SELECTING CHEBYSHEV'S NORM AS A SECONDARY GOAL FOR RANKING IN THE PRESENCE OF SYMMETRY FACTOR IN DEA

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In data envelopment analysis (DEA), the cross-efficiency evaluation method describes a cross-efficiency matrix in which the units are self and peer evaluated. The problem that may reduce the usage of the cross-efficiency evaluation method is that it cannot be unique for the cross-efficiency of scores, because of the alternate optima. In this way it stands to define logically secondary goals and introduce it to cross-efficiency appraisement. Here we propose the symmetric weight assignment technique (SWAT) which does not influence feasibility and rewards decision making units (DMUs)and then we conclude a symmetric selection of weights. At the end a numerical problem is investigated by our proposed method and its results are collated with former methods.

Keywords: Data envelopment analysis, Ranking, Chebyshev' s norm.
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1. Introduction

Data envelopment analysis (DEA) is a linear programming that measures the relative efficiency of decision making units (DMUs) and has been far developed [5]. The best condition for DEA models is their having a unique efficiency score. But there are no restrictions on value weight that can be on any individual input or output relative to others. DMUs choose weights to make themselves appear more efficient relative to other DMUs. So each DMUs may choose all of its weights on some variables. Therefore, many studies have focused on approaches to restrict the flexibility of weights. The exact method for determining weight restrictions is based on a particular application or expert opinion about the relative significance of the variables while preserving linearity and affecting the feasibility region. Recently Dimitrov and Sutton [8] have proposed a model for restricting weights with the aim of deploying symmetry in weight allocation. In this paper we proposed the *chebyshev's* norm as a secondary goal in DEA cross-efficiency evaluation.The cross-evaluation method ranks *DMUs* using cross-efficiency scores in the study of Sexton [12]. The basic opinion of cross-evaluation is to use DEA in a peer-evaluation in lieu of a self-evaluation mode. A problem that may reduce the usefulness of cross-efficiency evaluation method is the non-uniqueness of DEA optimal, because cross-efficiency scores obtained from the classic DEA are not

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usually unique. Therefore, researchers recommend the use of secondary goals to deal with the non-uniqueness issue [12] and Doyle and Green [7] proposed aggressive and benevolent model formulation. Liang et al. [11] developed Doyle and Green' s [7] method. Wu et al. [14] extended a new method based on rank priority as secondary goal. In this paper, we propose a secondary goal for cross-efficiency evaluation. By selecting symmetric weight by $DMUs$, giving and solving a numerical example by our proposed method, we compare it with alternative approaches. The remainder of this paper has the following structure: in section 2, we describe the background. Section 3 presents our method as a secondary goal in DEA cross-efficiency evaluation. Section 4 illustrates the proposed method using a numerical example and collates it with former methods. At the end, conclusions are presented in section 5.

2. Background

2.1 DEA models

Data envelopment analysis (DEA) was introduced as a method of measuring relative efficiency of a group of similar decision making units $(DMUs)$. Consider n decision making units DMU_j (j=1,...,n). X_j is a vector containing the values for the input variables of DMU_j and similarly Y_j is a vector containing the values for the output variables of DMU_j. X_o and Y_o are the input and output vectors for the DMU under evaluation. U is a vector for the output weight and V is a vector for the input weight. The relative efficiency score of DMU_o under the CCR model is given by the following optimization problem:

$$\begin{aligned} Max \quad & \frac{U^T Y_o}{V^T X_o} \\ S.t. \quad & \frac{U^T Y_j}{V^T X_j} \le 1, \qquad j = 1,...,n \\ & U \ge 0, \quad V \ge 0. \end{aligned} \tag{1}$$

We know that DEA model (1) is equivalent to the following output-oriented formulation as described in Charnes et al. [5].

$$\begin{aligned} Min \quad & V^T X_o \\ S.t. \quad & U^T Y_o = 1 \\ & -V^T X_j + U^T Y_j \le 0, \quad j = 1,...,n \\ & U \ge 0, \quad V \ge 0. \end{aligned} \tag{2}$$

2.2 Weight restrictions

One of the intense limitations of usual DEA models is their weight flexibility, allowing a *DMU* to search maximum efficiency by selecting a composition of weights that either is implausible because it ignores one or more variables, or is unacceptable because it is consistent with the expert judgment available to Decision Making. So weight flexibility results in two *DMUs* having equal efficiency scores, one with all its weight on one variable and the other with its weight symmetric to all variables. Therefore, this problem has led to the development of weight restriction DEA models [2]. We considered a lower and an upper bound on outputs or inputs as follows:

$$\begin{aligned} a_r \le u_r y_{rj} \le b_r \quad & \forall r, j \\ c_i \le v_i x_{ij} \le d_i \quad & \forall i, j. \end{aligned} \tag{3}$$

But by adding these restrictions to models, the programs will often be infeasible. So, Dimitrov and Sutton [8] have proposed a model that has not the problem, but rewards *DMUs* that make a symmetric choice of weights. The total measure of symmetry is relative to the value of each output dimension all; other output dimensions and for input variables the measure of symmetry is relative to each input dimension with all other input dimensions:

$$\begin{aligned} |u_i y_{oi} - u_j y_{oj}| &= z_{ij} \\ |v_i x_{oi} - v_j x_{oj}| &= t_{ij} \end{aligned} \tag{4}$$

which z_{ij} in (4) is the difference in symmetry between output dimension i and dimension j for the DMU under evaluation and T_{ij} in (4) is the difference in symmetry between input dimension i and dimension j for the DMU under evaluation. As we would like to reward symmetry, if we suppose that $Z = Max\{z_{ij} \mid \forall i, j = 1,..., n\}$, $|u_i y_{oi} - u_j y_{oj}| = z_{ij} \le Z, \ \forall i, j$ and $T = Max\{t_{ij} \mid \forall i, j = 1,..., m\}$ so, $|v_i x_{oi} - v_j x_{oj}| = t_{ij} \le T, \ \forall i, j.$ Then we effectively reward symmetry with a symmetry scaling factor $\beta \ge 0.$ Adding the symmetry constraint to objective function rewrites (2) to:

$$
\begin{aligned}
&Min \quad V^T X_o + \beta(Z+T) \\
&S.t. \quad U^T Y_o = 1 \\
&\qquad -V^T X_j + U^T Y_j \le 0, \quad j = 1,...,n \\
&\qquad |u_i y_{oi} - u_j y_{oj}| = Z \qquad i, j = 1,...,s \\
&\qquad |v_i x_{oi} - v_j x_{oj}| = T \qquad i, j = 1,...,m \\
&\qquad U \ge 0, \quad V \ge 0.
\end{aligned} \tag{5}
$$

Note that (5) is not linear. But by changing the absolute value function in model (5) we can change it to linear forms.

$$
\begin{aligned}
&Min \quad V^T X_o + \beta(Z+T) \\
&S.t. \quad U^T Y_o = 1 \\
&\qquad -V^T X_j + U^T Y_j \le 0, \quad j = 1,...,n \\
&\qquad u_i y_{oi} - u_j y_{oj} \le Z, \qquad i, j = 1,...,s \\
&\qquad -u_i y_{oi} + u_j y_{oj} \le Z, \quad i, j = 1,...,s \\
&\qquad v_i x_{oi} - v_j x_{oj} \le T, \qquad i, j = 1,...,m \\
&\qquad -v_i x_{oi} - x_j x_{oj} \le T, \quad i, j = 1,...,m \\
&\qquad U \ge 0, \quad V \ge 0.
\end{aligned} \tag{6}
$$

Instead of having an explicit bound, we introduce the symmetry scaling factor β as non-negative importance factor in model (6) which can be used for the output-oriented formulation.

2.3 Cross-efficiency evaluation

The cross-evaluation matrix was first developed by Sexton et al. [12]. The matrix is calculated using the standard DEA model (model (1)) for any DMU_o under evaluation, the efficiency score θ_{oo}^* under the CCR model is given by model (1). The cross-efficiency of DMU_j using the weights that DMU_o has selected in

model (1) or (2), is then:

$$\theta_{oj} = \frac{U_o^* Y_j}{V_o^* X_j}. \tag{7}$$

Where (U_o^*, V_o^*) signs optimal values in model (1) or (2), when DMU_o is evaluated. For DMU_j $(j = 1,...,n)$, the average of all θ_{oj} (o =1,...,n), is referred to as the cross-efficiency score for DMU_j. The optimal weights obtained from model (1) or (2) may not be unique. So, to resolve this ambiguity, we offer Chebyshev' s norm as a secondary goal in cross-efficiency evaluation.

3. Implementation of a secondary goal in Sexton method

As mentioned before, the cross-efficiency scores, obtained from model (1) or (2) are not unique and may have an alternate optima. So the need for establishing a secondary goal or criterion that can be used for choosing weights by selections from optima solution for multipliers in model (2) can improve the model. Using the weight restriction we propose a secondary goal by Chebyshev' s norm in cross-efficiency method. The advantages of this method compared to with the method of G.R. Jahanshahloo et al. [12]. is decreasing the variables from s^2 to one and from m^2 to one, according to the following algorithm:

Step 1: Determine the efficiencies, θ_o (o=1,...,n) for all *DMUs* after solving model (2).

Step 2: After obtaining the efficiencies of all *DMUs*, we can select the solutions via the secondary goal for each DMU as follows:

$$\begin{aligned}
Min \quad & Z_o + T_o && (a)\\
S.t. \quad & V_o X_o = \theta_o && (b)\\
& U_o^t Y_j - V_o^t X_j \le o, && j = 1,...,n \quad (c)\\
& u_{oi} y_{oi} - u_{oj} y_{oj} \le Z_o, && \forall i, j \quad (d)\\
& -u_{oi} y_{oi} + u_{oj} y_{oj} \le Z_o, && \forall i, j \quad (e)\\
& v_{oi} x_{oi} - v_{oj} x_{oj} \le T_o, && \forall i, j \quad (f)\\
& -v_{oi} x_{oi} + v_{oj} x_{oj} \le T_o, && \forall i, j \quad (g)\\
& U_o \ge d, \quad V_o \ge d && (h).
\end{aligned} \tag{8}$$

Where (a), (b), (c), and (f) in model (8) present the optimal solution set in model (2). Our goal is to select the symmetric weight though optimal solutions by adding constraints (d), (e), (f), and (g) and minimizing Z and T.

In model (8) we introduced a method to reward by symmetry selecting weights. That is a suitable approach because weights are centralized on only one variable. In the proposed method we chose symmetry weight with Chebyshev' s norm as a secondary goal in DEA cross-efficiency evaluation.

Step 3: The cross-efficiency for any DMU_j using the weights that DMU_o has chosen in model (8), is then

$$\theta_{oj} = \frac{U_o^* Y_j}{V_o^* X_j}. \tag{9}$$

The new cross-efficiency score for DMU_j is as follows:

$$\theta_j = \frac{1}{n}\sum_{o=1}^{n}\theta_{oj}. \tag{10}$$

We can use the model used in section 2.2, instead of step 1 and 2 as follows:

$$\begin{aligned}
Min \quad & V_o^T X_o + \beta(Z_o + T_o) \\
S.t. \quad & U_o^T Y_o = 1 \\
& -V_o^T X_j + U_o^T Y_j \le 0, \quad j = 1,...,n \\
& u_{oi} y_{oi} - u_j y_{oj} \le Z_o, \quad i, j = 1,...,s \\
& -u_{oi} y_{oi} - u_j y_{oj} \le Z_o, \quad i, j = 1,...,s \\
& v_{oi} x_{oi} - v_j x_{oj} \le T_o, \quad i, j = 1,...,m \\
& -v_i x_{oi} - v_j x_{oj} \le T_o, \quad i, j = 1,...,m \\
& U_o \ge 0, \quad V_o \ge 0.
\end{aligned} \tag{11}$$

This model explicitly rewards DMUs that make a symmetric selection of weights. The β is the symmetry scaling as a non-negative importance factor. This parameter determines how much a particular DMU will be penalized for a symmetric selection of virtual weights. The ideal β value is the decision maker.

4 Numerical example

Sexton et al. [12] considered a case of six nursery homes reported in Table 1 as input and output data for a given year as follows:

StHr (x_1): staff hours per day, including nurses, physicians, etc.

Supp (x_2) : supplies per day, measured in thousands of dollars.

MCPD (y_1): total medicare-plus medicaid-reimbursed patient days (0000).

PPPD (y_2): total privately paid patient days (0000) [11].

Table 1

Nursing home data

DMU	$\frac{input}{StHr(x_1)}$	$\frac{input}{Supp(x_2)}$	$\frac{output}{MCPD(y_1)}$	$\frac{output}{PPPD(y_2)}$
A	1.5	0.2	1.4	0.35
B	4	0.7	1.4	2.1
C	3.2	1.2	4.2	1.05
D	5.2	2	2.8	4.2
E	3.5	1.2	1.9	2.5
F	3.2	0.7	1.4	1.5

Table 2 presents the results of the ranking for model (8) then the results are collated with former methods. With the results of Alder et al. [1] and Liang et al .[11], Table 3 presents the results of the ranking for model (11) and Table 4 presents the results of the ranking for Dimitrov and Sutton' s model [8] and shows that Table 4 for $\beta \geq 1$ are different from other methods. In this example we present that cross-efficiency evaluation with different secondary goals that have different rankings. For example, the results of both cross-efficiency and Dimitrov and Sutton ' s model are different from other methods.

Table 2

The ranking for model (8) and comparison with other methods.

DMU	*A*	*B*	*C*	*D*	*E*	*F*
model(8)(rank)	1	4	2	3	5	6
CCR	1	1	1	1	5	6
Additive[6]	1	1	1	1	5	6
BCC[4]	1	1	1	1	1	6
Supper-efficiency[3]	1	2	3	4	5	6
Statistical-based model(CCA)[9]	1	2	3	4	5	6
Statistical-based model(DR/DEA)[13]	1	5	2	3	4	6
Cross-efficiency-aggresive	1	2	5	2	4	6
Cross-efficiency-benevolent	1	4	5	1	3	6
Liang et al,smodel	1	4	5	1	3	6

Table 3

The results of cross-efficiency score for model (11).

	A	B	C	D	E	F
$\beta = 0$	4	1	5	2	3	6
$\beta = 0.05$	1	4	5	2	3	6
$\beta = 0.1$	1	4	5	2	3	6
$\beta = 0.5$	1	5	4	2	3	6
$\beta = 1$	1	4	2	3	5	6
$\beta \geq 1.5$	1	4	2	3	5	6

According to the results that have been presented in Table 2 and 3 we see that for $\beta \geq 1$ the results of ranking in Table 3 is the same as model (8). It means that the ranking DMUs in both methods are the same. From another angle for $\beta = 0.1$ the obtained ranking from Cross-efficiency-benevolent [7] and Liang et al. [11] are the same, too. Also in Table 3 we see that, by increasing the value of β, the rank of DMUA and DMUC have increased and the rank of DMUD and DMUE have decreased.

Table 4:

The results of the ranking for Dimitrov and Sutton' s model [8].

	A	*B*	*C*	*D*	*E*	*F*
$\beta = 0$	1	1	1	1	5	6
$\beta = 0.05$	1	4	2	3	5	6
$\beta = 0.1$	1	4	2	3	5	6
$\beta = 0.5$	1	5	2	3	4	6
$\beta = 1$	1	6	3	4	2	5
$\beta \geq 1.5$	1	6	3	4	2	5

The rank of DMUF is 6 in all methods but in Dimitrov and Sutton' s [8] model by increasing the value of β, the rank of DMUF has improved. For $\beta = 0$ the results of Dimitrov and Sutton' s model [8] are exactly the same as CCR model. For $\beta = 0.05$ the results of Dimitrov and Sutton' s and (8) and (11) models, are the same as each other. For $\beta = 0.5$ the results of Dimitrov and Sutton' s model [8] are exactly the same as Statistical-based model (DR / DEA) [13].

5. Conclusions

The problem that may reduce the usage of the cross-efficiency evaluation method is that it cannot be unique for the cross-efficiency of scores, because of the existence of alternate optima and because DEA weights are not unique generally. So this paper recommends a new secondary goal by Chebyshev's norm based on symmetric weights selections. The advantage of our proposed method in comparison with the former method is decreasing the variables from s^2 to one and from m^2 to one. We can use β_{ij}, instead of β, for denoting the relationship between every dimensional pair. We propose a method for applying symmetric weight assignment technique (SWAT) [8] for cross-efficiency evaluation [12]. We increased symmetric constraints into output weights to reward symmetric output and to preserve its linearity.

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