) + \gamma + (s^2 + s)\mu \leq 1$ . Then  $T$  has a unique fixed point in  $X$ .

**Theorem 1.5** (see theorem 3.1.8 in [19]). *Let  $(X, d)$  be a complete  $b$ -metric space with constant  $s \geq 1$  and let  $T : X \rightarrow X$  be such that*

$$d(T(x), T(y)) \leq k \max \left\{ d(x, y), d(x, T(x)), d(y, T(y)), \frac{d(x, T(y)) + d(y, T(x))}{2s} \right\}$$

*for every  $x, y \in X$ , where  $0 \leq ks < 1$ . Then  $T$  has a unique fixed point in  $X$ .*

**Corollary 1.1** ( see corollary 2.3 in [23]). *Let  $(X, d)$  be a complete  $b$ -metric space with constant  $s \geq 1$  and let  $T, S : X \rightarrow X$  be two mappings. Suppose that there exist  $a > 0$  such that*

$$d(Tx, Sy) \leq \frac{1}{s+a} \max \left\{ d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2s}(d(x, Sy) + d(Tx, y)) \right\}$$

*$\forall x, y \in X$ . Then  $T$  and  $S$  have a unique common fixed point in  $X$ .*

**Corollary 1.2** ( see corollary 2.4 in [23]). *Let  $(X, d)$  be a complete  $b$ -metric space with constant  $s \geq 1$  and let  $T, S : X \rightarrow X$  be two mappings. Suppose the following inequality*

$$d(Tx, Sy) \leq \frac{1}{s^2} \max \left\{ d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2s}(d(x, Sy) + d(Tx, y)) \right\}$$

*$\forall x, y \in X$ . Then  $T$  and  $S$  have a unique common fixed point in  $X$ .*

## 2. Main results

**Definition 2.1.** *Let  $s \geq 1$ , and  $\mathcal{F}_s$  be the set of all continuous functions*

*$\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$  such that:*

*$(\phi_1) : \phi$  is nondecreasing in variable  $t_1$  and non increasing in variables  $t_2, t_3, t_4, t_5, t_6$ .*

*$(\phi_2) : \exists r \in [0, \frac{1}{s}[$ , such that  $\forall u, v \geq 0$  :*

*$\phi(u, v, v, u, s(u+v), 0) \leq 0$  or  $\phi(u, v, u, v, 0, s(u+v)) \leq 0$  or  $\phi(u, 0, 0, v, sv, 0) \leq 0$  or*

*$\phi(u, 0, v, 0, 0, sv) \leq 0 \implies u \leq rv$ .*

*Moreover, if  $\forall u \geq 0$  such that  $\phi(u, u, 0, 0, u, u) \leq 0 \implies u = 0$ , we say that  $\phi$  check  $(\phi_3)$ .*

**Example 2.1.**  $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - (\alpha t_2 + \beta t_3 + \gamma t_4)$ . with  $\alpha, \beta, \gamma \geq 0$ ;  $\alpha + \beta + \gamma < \frac{1}{s}$ .

**Example 2.2.**  $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - (\alpha t_2 + \beta t_3 + \gamma t_4)$ . with  $\beta + s(\alpha + \gamma) < 1$ .

**Example 2.3.**  $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - rt_2$  with  $rs < 1$ .

**Example 2.4.**

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - (\alpha t_2 + \beta t_3 + \gamma t_4 + \mu[t_5 + t_6])$$

*with  $\alpha, \beta, \gamma, \mu \geq 0$  and  $s(\alpha + \beta) + \gamma + (s^2 + s)\mu \leq 1$ .*

**Example 2.5.**

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - r \max \left\{ t_2, t_3, t_4, \frac{t_5 + t_6}{2s} \right\}$$

*where  $0 \leq r < 1$ , with  $rs < 1$ .*

**Example 2.6.**

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \frac{1}{s+a} \max \left\{ t_2, t_3, t_4, \frac{t_5 + t_6}{2s} \right\} \quad \text{with } a > 0.$$

**Example 2.7.**

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \frac{1}{s^2} \max \left\{ t_2, t_3, t_4, \frac{t_5 + t_6}{2s} \right\}.$$

**Theorem 2.1.** Let  $(X, d)$  be a complete  $b$ -metric space with constant  $s$ . We suppose that  $d$  is continuous with respect to one of its variables,  $F, G : X \rightarrow C(X)$  and  $\phi \in \mathcal{F}_s$  such that

$$\phi(H(Fx, Gy), d(x, y), d(x, Fx), d(y, Gy), d(x, Gy), d(y, Fx)) \leq 0 \quad (1).$$

Then  $F$  and  $G$  have a common fixed point  $x \in X$ .

Moreover, if  $x$  is absolutely fixed for  $F$  or  $G$  ( which means that  $F(x) = \{x\}$  or  $G(x) = \{x\}$  ) and  $\phi$  check  $(\phi_3)$ , then the fixed point is unique.

For the proof of this theorem we need two lemmas.

**Lemma 2.1.** In a  $b$ -metric space  $(X, d)$ , if the function  $d$  is continuous with respect to one of its variable, then it is continuous with respect to the other.

*Proof.* Suppose that  $d$  is continuous with respect to the first variable, and let  $(y_n)$  be a sequence of elements of  $X$  such that  $(y_n)$  is  $b$ -convergent to  $y \in X$ . Then since  $d$  is symmetric we have for all  $x \in X$  :

$$\lim_{n \rightarrow \infty} d(x, y_n) = \lim_{n \rightarrow \infty} d(y_n, x) = d(y, x) = d(x, y).$$

□

**Lemma 2.2.** Let  $(X, d)$  be a  $b$ -metric space and let  $A \subset C(X)$ , if  $d$  is continuous with respect to one of its variables, then for all  $x \in X$ , there exists  $y_0 \in A$  such that

$$d(x, A) = \inf_{y \in A} d(x, y) = d(x, y_0).$$

*Proof.* We have  $d(x, A) = \inf_{y \in A} d(x, y)$ , so for every  $n \in \mathbb{N}^*$  there exists  $x_n \in A$  such that

$$d(x, A) - \frac{1}{n} < d(x, A) \leq d(x, x_n) < d(x, A) + \frac{1}{n}.$$

Since  $A$  is compact,  $(x_n)$  has a subsequence, also noted  $(x_n)$ , which is  $b$ -convergent to  $x_0 \in A$ . So :

$$|d(x, x_n) - d(x, A)| < \frac{1}{n} \rightarrow 0 \quad \text{when } n \rightarrow \infty.$$

from where  $\lim_{n \rightarrow \infty} d(x, x_n) = d(x, A)$ , so since  $d$  is continuous we deduce that

$$\lim_{n \rightarrow \infty} d(x, x_n) = d(x, x_0),$$

hence from the uniqueness of the limit in a  $b$ -metric space, we have  $d(x, x_0) = d(x, A)$ . □

*Proof. of the main result.*

### **Existence.**

Let  $x_0 \in X$  and  $x_1 \in Fx_0$ , let's show that there exists  $x_2 \in Gx_1$  such that

$$d(x_1, x_2) \leq H(Fx_0, Gx_1).$$

Since  $x_1 \in Fx_0$ , we have :

$$d(x_1, Gx_1) \leq e(Fx_0, Gx_1) \leq H(Fx_0, Gx_1).$$

And we have  $Gx_1$  is compact and  $d$  is continuous with respect to one of its variables, so according to lemma 2.2 there exists  $x_2 \in Gx_1$  such that :

$$d(x_1, x_2) \leq H(Fx_0, Gx_1).$$

In the same, since

$$\begin{aligned} d(x_2, Fx_2) &\leq e(Gx_1, Fx_2) \\ &\leq H(Gx_1, Fx_2) \end{aligned}$$

and  $Fx_2$  is compact, there exist  $x_3 \in Fx_2$  such that

$$d(x_3, x_2) \leq H(Fx_2, Gx_1).$$

In the same there exist  $x_4 \in Gx_3$  such that

$$d(x_3, x_4) \leq H(Gx_3, Fx_2).$$

By recurrence, we construct a sequence  $(x_n)$  such that  $x_{2n+1} \in Fx_{2n}$ , and  $x_{2n+2} \in Gx_{2n+1}$  which satisfies :

$$d(x_{2n+1}, x_{2n}) \leq H(Fx_{2n}, Gx_{2n-1}) \quad \text{and} \quad d(x_{2n+2}, x_{2n+1}) \leq H(Fx_{2n}, Gx_{2n+1}).$$

According to  $(1)$  we have :

$$\phi(H(Fx_{2n}, Gx_{2n-1}), d(x_{2n}, x_{2n-1}), d(x_{2n}, Fx_{2n}), d(x_{2n-1}, Gx_{2n-1}), d(x_{2n}, Gx_{2n-1}), d(x_{2n-1}, Fx_{2n})) \leq 0.$$

Now using  $(\phi_1)$ , we deduce that

$$\phi(H(Fx_{2n}, Gx_{2n-1}), d(x_{2n}, x_{2n-1}), H(Fx_{2n}, Gx_{2n-1}), d(x_{2n-1}, x_{2n}), 0, s[d(x_{2n-1}, x_{2n}) + H(Fx_{2n}, Gx_{2n-1})]) \leq 0.$$

So according to  $(\phi_2)$ , we have

$$H(Fx_{2n}, Gx_{2n-1}) \leq rd(x_{2n}, x_{2n-1})$$

hence

$$d(x_{2n+1}, x_{2n}) \leq rd(x_{2n}, x_{2n-1}) \quad (*).$$

In the same way, we have

$$\phi(H(Fx_{2n}, Gx_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n}, Fx_{2n}), d(x_{2n+1}, Gx_{2n+1}), d(x_{2n}, Gx_{2n+1}), d(x_{2n+1}, Fx_{2n})) \leq 0.$$

Now using  $(\phi_1)$ , we deduce that

$$\phi(H(Fx_{2n}, Gx_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), H(Fx_{2n}, Gx_{2n+1}), s[d(x_{2n}, x_{2n+1}) + H(Fx_{2n}, Gx_{2n+1})], 0) \leq 0.$$

So according to  $(\phi_2)$ , we have

$$H(Fx_{2n}, Gx_{2n+1}) \leq rd(x_{2n}, x_{2n+1})$$

hence

$$d(x_{2n+2}, x_{2n+1}) \leq rd(x_{2n}, x_{2n+1}) \quad (**).$$

So for everything  $n \in \mathbb{N}^*$   $d(x_n, x_{n+1}) \leq rd(x_{n-1}, x_n)$ , and consequently

$$\begin{aligned} d(x_n, x_{n+1}) &\leq rd(x_{n-1}, x_n) \\ &\leq r^2 d(x_{n-2}, x_{n-1}) \\ &\leq r^3 d(x_{n-3}, x_{n-2}) \\ &\leq \dots \\ &\leq r^n d(x_0, x_1). \end{aligned}$$

Now we have to show that  $(x_n)$  is a Cauchy sequence. Let  $m, n \in \mathbb{N}$ , such that  $n < m$ , then

$$\begin{aligned} d(x_n, x_m) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + \\ &\dots + s^{m-n-1} d(x_{m-2}, x_{m-1}) + s^{m-n-1} d(x_{m-1}, x_m) \end{aligned}$$

On the other hand we have :

$$\begin{aligned}
d(x_n, x_m) &\leq sr^n d(x_0, x_1) + s^2 r^{n+1} d(x_0, x_1) + s^3 r^{n+2} d(x_0, x_1) + \\
&\quad \dots + s^{m-n-1} r^{m-2} d(x_0, x_1) + s^{m-n-1} r^{m-1} d(x_0, x_1) \\
&\leq sr^n (1 + (sr) + (sr)^2 + \dots + (sr)^{m-n-2} + s^{m-n-2} r^{m-n-1}) d(x_0, x_1) \\
&= sr^n \left( \frac{1 - (sr)^{m-n-1}}{1 - sr} + s^{m-n-2} r^{m-n-1} \right) d(x_0, x_1) \\
&\leq r^n \left( \frac{s}{1 - sr} + (sr)^{m-n-1} \right) d(x_0, x_1).
\end{aligned}$$

from where  $\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$ . Then  $(x_n)$  is a Cauchy sequence. As the  $b$ -metric space  $(X, d)$  is complete, there exists  $x \in X$  such that  $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ . Next we show that  $x \in Fx$  and  $x \in Gx$ , indeed, by (1) we have

$$\begin{aligned}
&\phi(H(Fx_{2n}, Gx), d(x_{2n}, x), d(x_{2n}, Fx_{2n}), d(x, G(x)), d(x_{2n}, Gx), d(x, Fx_{2n})) \leq 0 \\
\Rightarrow &\phi(H(Fx_{2n}, Gx), d(x_{2n}, x), d(x_{2n}, x_{2n+1}), d(x, Gx), d(x_{2n}, Gx), d(x, x_{2n+1})) \leq 0 \\
\Rightarrow &\phi(H(Fx_{2n}, Gx), d(x_{2n}, x), d(x_{2n}, x_{2n+1}), d(x, Gx), s[d(x_{2n}, x) + d(x, Gx)], d(x, x_{2n+1})) \leq 0.
\end{aligned}$$

letting  $n \rightarrow \infty$  we obtain

$$\phi\left(\liminf_{n \rightarrow \infty} H(Fx_{2n}, Gx), 0, 0, d(x, Gx), sd(x, Gx), 0\right) \leq 0.$$

Now using the fact that  $H(Fx_{2n}, Gx_{2n+1}) \leq rd(x_{2n}, x_{2n+1})$ ,  $x_{2n+2} \in Gx_{2n+1}$  and

$$\begin{aligned}
H(Fx_{2n}, Gx) &\leq sH(Fx_{2n}, \{x_{2n+2}\}) + sH(\{x_{2n+2}\}, Gx) \\
&\leq sH(Fx_{2n}, Gx_{2n+1}) + s^2 H(\{x_{2n+2}\}, \{x\}) + s^2 H(\{x\}, Gx) \\
&\leq srd(x_{2n}, x_{2n+1}) + s^2 d(x_{2n+2}, x) + s^2 H(\{x\}, Gx),
\end{aligned}$$

we deduce that the sequence  $(H(Fx_{2n}, Gx))_n$  is bounded. Then by  $(\phi_2)$ , we have

$$\liminf_{n \rightarrow \infty} H(Fx_{2n}, Gx) \leq rd(x, Gx). \quad (2)$$

On the other hand we show that  $d(x, Gx) = 0$ . Suppose that  $d(x, Gx) > 0$ , then

$$d(x, Gx) \leq s[d(x, x_{2n+1}) + d(x_{2n+1}, Gx)] \leq s[d(x, x_{2n+1}) + H(Fx_{2n}, Gx)],$$

by (2) we have

$$\begin{aligned}
d(x, Gx) &\leq \liminf_{n \rightarrow \infty} s[d(x, x_{2n+1}) + H(Fx_{2n}, Gx)] \\
&= srd(x, Gx) \\
&< d(x, Gx)
\end{aligned}$$

which is a contradiction. Hence  $d(x, Gx) = 0$  and consequently  $x \in Gx$ , also we have  $x \in Fx$ .

Indeed. by (1) we have

$$\begin{aligned}
&\phi(H(Fx, Gx_{2n-1}), d(x, x_{2n-1}), d(x, Fx), d(x_{2n-1}, Gx_{2n-1}), d(x, Gx_{2n-1}), d(x_{2n-1}, Fx)) \leq 0 \\
\Rightarrow &\phi(H(Fx, Gx_{2n-1}), d(x, x_{2n-1}), d(x, Fx), d(x_{2n-1}, x_{2n}), d(x, x_{2n}), d(x_{2n-1}, Fx)) \leq 0 \\
\Rightarrow &\phi(H(Fx, Gx_{2n-1}), d(x, x_{2n-1}), d(x, Fx), d(x_{2n-1}, x_{2n}), d(x, x_{2n}), s[d(x_{2n-1}, x) + d(x, Fx)]) \leq 0.
\end{aligned}$$

letting  $n \rightarrow \infty$  we obtain

$$\phi\left(\liminf_{n \rightarrow \infty} H(Fx, G(x_{2n-1})), 0, d(x, F(x)), 0, 0, sd(x, F(x))\right) \leq 0.$$

Then by  $(\phi_2)$ , we have

$$\liminf_{n \rightarrow \infty} H(Fx, Gx_{2n-1}) \leq rd(x, Fx). \quad (3)$$

On the other hand we have

$$\begin{aligned} d(x, Fx) &\leq s[d(x, x_{2n}) + d(x_{2n}, Fx)] \\ &\leq s[d(x, x_{2n}) + H(Fx, Gx_{2n-1})], \end{aligned}$$

by (3) we have

$$\begin{aligned} d(x, Fx) &\leq \liminf_{n \rightarrow \infty} s[d(x, x_{2n}) + H(Fx, Gx_{2n-1})] \\ &= srd(x, Fx) \\ &< d(x, Fx). \end{aligned}$$

which is a contradiction if  $d(x, Fx) > 0$ . Hence  $d(x, F(x)) = 0$  and consequently  $x \in F(x)$ .

**Unicity.** Suppose that  $F(x) = \{x\}$  and  $\phi$  check the  $(\phi_3)$  and  $y \in X$  is an other common fixed point of  $F$  and  $G$ , then by (1) we have

$$\phi(H(Fx, Gy), d(x, y), d(x, Fx), d(y, Gy), d(x, Gy), d(y, Fx)) \leq 0,$$

consequently

$$\phi(H(x, Gy), d(x, y), d(x, x), d(y, Gy), d(x, Gy), d(y, x)) \leq 0,$$

so

$$\phi(d(x, y), d(x, y), 0, 0, d(x, y), d(y, x)) \leq 0.$$

By  $(\phi_3)$  we have  $d(x, y) = 0$ , then  $x = y$ . So  $x$  is the unique common fixed point of  $F$  and  $G$ .  $\square$

As a consequence of theorem 2.1, if  $F = G = T$ , then we obtain the following corollary

**Corollary 2.1.** *Let  $(X, d)$  be a complete b-metric space with constant  $s$ . We suppose that  $d$  is continuous with respect to noe of its variables,  $T : X \longrightarrow C(X)$  and  $\phi \in \mathcal{F}_s$  such that*

$$\phi(H(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

*Then  $T$  has a fixed point  $x \in X$ . Moreover, if  $x$  is absolutely fixed and  $\phi$  check  $(\phi_3)$ , then the fixed point is unique.*

**Example 2.8.** *Let  $(X = [0, +\infty[, d)$  be a complete b-metric space with constant  $s = 2$ ,  $d(x, y) = (x - y)^2$ . We define  $T : X \longrightarrow C(X)$ , by*

$$T(x) = \begin{cases} 0 & \text{if } x < 2 \\ \left[0, \frac{x}{1+x}\right] & \text{otherwise} \end{cases}$$

*we prove that  $T$  check  $H(Tx, Ty) \leq \frac{1}{3} \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \left[ \frac{d(x, Ty) + d(y, Tx)}{2s} \right] \right\}$ . Indeed, we have the following situations:*

1) *If  $x, y \in [0, 2[$ , then*

$$H(Tx, Ty) = 0 \leq \frac{1}{3} \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \left[ \frac{d(x, Ty) + d(y, Tx)}{2s} \right] \right\}.$$

2) If  $x \in [0, +\infty[$ ,  $y \in [2, +\infty[$ , then

$$\begin{aligned}
 H(Tx, Ty) &= \left( \frac{y}{y+1} \right)^2 \\
 &\leq \frac{4}{3} \left( \frac{y}{y+1} \right)^2 \\
 &\leq \frac{1}{3} \left( \frac{y^2}{y+1} \right)^2 \\
 &= \frac{1}{3} d(y, Ty) \\
 H(Tx, Ty) &\leq \frac{1}{3} \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \left[ \frac{d(x, Ty) + d(y, Tx)}{2s} \right] \right\}.
 \end{aligned}$$

All the conditions of theorem 2.1 are satisfied with  $\phi$  as in example 2.5, then 0 is the unique absolutely fixed point of  $T$ .

Now if we replace  $H(.,.)$  by  $\delta(.,.)$ , then we obtain the following result :

**Theorem 2.2.** Let  $(X, d)$  be a complete  $b$ -metric space with constant  $s$ ,  $F, G : X \rightarrow CB(X)$  and  $\phi \in \mathcal{F}_s$  such that

$$\phi(\delta(Fx, Gy), d(x, y), d(x, Fx), d(y, Gy), d(x, Gy), d(y, Fx)) \leq 0. \quad (4)$$

Then  $F$  and  $G$  have a common fixed point. Moreover if  $\phi$  check  $(\phi_3)$ , then the fixed point is unique.

*Proof.* **Existence.**

Let  $x_n \in X$  such that  $x_{2n+1} \in Fx_{2n}$  and  $x_{2n} \in Gx_{2n-1}$ , then by (4), we have :

$$\phi(\delta(Fx_{2n}, Gx_{2n-1}), d(x_{2n}, x_{2n-1}), d(x_{2n}, Fx_{2n}), d(x_{2n-1}, Gx_{2n-1}), d(x_{2n}, Gx_{2n-1}), d(x_{2n-1}, Fx_{2n})) \leq 0.$$

and

$$\phi(\delta(Fx_{2n}, Gx_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n}, Fx_{2n}), d(x_{2n+1}, Gx_{2n+1}), d(x_{2n}, Gx_{2n+1}), d(x_{2n+1}, Fx_{2n})) \leq 0.$$

So using the same argument as in theorem 2.1, we deduce that  $x_n$  is a Cauchy sequence and converges to the common fixed point of  $F$  and  $G$ .

**Unicity.** Suppose that  $\phi$  check  $(\phi_3)$ . Let  $y \in X$  be an other common fixed point of  $F$  and  $G$ , then by (4) we have

$$\phi(\delta(Fx, Gy), d(x, y), d(x, Fx), d(y, Gy), d(x, Gy), d(y, Fx)) \leq 0,$$

so

$$\phi(\delta(Fx, Gy), \delta(Fx, Gy), 0, 0, \delta(Fx, Gy), \delta(Fx, Gy)) \leq 0,$$

hence By  $(\phi_3)$  we have  $\delta(Fx, Gy) = 0$ , so  $Fx = Gy = \{x\} = \{y\}$  and  $x$  is the unique common absolutely fixed point of  $F$  and  $G$ .  $\square$

-If  $F = f$  and  $G = g$  are single valued mappings, then by theorem 2.2 we obtain the following corollary

**Corollary 2.2.** Let  $(X, d)$  be a complete  $b$ -metric space with constant  $s$ ,  $f, g : X \rightarrow X$  and  $\phi \in \mathcal{F}_s$  such that

$$\phi(d(f(x), g(y)), d(x, y), d(x, f(x)), d(y, g(y)), d(x, g(y)), d(y, f(x))) \leq 0,$$

then  $f$  and  $g$  have a common fixed point. Moreover if  $\phi$  check  $(\phi_3)$ , then the fixed point is unique.

If  $f = g = T$  in corollary 2.2, then we obtain the following corollary



**Corollary 2.3.** *Let  $(X, d)$  be a complete  $b$ -metric space with constant  $s$ ,  $T : X \rightarrow X$  and  $\phi \in \mathcal{F}_s$  such that*

$$\phi(d(T(x), T(y)), d(x, y), d(x, T(x)), d(y, T(y)), d(x, T(y)), d(y, T(x))) \leq 0,$$

*then  $T$  has a fixed point  $x \in X$ . Moreover, if  $\phi$  check  $(\phi_3)$ , then the fixed point is unique.*

### 3. Consequences of the main result

From corollary 2.3 and example 2.1 we obtain theorem 1.1

From corollary 2.3 and example 2.2 we obtain theorem 1.2

From corollary 2.3 and example 2.3 we obtain theorem 1.3

From corollary 2.3 and example 2.4 we obtain theorem 1.4

From corollary 2.3 and example 2.5 we obtain theorem 1.5

From corollary 2.2 and example 2.6 we obtain corollary 1.1

From corollary 2.2 and example 2.7 we obtain corollary 1.2

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