LIE SYMMETRY ANALYSIS AND CONSERVATION LAWS FOR TIME FRACTIONAL COUPLED WHITHAM-BROER-KAUP EQUATIONS

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In the current work, time fractional coupled Whitham-Broer-Kaup equations which describes the anomalous bidirectional propagation of long waves in shallow water is investigated. A Lie symmetry analysis is formulated and used to the governing model. The symmetry reductions of the model are constructed and the system of nonlinear time fractional partial differential equations is similarity reduced to a system of nonlinear frcational ordinary differential equations in Erdelyi-Kober derivative sense. Moreover, the resultant symmetry generators are used to calculate conserved vectors for the time fractional problem. Two different kinds of conservation laws of the problem have been constructed.

Keywords: Lie symmetry analysis; Time fractional coupled Whitham-Broer-Kaup equations; Erdelyi-Kober operators; Conservation laws

1. Introduction

In recent decades theory of fractional calculus [1, 2] has gained notable attention of many researchers in science and engineering due to its high ability for describing various complicated natural phenomena. Many of anomalous phenomena and complex processes in natural science have been successfully described by using the theory of fractional calculus and mathematically modeled as fractional differential or integral equations[3, 4, 5, 6]. The main advantage of fractional modeling is that they are excellent tools to appropriately characterize hereditary and inherent memory properties of the anomalous phenomena. In recent years several semi-analytical methods and numerical techniques have been developed and employed for solving and investigating various practical problems which have been mathematically modeled as fractional equations [7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

In this study the following important mathematical model, the time fractional Whitham-Broer-Kaup equations, which describes the anomalous bidirectional propagation of long waves in shallow water will be investigated[17].

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$$\begin{aligned}
&\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + \frac{\partial v}{\partial x} + u\frac{\partial u}{\partial x} + q\frac{\partial^2 u}{\partial x^2} = 0,\\
&\frac{\partial^{\alpha} v}{\partial t^{\alpha}} + \frac{\partial (uv)}{\partial x} + p\frac{\partial^3 u}{\partial x^3} - q\frac{\partial^2 v}{\partial x^2} = 0,
\end{aligned} \tag{1}$$

where $u(x,t)$ and $v(x,t)$ denote the fluid velocity along the horizontal direction and vertical displacement of the fluid from its equilibrium position, respectively. The constants p and q $((p,q)\neq(0,0))$ are diffusion coefficients and $\frac{\partial^{\alpha}}{\partial t^{\alpha}}(.)$ denotes the Riemann-Liouville partial derivative operator of order α $(0<\alpha\leq 1)$ with respect to time component, t, which is defined as follows :

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}(u(x,t)) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\dfrac{\partial^n}{\partial t^n}\displaystyle\int_0^t (t-\tau)^{m-\alpha-1}u(x,\tau)d\tau, & m-1<\alpha<m,\ m\in \mathrm{N},\\ \dfrac{\partial^n}{\partial t^n}(u(x,t)), & \alpha = n\in \mathrm{N}, \end{cases}$$

where $\Gamma(.)$ is the gamma function. For $\alpha=1$ the anomalous bidirectional propagation model (1) is reduced to the classical Whitham-Broer-Kaup problem. Several numerical and semi-analytical methods have been formulated and employed to investigate the classical Whitham-Broer-Kaup problem[18, 19, 20, 21, 22]. The classical Whitham-Broer-Kaup (WBK) equation has been widely used to study of solitary wave theory in shallow water. However, the classical Whitham-Broer-Kaup (WBK) equation is not adequate to describe complicated mechanism of propagation of shallow water waves in porous medium, such as tsunami wave propagation. In these situations the temporal derivative in the WBK equation and corresponds to the variation in the flux, $\frac{\partial}{\partial t}$, should be replaced with fractional time derivative, $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$, which properly describes variation in the flux through the fractal boundary, where α denotes the fractal dimensions of the porous medium [23]. Saha Ray [17] has discussed the time-fractional WBK equations (1) and approximated the traveling wave solutions of the model in their generalized Taylor expansion forms. In [24] a semi-analytical technique, called the residual power series method is formulated and used to approximate traveling solutions of the model (1). By choosing $p=1$ and $q=0$ in the (1) the so-called time fractional modified Boussinesq system is concluded. While if $p=0$ and

$q \neq 0$ another special case of the time fractional coupled Whitham-Broer-Kaup equations, called time fractional coupled approximate long wave equations is obtained. In [25] the time-space fractional Whitham-Broer-Kaup equations has been investigated. Recently, Amjad Ali et.al employed a numerical method based on the Adomian decomposition method coupled with the Laplace transforms to calculate the approximate traveling wave solutions of the time-space fractional coupled Whitham-Broer-Kaup equations[26].

In recent years, Lie symmetry analysis has been developed and widely used to deal with several types of complicated nonlinear differential problems[27, 28, 29, 30]. Buckwar and Luchko in [31] computed invariant solutions of fractional differential equations by employing the scaling transformations. Moreover, a symmetry group of scaling transformations for time-space fractional partial differential equation is extracted in [32]. Gazizov et al. introduced a prolongation formula for the Riemann-Liouville fractional derivative operator[33]. In [34] they utilized the proposed Lie point symmetry method for solving the nonlinear time fractional diffusion problem. Recently, Hashemi et al. formulated and used the Lie symmetry approach for investigating various types of the fractional differential equations[35,36,37,38,39,40,41]. Singla and Gupta extracted a Lie point symmetry analysis for systems of coupled time fractional partial differential equations [42,43]. Moreover, they developed the Lie symmetry approach for space-time fractional systems of partial differential equations [44, 45]. In this study, the expanded Lie symmetry approach proposed by Singla and Gupta is utilized for symmetry analysis and similarity reductions of the time fractional Whitham-Broer-Kaup equations (1).

The idea of conservation law plays an important role to analyze the fundamental properties of the physical models[46]. Relation between symmetries of differential equations and conservation laws is explained by the Noether's theorem [47]. The Noether's theorem is valid for differential equations having Lagrangians. The classical Noether's theorem have been generalized and employed to find conservation laws for several fractional differential equations having fractional Lagrangians [48,49,50,51]. Ibragimov in [52] extracted a new generalized conservation theorem based on the adjoint equations for the nonlinear differential equations not having Lagrangians. Lukashchuk has used the new conservation theorem to find conservation laws for time fractional subdiffusion and diffusion-wave equations[53]. Gazizov et al. found conservation laws for the time-fractional Kompaneets equations based on the generalization of fractional Noether's operator [54]. Very recently, Singla and Gupta in [55,56] have extracted the fractional Noether's operators to calculate conserved vectors of the time and space-time fractional nonlinear systems of partial differential equations, respectively. Majlesi et.al. performed a Lie symmetry analysis on a coupled system of time fractional Jaulent-Miodek equations and constructed the

conservation laws of the problem [57]. In the current work the generalization of fractional Noether's operator proposed by Singla and Gupta [55] is employed for calculating conserved vectors for the governing system of nonlinear time fractional partial differential equation (1).

2. Lie symmetry analysis and similarity reductions of the time fractional Whitham-Broer-Kaup system

In this section an invariant analysis for the time fractional Whitham-Broer-Kaup system (1) would be presented. Moreover, the symmetry reductions of the governing time fractional nonlinear system based on the symmetry groups are investigated. For this purpose, firstly the main details of Lie symmetry analysis for systems of time fractional PDEs are briefly described.

2.1 Description of the Lie symmetry analysis for systems of time fractional PDEs

Here we consider a coupled system of two time fractional nonlinear PDEs as follows[42]:

$$\begin{aligned} \Delta_1 &\equiv \frac{\partial^\alpha u}{\partial t^\alpha} - F(x,t,u,v,u_x,v_x,) = 0, \\ \Delta_2 &\equiv \frac{\partial^\alpha v}{\partial t^\alpha} - G(x,t,u,v,u_x,v_x,) = 0, \end{aligned} \tag{2}$$

where $\{u(x,t),v(x,t)\}$ and $\{x,t\}$ are dependent and independent variables respectively. $\alpha > 0$ is a real number and $\frac{\partial^\alpha}{\partial t^\alpha}(.)$ denotes the Riemann-Liouville partial derivative operator and subscripts denote integer partial derivatives. According to the Lie symmetry analysis, assume the time fractional system (2) is invariant under the following one parameter Lie group of transformations :

$$\begin{aligned} \bar{x} &= x + \varepsilon\xi(x,t,u,v) + O(\varepsilon^2), \\ \bar{t} &= t + \varepsilon\tau(x,t,u,v) + O(\varepsilon^2), \\ \bar{u} &= u + \varepsilon\eta(x,t,u,v) + O(\varepsilon^2), \\ \bar{v} &= v + \varepsilon\phi(x,t,u,v) + O(\varepsilon^2), \\ \frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon\eta^{\alpha,t} + O(\varepsilon^2), \\ \frac{\partial^\alpha \bar{v}}{\partial \bar{t}^\alpha} &= \frac{\partial^\alpha v}{\partial t^\alpha} + \varepsilon\phi^{\alpha,t} + O(\varepsilon^2), \end{aligned} \tag{3}$$

$$\frac{\partial^j \bar{u}}{\partial \bar{x}^j} = \frac{\partial^j u}{\partial x^j} + \varepsilon \eta^{j,x} + O(\varepsilon^2), \quad j = 1,2,3,$$
$$\frac{\partial^j \bar{v}}{\partial \bar{x}^j} = \frac{\partial^j v}{\partial x^j} + \varepsilon \phi^{j,x} + O(\varepsilon^2), \quad j = 1,2,3,,$$

where ε is a parameter, (ξ, τ, η, ϕ) is the set of infinitesimals and $\eta^{\alpha,t}, \phi^{\alpha,t}$ and $\eta^{j,x}, \phi^{j,x}, (j = 1,2,)$ are extended infinitesimals of order α and j respectively. The infinitesimal generator of the time fractional system (2) is a vector field as follows:

$$V = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \eta(x,t,u,v)\frac{\partial}{\partial u} + \phi(x,t,u,v)\frac{\partial}{\partial v}. \quad (4)$$

The vector field (4) generates a symmetry of (2) provided that it admits the following infinitesimal invariance criteria:

$$Pr^{\alpha} V(\Delta_1)|_{\Delta_1=0,\Delta_2=0} = 0,$$
$$Pr^{\alpha} V(\Delta_2)|_{\Delta_1=0,\Delta_2=0} = 0, \quad (5)$$

where $Pr^{\alpha} V$ is the generalized fractional prolongation operator[42,43] :

$$Pr^{\alpha,m,n} V = V + \eta^{\alpha,t}\frac{\partial}{\partial u_t^{(\alpha)}} + \eta^{1,x}\frac{\partial}{\partial u_x} + + \eta^{m,x}\frac{\partial}{\partial u_{mx}} + \phi^{\alpha,t}\frac{\partial}{\partial v_t^{(\alpha)}} + \phi^{1,x}\frac{\partial}{\partial v_x} + + \phi^{n,x}\frac{\partial}{\partial v_{nx}}, \quad (6)$$

where $m, n \in \mathrm{N}$ denote orders of partial differential equations (2). In the above relation the integer order extended infinitesimals $\eta^{j,x}$ and $\phi^{j,x}$ are defined as follows:

$$\eta^{j,x} = D_x \eta^{j-1,x} - (D_x \xi)\frac{\partial^j u}{\partial x^j} - (D_x \tau)\frac{\partial}{\partial t}\left(\frac{\partial^{j-1} u}{\partial x^{j-1}}\right),$$
$$\phi^{j,x} = D_x \phi^{j-1,x} - (D_x \xi)\frac{\partial^j v}{\partial x^j} - (D_x \tau)\frac{\partial}{\partial t}\left(\frac{\partial^{j-1} v}{\partial x^{j-1}}\right), \quad j = 1,2,,$$

where D_x denotes the total derivative operator with respect to x

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + + v_x\frac{\partial}{\partial v} + v_{xx}\frac{\partial}{\partial v_x} + \cdots,$$

Moreover the α-order extended infinitesimal $\eta^{\alpha,t}$ is defined as follows:

$$\eta^{\alpha,t} = D_t^{\alpha}\eta + \xi D_t^{\alpha}(u_x) - D_t^{\alpha}(\xi u_x) + D_t^{\alpha}(D_t(\tau)u) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u),$$

where D_t^{α} is the total fractional derivative with respect to t. Using the generalized Leibniz rule [2] and generalized chain rule [5] and after some simplifications the α-order extended infinitesimal $\eta^{\alpha,t}$ is given in the following form (see for more details [42, 43]) :

$$\eta^{\alpha,t} = \frac{\partial^{\alpha}\eta}{\partial t^{\alpha}} + (\eta_u - \alpha D_t(\tau))\frac{\partial^{\alpha}u}{\partial t^{\alpha}} - u\frac{\partial^{\alpha}\eta_u}{\partial t^{\alpha}} + (\eta_v \frac{\partial^{\alpha}v}{\partial t^{\alpha}} - v\frac{\partial^{\alpha}\eta_v}{\partial t^{\alpha}})$$
$$+\sum_{i=1}^{\infty}[\binom{\alpha}{i}\frac{\partial^i\eta_u}{\partial t^i} - \binom{\alpha}{i+1}D_t^{i+1}(\tau)]D_t^{\alpha-i}(u) + \sum_{i=1}^{\infty}\binom{\alpha}{i}\frac{\partial^i\eta_v}{\partial t^i}D_t^{\alpha-i}(v)$$
$$-\sum_{i=1}^{\infty}\binom{\alpha}{i}D_t^i(\xi)D_t^{\alpha-i}(u_x) + \varsigma_1 + \varsigma_2, \qquad (7)$$

where D_t denotes the total derivative operator with respect to t and

$$\varsigma_1 = \sum_{i=2}^{\infty}\sum_{j=2}^{i}\sum_{r=2}^{j}\sum_{s=0}^{r-1}\binom{\alpha}{i}\binom{i}{j}\binom{r}{s}\frac{t^{i-\alpha}}{r!\Gamma(i-\alpha+1)}(-1)^s u^s \frac{\partial^j(u^{r-s})}{\partial t^j}\frac{\partial^{i-j+r}\eta}{\partial t^{i-j}\partial u^r},$$

$$\varsigma_2 = \sum_{i=2}^{\infty}\sum_{j=2}^{i}\sum_{r=2}^{j}\sum_{s=0}^{r-1}\binom{\alpha}{i}\binom{i}{j}\binom{r}{s}\frac{t^{i-\alpha}}{r!\Gamma(i-\alpha+1)}(-1)^s v^s \frac{\partial^j(v^{r-s})}{\partial t^j}\frac{\partial^{i-j+r}\eta}{\partial t^{i-j}\partial v^r}.$$

Similarity, the α-order extended infinitesimal $\phi^{\alpha,t}$ can be simplified as follows :

$$\phi^{\alpha,t} = \frac{\partial^{\alpha}\phi}{\partial t^{\alpha}} + (\phi_v - \alpha D_t(\tau))\frac{\partial^{\alpha}v}{\partial t^{\alpha}} - v\frac{\partial^{\alpha}\phi_v}{\partial t^{\alpha}} + (\phi_u \frac{\partial^{\alpha}u}{\partial t^{\alpha}} - u\frac{\partial^{\alpha}\phi_u}{\partial t^{\alpha}})$$
$$+\sum_{i=1}^{\infty}[\binom{\alpha}{i}\frac{\partial^i\phi_v}{\partial t^i} - \binom{\alpha}{i+1}D_t^{i+1}(\tau)]D_t^{\alpha-i}(v) + \sum_{i=1}^{\infty}\binom{\alpha}{i}\frac{\partial^i\phi_u}{\partial t^i}D_t^{\alpha-i}(u)$$
$$-\sum_{i=1}^{\infty}\binom{\alpha}{i}D_t^i(\xi)D_t^{\alpha-i}(v_x) + \rho_1 + \rho_2, \qquad (8)$$

where

$$\rho_1 = \sum_{i=2}^{\infty}\sum_{j=2}^{i}\sum_{r=2}^{j}\sum_{s=0}^{r-1}\binom{\alpha}{i}\binom{i}{j}\binom{r}{s}\frac{t^{i-\alpha}}{r!\Gamma(i-\alpha+1)}(-1)^s u^s \frac{\partial^j(u^{r-s})}{\partial t^j}\frac{\partial^{i-j+r}\phi}{\partial t^{i-j}\partial u^r},$$

$$\rho_2 = \sum_{i=2}^{\infty}\sum_{j=2}^{i}\sum_{r=2}^{j}\sum_{s=0}^{r-1}\binom{\alpha}{i}\binom{i}{j}\binom{r}{s}\frac{t^{i-\alpha}}{r!\Gamma(i-\alpha+1)}(-1)^s v^s \frac{\partial^j(v^{r-s})}{\partial t^j}\frac{\partial^{i-j+r}\phi}{\partial t^{i-j}\partial v^r},$$

From the above relations it can be easily concluded that whenever infinitesimals ϕ and η are linear with respect to each dependent variables u and v, the expressions $\varsigma_1, \varsigma_2, \rho_1$ and ρ_2 vanish identically.

2.2 The symmetry reductions of the time fractional Whitham-Broer-Kaup system

Here, the proposed Lie symmetry analysis is formulated and employed to reduce the problem (1). For this purpose, by imposing the prolongations (6) $Pr^{\alpha,2,1}V\,(\Delta_1)$ and $Pr^{\alpha,3,2}V\,(\Delta_2)$ to problem (1) under group of transformations (3), the following invariance criterions are concluded:

$$[\eta^{\alpha,t}+\phi^{1,x}+\eta u_x+u\eta^{1,x}+q\eta^{2,x}]_{\Delta_1=0,\Delta_2=0}=0,$$
$$[\phi^{\alpha,t}+p\eta^{3,x}+u\phi^{1,x}+v\eta^{1,x}+\eta v_x+\phi u_x-q\phi^{2,x}]_{\Delta_1=0,\Delta_2=0}=0. \qquad (9)$$

By substituting the integer and α order extended infinitesimals into (9) and equating the coefficients of various powers and partial derivatives of independent variables u and v to zero, a set of fractional and classical partial differential equations with respect to the variables ξ,τ,η and ϕ is obtained. By solving the resultant system of differential equations symbolically the following values for infinitesimals functions are computed:

$$\xi=c_1+c_2\alpha x,\quad \tau=c_3+2tc_2,\quad \eta=-\alpha c_2 u,\quad \phi=-2\alpha c_2 v, \qquad (10)$$

where c_1,c_2 and c_3 are free constant parameters. Notice that the lower limit of the Riemann-Liouville partial derivative operator is fixed, so to ensure that it is invariant under group of transformations (3), the following initial condition should be satisfied.

$$\tau(x,t,u,v)|_{t=0}=0. \qquad (11)$$

So $c_3 = 0$ and the following sets of the infinitesimal generators are obtained:

$$V_1=\alpha x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t}-\alpha u\frac{\partial}{\partial u}-2\alpha v\frac{\partial}{\partial v},\qquad V_2=\frac{\partial}{\partial x}. \qquad (12)$$

Clearly, the above vector fields form a closed Lie algebra:

$$[V_1,V_1]=[V_2,V_2]=0,\qquad [V_1,V_2]=-\alpha V_2=-[V_2,V_1].$$

For the infinitesimal generator V_1, the following characteristic equations is concluded :

$$\frac{dx}{\alpha x}=\frac{dt}{2t}=\frac{du}{-\alpha u}=\frac{dv}{-2\alpha v}.$$

By solving the above characteristic equations the following similarity variable and invariant solutions are obtained.

$$z=xt^{-\frac{\alpha}{2}},\quad u(x,t)=t^{-\frac{\alpha}{2}}F(z),\quad v(x,t)=t^{-\alpha}G(z), \qquad (13)$$

where z and $F(z),G(z)$ are new independent variable and dependent variables respectively.

Now, the reduction form of the time fractional system (1) with respect to the above presented symmetry generators is given by the following theorem.

Theorem 1.Corresponding to the infinitesimal generator V_1, the similarity variable $z = xt^{-\frac{\alpha}{2}}$ with the similarity transformations $u(x,t) = t^{-\frac{\alpha}{2}}F(z)$ and $v(x,t) = t^{-\alpha}G(z)$, reduced the time fractional system of partial differential equations (1) to the following system of fractional ordinary differential of equations:

$$(P_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha}F)(z) = -G'(z) + F(z)F'(z) + qF''(z),$$
$$(P_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha}G)(z) = -pF'''(z) + (F(z)G(z))' - qG''(z), \qquad (14)$$

where the left-hand side Erdelyi-Kober fractional differential operator, $(P_{\varsigma}^{\rho,\alpha})$, is defined as[58] :

$$(P_{\varsigma}^{\rho,\alpha}H)(z) = \prod_{j=0}^{m-1}(\rho + j - \frac{1}{\varsigma}z\frac{d}{dz})(K_{\varsigma}^{\rho+\alpha,m-\alpha}H)(z), \quad z > 0,\ \alpha > 0, \qquad (15)$$

where

$$m = \begin{cases} [\alpha]+1 & \text{if } \alpha \notin \mathrm{N}, \\ \alpha & \text{if } \alpha \in \mathrm{N}, \end{cases}$$

and

$$(K_{\varsigma}^{\rho,\alpha}H)(z) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^{\infty}(\iota-1)^{\alpha-1}\iota^{-(\rho+\alpha)}H(z\iota^{\frac{1}{\varsigma}})d\iota & \text{if } \alpha > 0 \\ H(z) & \text{if } \alpha = 0 \end{cases}$$

***Proof*:** According to the similarity solutions (13), for $n-1 < \alpha < n$ $(n \in \mathrm{N})$ the α-order Riemann-Liouville time fractional derivative for $u(x,t)$ is given as follows:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_0^t (t-s)^{n-\alpha-1}s^{-\frac{\alpha}{2}}F(xs^{-\frac{\alpha}{2}})ds, \qquad (16)$$

Now by using the change of variable, $\iota = \frac{t}{s}$, the relation (16) is transformed to following form :

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\partial^n}{\partial t^n}[\frac{t^{n-\frac{3}{2}\alpha}}{\Gamma(n-\alpha)}\int_1^{\infty}(\iota-1)^{n-\alpha-1}\iota^{-(n-\frac{3\alpha}{2}+1)}F(z\iota^{\frac{\alpha}{2}})d\iota], \qquad (17)$$

by comparing with the Erdelyi-Kober fractional differential operator, the above relation can be presented as follows:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{n}}{\partial t^{n}}[t^{n-\frac{3}{2}\alpha}(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}F)(z)]. \tag{18}$$

So we have,

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{n-1}}{\partial t^{n-1}}(\frac{\partial}{\partial t}[t^{n-\frac{3}{2}\alpha}(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}F)(z)])$$

$$=\frac{\partial^{n-1}}{\partial t^{n-1}}[(n-\frac{3}{2}\alpha)t^{n-\frac{3}{2}\alpha-1}(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}F)(z)+t^{n-\frac{3}{2}\alpha}\frac{d}{dz}(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}F)(z)\frac{\partial z}{\partial t}]$$

$$=\frac{\partial^{n-1}}{\partial t^{n-1}}[t^{n-\frac{3}{2}\alpha-1}(n-\frac{3}{2}\alpha+t\frac{\partial z}{\partial t}\frac{d}{dz})(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}F)(z)]$$

$$=\frac{\partial^{n-1}}{\partial t^{n-1}}[t^{n-\frac{3}{2}\alpha-1}(n-\frac{3}{2}\alpha-\frac{\alpha}{2}z\frac{d}{dz})(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}F)(z)]. \tag{19}$$

Following the above process for $(n-1)$ times, we easily obtain:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{n}}{\partial t^{n}}[t^{n-\frac{3}{2}\alpha}(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}F)(z)]$$

$$=t^{-\frac{3}{2}\alpha}\prod_{j=0}^{n-1}(1-\frac{3}{2}\alpha+j-\frac{\alpha}{2}z\frac{d}{dz})(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}F)(z)$$

$$=t^{-\frac{3}{2}\alpha}(P_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha}F)(z). \tag{20}$$

In similar manner, the α-order Riemann-Liouville time fractional derivative for $v(x,t)$ with respect to the similarity transforms (13) can be expressed as follows:

$$\frac{\partial^{\alpha}v}{\partial t^{\alpha}}=t^{-2\alpha}(P_{\frac{2}{\alpha}}^{1-2\alpha,\alpha}G)(z). \tag{21}$$

Moreover for $\alpha=n\in\mathrm{N}$ we have:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{n}u}{\partial t^{n}}=\frac{\partial^{n}}{\partial t^{n}}(t^{-\frac{n}{2}}F(z))$$

$$=\frac{\partial^{n-1}}{\partial t^{n-1}}[\frac{\partial}{\partial t}(t^{-\frac{n}{2}}F(z))]$$

$$=\frac{\partial^{n-1}}{\partial t^{n-1}}[t^{-\frac{n}{2}-1}(-\frac{n}{2}-\frac{n}{2}z\frac{d}{dz})F(z)]$$

$$\vdots$$

$$=t^{-\frac{3}{2}n}\prod_{j=0}^{n-1}(1-\frac{3}{2}n+j-\frac{n}{2}z\frac{d}{dz})F(z)$$

$$= t^{-\frac{3}{2}n}(P_{\frac{2}{n}}^{1-\frac{3n}{2},n}F)(z),$$

and similarly, for $\alpha = n \in \mathrm{N}$ we have:

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial^n v}{\partial t^n} = t^{-2n}(P_{\frac{2}{n}}^{1-2n,n}G)(z).$$

Now by substituting the proposed similarity transformations (13) in the main problem (1) and from relations (20) and (21) the reduced system of fractional ordinary equations (14) is concluded.

Moreover, for the infinitesimal generator V_2, by solving the associated characteristic equations the invariant solutions of the problem (1) are obtained as $u(x,t) = f(t)$ and $v(x,t) = g(t)$. Substituting the above obtained group of invariant solutions in the governing problem (1), the following reduced system is concluded:

$$\begin{cases} \dfrac{\partial^\alpha f(t)}{\partial t^\alpha} = 0 \\ \dfrac{\partial^\alpha g(t)}{\partial t^\alpha} = 0 \end{cases}$$

Integrating both sides of the above system, the following invariant solutions are arrived:

$$f(t) = \frac{\kappa_1}{\Gamma(\alpha)}t^{\alpha-1}, \qquad g(t) = \frac{\kappa_2}{\Gamma(\alpha)}t^{\alpha-1}, \tag{22}$$

where κ_1 and κ_2 are arbitrary constants.

3. Conservation laws of the time fractional Whitham-Broer-Kaup system

In this section a new approach which has been developed and implemented to the time-fractional system of PDEs [55] is employed to construct conservation laws for the time fractional Whitham-Broer-Kaup system (1). According to the approach proposed by Singla and Gupta[55], the formal Lagrangian for the time fractional PDEs system (1) is considered as follows:

$$\begin{aligned} \mathrm{L} = {} & f(x,t)\left[\frac{\partial^\alpha u}{\partial t^\alpha} + v_x(x,t) + u(x,t)u_x(x,t) + qu_{xx}(x,t)\right] \\ & + g(x,t)\left[\frac{\partial^\alpha v}{\partial t^\alpha} + pu_{xxx}(x,t) + u(x,t)v_x(x,t) + u_x(x,t)v(x,t) - qv_{xx}(x,t)\right], \end{aligned} \tag{23}$$

where $f(x,t)$ and $g(x,t)$ are newly introduced dependent variables. The adjoint equations for the formal Lagrangian operator (23) are calculated as follows:

$$\frac{\delta \mathrm{L}}{\delta u} = (D_t^{\alpha})^* f - u(x,t) f_x(x,t) - v(x,t) g_x(x,t) + q f_{xx}(x,t) - p g_{xxx}(x,t) = 0, \quad (24)$$

and

$$\frac{\delta \mathrm{L}}{\delta v} = (D_t^{\alpha})^* g - f_x(x,t) - g_x(x,t) u(x,t) - q g_{xx}(x,t) = 0, \quad (25)$$

where the Euler-Lagrange operator, $\frac{\delta}{\delta \omega}$, is defined as :

$$\frac{\delta}{\delta \omega} = \frac{\partial}{\partial \omega} + (D_t^{\alpha})^* \frac{\partial}{\partial (D_t^{\alpha} \omega)} + \sum_{k=1}^{\infty} (-1)^k D_{d_1} D_{d_2} D_{d_k} \frac{\partial}{\partial (\omega)_{d_1, d_2, , d_k}},$$

where $(D_t^{\alpha})^*$ denotes the adjoint operator for the Riemann-Liouville fractional operator (D_t^{α}) and equals to the standard right-hand sided Caputo fractional differential operator [55]. According to [55], the time fractional system of equations(2.5) is said to be nonlinearly self-adjoint whenever by introducing new variables $f = \psi(x,t,u,v)$ and $g = \varphi(x,t,u,v)$, where at least one of them is non-zero and substituting them and their related partial derivatives with respect to x into adjoint equations (24) and (25), the resultant equations are satisfied for all solutions of the governing problem (1). It means that for nonlinear self-adjointness of (1) the following conditions should be held :

$$\frac{\delta \mathrm{L}}{\delta u} = \mu_1 \left(\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + v_x + u u_x + q u_{xx}\right) + \mu_2 \left(\frac{\partial^{\alpha} v}{\partial t^{\alpha}} + p u_{xxx} + u v_x + v u_x - q v_{xx}\right),$$

$$\frac{\delta \mathrm{L}}{\delta v} = \mu_3 \left(\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + v_x + u u_x + q u_{xx}\right) + \mu_4 \left(\frac{\partial^{\alpha} v}{\partial t^{\alpha}} + p u_{xxx} + u v_x + v u_x - q v_{xx}\right),$$

where $\mu_i, (i = 1,2,3,4)$ are unknown coefficients. The above relations are given as follows:

$$(D_t^{\alpha})^* \psi - u(x,t)(\psi_x + \psi_u u_x + \psi_v v_x) - v(x,t)(\varphi_x + \varphi_u u_x + \varphi_v v_x)$$

$$+ q(\psi_{xx} + 2\psi_{x,u} u_x + 2\psi_{x,v} v_x + \psi_{u,u} u_x^2 + 2\psi_{u,v} u_x v_x + \psi_u u_{xx} + \psi_{v,v} v_x^2 + \psi_v v_{x,x})$$

$$- p[\varphi_{x,x,x} + 3\varphi_{u,u,v} u_x^2 v_x + 3\varphi_{v,v} v_{xx} v_x + 3\varphi_{u,v,v} u_x v_x^2 + 3\varphi_{u,u} u_x u_{x,x} + 3\varphi_{u,v} u_{x,x} v_x$$

$$+ 6\varphi_{x,u,v} u_x v_x + 3\varphi_{u,v} u_x v_{x,x} + 3\varphi_{x,x,u} u_x + 3\varphi_{x,x,v} v_x + 3\varphi_{x,u} u_{x,x} + 3\varphi_{x,v} v_{x,x}$$
$$+ \varphi_u u_{xxx} + \varphi_v v_{xxx} + 3\varphi_{x,u,u} u_x^2 + 3\varphi_{x,v,v} v_x^2 + \varphi_{u,u,u} u_x^3 + \varphi_{v,v,v} v_x^3]$$
$$= \mu_1 \left(\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + v_x + u u_x + q u_{xx}\right) + \mu_2 \left(\frac{\partial^{\alpha} v}{\partial t^{\alpha}} + p u_{xxx} + u v_x + v u_x - q v_{xx}\right), \quad (26)$$

and $(D_t^{\alpha})^* \varphi - (\psi_x + \psi_u u_x + \psi_v v_x) - (\varphi_x + \varphi_u u_x + \varphi_v v_x) u - q(\varphi_{xx} + 2\varphi_{x,u} u_x$

$$+2\varphi_{x,v}v_x+\varphi_{u,u}u_x^2+2\varphi_{u,v}u_xv_x+\varphi_u u_{xx}+\varphi_{v,v}v_x^2+\varphi_v v_{x,x})$$

$$=\mu_3(\frac{\partial^\alpha u}{\partial t^\alpha}+v_x+uu_x+qu_{xx})+\mu_4(\frac{\partial^\alpha v}{\partial t^\alpha}+pu_{xxx}+uv_x+vu_x-qv_{xx}). \quad (27)$$

Now by balancing the coefficients of the different powers of dependent variables u and v and their related partial derivatives in both sides of (26) and (27) a systems of partial differential equations should be concluded. Solving the resultant system of differential equations analytically, the following results is obtained:

$$\mu_i=0,\quad i=1,2,3,4,$$
$$\psi(x,t,u,v)=f(x,t)=A,\qquad \varphi(x,t,u,v)=g(x,t)=B, \quad (28)$$

where A and B are free constants. This confirms the nonlinear self-adjointness of the problem (1). Corresponding to each vector field $V_i=\xi_i\partial_x+\tau_i\partial_t+\eta_i\partial_u+\phi_i\partial_v$ there exist Lie characteristic functions which are defined as follows:

$$W_i^u=\eta_i-\xi_i u_x-\tau_i u_t,\qquad W_i^v=\phi_i-\xi_i v_x-\tau_i v_t.$$

So Corresponding to the vector fields (12) for the time fractional Whitham-Broer-Kaup problem the following Lie characteristic functions are concluded, respectively:

$$W_1^u=-\alpha u-\alpha x u_x-2tu_t,\qquad W_1^v=-2\alpha v-\alpha x v_x-2tv_t, \quad (29)$$

and

$$W_2^u=-u_x,\qquad W_2^v=-v_x. \quad (30)$$

Now by using the above constructed Lie characteristic functions, conserved vectors corresponding to the governing problem should be calculated. A vector $C=(C^x,C^t)$ is said to be a conserved vector for the problem (1) if the following conservation equation is held for it,

$$D_t(C^t)+D_x(C^x)_{(1)}=0. \quad (31)$$

Clearly in the governing system of fractional partial differential equations (1), all of the partial derivatives with respect to the independent variable x are of integer-orders, so the x components of the conserved vectors are obtained based on the following classical forms [52]:

$$C_i^x=(W_i^u\frac{\delta \mathrm{L}}{\delta u_x}+D_x(W_i^u)\frac{\delta \mathrm{L}}{\delta u_{xx}}+D_x^2(W_i^u)\frac{\delta \mathrm{L}}{\delta u_{xxx}})+(W_i^v\frac{\delta \mathrm{L}}{\delta v_x}+D_x(W_i^v)\frac{\delta \mathrm{L}}{\delta v_{xx}}). \quad (32)$$

Moreover, an extended formula to compute the t -component of conserved vector for the time fractional system of equations are introduced as follows [55] :

$$C_i^t=\sum_{k=0}^{n-1}(-1)^k[D_t^{\alpha-1-k}(W_i^u)D_t^k(\frac{\partial \mathrm{L}}{\partial(D_t^\alpha u)})+D_t^{\alpha-1-k}(W_i^v)D_t^k(\frac{\partial \mathrm{L}}{\partial(D_t^\alpha v)})]$$

$$-(-1)^n[\mathrm{J}\ (W_i^{\ u}, \frac{\partial \mathrm{L}}{\partial(D_t^{\alpha}u)}) + \mathrm{J}\ (W_i^{\ v}, \frac{\partial \mathrm{L}}{\partial(D_t^{\alpha}v)})], \tag{33}$$

where $n = [\alpha]+1$ and $\mathrm{J}\ (f, g)$ denotes an integral transform which defined as follows:

$$\mathrm{J}\ (f, g) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^p \frac{f(x,s)g(x,r)}{(r-s)^{\alpha+1-n}} drds.$$

Letting $A = 1$ and $B = 1$ and by substituting Lie characteristic functions (29) and (30) into relations (32) and (33), the following conserved vectors for the time fractional problem (1) is obtained.

Case1. $0 < \alpha < 1$, conserved vectors are given as follows:

$$\begin{aligned}
C_1^x &= -u(\alpha u + \alpha x u_x + 2tu_t + 2\alpha v + \alpha x v_x + 2tv_t) - v(\alpha u + \alpha x u_x + 2tu_t) \\
&\quad -q(2\alpha u_x + \alpha x u_{xx} + 2tu_{tx} - 3\alpha v_x - \alpha x v_{xx} - 2tv_{tx}) \\
&\quad -p(3\alpha u_{xx} + \alpha u_{xxx} + 2tu_{txx}) - (2\alpha v + \alpha x v_x + 2tv_t), \\
C_1^t &= \alpha(-I_t^{1-\alpha}(u) + 2I_t^{1-\alpha}(v)) - \alpha x(I_t^{1-\alpha}(u_x) + I_t^{1-\alpha}(v_x)) + 2I_t^{1-\alpha}(tv_t) - 2I_t^{1-\alpha}(tu_t),
\end{aligned} \tag{34}$$

$$\begin{aligned}
C_2^x &= u(-u_x - v_x) - vu_x - v_x - q(u_{xx} + v_{xx}) - pu_{xxx}, \\
C_2^t &= -I_t^{1-\alpha}(u_x) - I_t^{1-\alpha}(v_x).
\end{aligned}$$

Case2. $1 < \alpha < 2$, conserved vectors are given as follows:

$$\begin{aligned}
C_1^x &= -u(\alpha u + \alpha x u_x + 2tu_t + 2\alpha v + \alpha x v_x + 2tv_t) - v(\alpha u + \alpha x u_x + 2tu_t) \\
&\quad -q(2\alpha u_x + \alpha x u_{xx} + 2tu_{tx} - 3\alpha v_x - \alpha x v_{xx} - 2tv_{tx}) \\
&\quad -p(3\alpha u_{xx} + \alpha u_{xxx} + 2tu_{txx}) - (2\alpha v + \alpha x v_x + 2tv_t), \\
C_1^t &= -\alpha(-D_t^{\alpha-1}(u) + 2D_t^{\alpha-1}(v)) - \alpha x(D_t^{\alpha-1}(u_x) + D_t^{\alpha-1}(v_x)) + 2D_t^{\alpha-1}(tv_t) - 2D_t^{\alpha-1}(tu_t),
\end{aligned} \tag{35}$$

$$\begin{aligned}
C_2^x &= u(-u_x - v_x) - vu_x - v_x - q(u_{xx} + v_{xx}) - pu_{xxx}, \\
C_2^t &= -D_t^{\alpha-1}(u_x) - D_t^{\alpha-1}(v_x).
\end{aligned}$$

4. Conclusions

In this paper, the Lie group analysis approach is used to study time fractional coupled Whitham-Broer-Kaup equations with Riemann-Liouville derivative operators. Based on the Lie symmetries analysis, the governing system of time fractional partial differential equations is similarity reduced to a system of nonlinear frcational ordinary differential equations in Erdelyi-Kober derivative

sense. The new conservation theorem based on the generalization of fractional Noether operators is used to calculate the conserved vectors and conservation laws of the model successfully.

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