

## DUALITY OF OPERATOR FRAMES IN BANACH SPACES

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In this paper, approximate dual operator frames in separable Banach spaces are introduced. The relationship between the duality of  $p$ -frames and the duality of  $(p, Y)$ -operator frames is characterized. The perturbations of dual operator frames and approximate dual operator frames are discussed. With approximate dual operator frames, we can approximatively reconstruct, or almost reconstruct the elements of Banach spaces. For a  $(p, Y)$ -operator frame, a sequence of approximate dual operator frames that converges to the dual operator frame always exists. We show that the direct sum of  $(p, Y)$ -operator frames can preserve the actual and approximate duality of operator frames.

**Keywords:** dual operator frames, approximate dual operator frames, direct sum.

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### 1. Introduction

Frames were first introduced by Duffin and Schaeffer [13] in nonharmonic Fourier analysis. However, research on frames was discontinued until 1986 when frames were applied to wavelet and Gabor transform [12]. Since then, frame theory has been extensively investigated [2, 3, 4, 5, 7, 11, 15, 16, 18]. A frame is a sequence  $\{f_i\}$  in a Hilbert space  $\mathcal{H}$  with existing positive constants, namely,  $A, B$  such that the following condition can be considered:

$$A\|f\| \leq \left( \sum_i |\langle f, f_i \rangle|^2 \right)^{\frac{1}{2}} \leq B\|f\|, \quad \forall f \in \mathcal{H}.$$

Given a frame  $\{f_i\}$  for  $\mathcal{H}$ , every element  $f$  in a Hilbert space  $\mathcal{H}$  can be represented as a linear combination

$$f = \sum_i c_i f_i.$$

Thus, frames can be viewed as the generalization of bases in Hilbert spaces. In terms of the redundancy of frames, coefficients, which are the essential distinction between frames and bases, can no longer be determined uniquely. To determine the coefficients  $\{c_i\}$ , we should obtain a dual frame  $\{g_i\}$  of  $\{f_i\}$  such that  $c_i = \langle f, g_i \rangle (\forall i)$ . Duality always exists for a frame  $\{f_i\}$  in Hilbert spaces, such as the canonical dual frame  $\{S^{-1}f_i\}$ , which is given by the frame operator  $S$ . Although duality always exists, dual frame calculation is complicated. Christensen and Laugesen [8] proposed methods to construct an approximate dual frame and thus solve such problem. An approximate dual frame associated with  $\{f_i\}$  is a sequence  $\{g_i\}$ , which satisfies the following condition:

$$\|f - \sum_i \langle f, g_i \rangle f_i\| \leq \varepsilon \|f\|, \quad \forall f \in \mathcal{H}$$

where  $0 < \varepsilon < 1$ . Thus,  $\sum_i \langle f, g_i \rangle f_i$  is closer to  $f$ , when  $\varepsilon$  is smaller, and this finding indicates that  $\{g_i\}$  is near the “true dual frame”.

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In 1991, Grochenig [14] considered atomic decompositions versus frames and introduced the concept of Banach frames. After 10 years, Aldroubi, Sung and Tang [1] introduced  $p$ -frames in Banach spaces. Christensen and Stoeva [9] also investigated the properties of  $p$ -frames in separable Banach spaces. A family  $\{x_i^*\} \subset X^*$  is called a  $p$ -frame for a Banach space  $X$  if constants  $A, B > 0$  exist, such that the following condition is satisfied:

$$A\|x\| \leq \left( \sum_i |\langle x_i^*, x \rangle|^p \right)^{\frac{1}{p}} \leq B\|x\|, \quad \forall x \in X. \quad (1)$$

Casazza et al. [6] generalized the concept of  $p$ -frames and introduced  $X_d$ -frames, where  $X_d$  is a BK-space. A family  $\{x_i^*\} \subset X^*$  is called an  $X_d$ -frame for the Banach space  $X$  if constants  $A, B > 0$  exist, such that the following is observed:

$$A\|x\| \leq \|\{\langle x_i^*, x \rangle\}\|_{X_d} \leq B\|x\|, \quad \forall x \in X.$$

Cao et al. [10] also introduced the concept of  $(p, Y)$ -operator frames, where  $Y$  is a Banach space and  $1 < p < \infty$ .  $(p, Y)$ -operator frames cover  $p$ -frames and  $X_d$ -frames. Let  $X, Y$  be Banach spaces. A family  $\{\Gamma_i\}$  of linear bounded operators from  $X$  to  $Y$  is a  $(p, Y)$ -operator frame if positive constants  $A, B$  exist, such that the following condition is satisfied:

$$A\|x\| \leq \left( \sum_i \|\Gamma_i x\|^p \right)^{\frac{1}{p}} \leq B\|x\|, \quad \forall x \in X. \quad (2)$$

In Banach space Setting, the duality of  $p$ -frame,  $X_d$ -frame or  $(p, Y)$ -operator frame does not necessarily exist, and this condition is an intractable issue for the series expansion of elements in Banach spaces. As such, we reduce our standards and determine the "approximate duality" to achieve the series expansion of elements in Banach spaces. For  $X_d$ -frames, two concepts of generalized dual Banach frames are given [17]: pseudo-dual Banach frames and approximate dual Banach frames. For a  $(p, Y)$ -operator frame  $\{\Gamma_i\}$  of a Banach space  $X$ , we aim to identify a sequence of bounded operators  $\{\Lambda_i\}$  from  $X^*$  to  $Y^*$ , following the given condition:

$$\|x - \sum_i \Lambda_i^* \Gamma_i x\| \leq \varepsilon \|x\|, \quad \forall x \in X,$$

where  $\varepsilon$  is a small positive number. In such a case, the following condition is obtained:

$$x \approx \sum_i \Lambda_i^* \Gamma_i x, \quad \forall x \in X.$$

Considering this condition and a previous work [17], we introduce the concept of approximate dual operator frames in our study. Our work demonstrates that a sequence of approximate dual operator frames converges to an actual dual operator frame. Even in Hilbert space setting, the sum or direct sum of two frames is no longer a frame. However, we discover that the direct sum of two operator frames is still an operator frame in their respective corresponding spaces. This paper also reveals that the direct sum can preserve the duality and approximate duality of operator frames in Banach space setting.

The paper is organized as follows. Section 2 provides some elementary definitions and results regarding  $p$ -frames and  $(p, Y)$ -operator frames. These definitions and results are essential for subsequent discussions. Section 3 discusses the approximate dual operator frames. The arguments mentioned in the preceding paragraphs are described on the basis of some important theorems. We then prove these theorems in detail.

## 2. Preliminary Definitions and Lemmas

Throughout this paper, let  $\mathbb{J}$  be the set of all natural numbers or a finite subset of all natural numbers,  $\mathbb{F}$  be the field  $\mathbb{C}$  of all complex numbers or the field  $\mathbb{R}$  of all real numbers and  $p, q$  be constants following the condition,  $1 < p, q < \infty$ . We use  $B(X, Y)$  to denote the

Banach space of all bounded linear operators from a Banach space  $X$  to another Banach space  $Y$ . If  $X = Y$ , then  $B(X) := B(X, X)$ . In this paper,  $I_X$  is always the identity operator on  $X$  (simply  $I$ ). Provided the existing canonical mappings  $J_X : X \rightarrow X^{**}$ ,  $J_Y : Y \rightarrow Y^{**}$ , we consider  $X, Y$  as a subspace of  $X^{**}, Y^{**}$  respectively. In this paper, all Banach spaces are separable.

If  $\{X_i\}_{i \in J}$  is a collection of Banach spaces, then we define the direct sum of Banach spaces as follows:

$$\oplus_p X_i = \{\{x_i\}_{i \in J} : x_i \in X_i (\forall i \in J), \|\{x_i\}_{i \in J}\|_p < \infty\},$$

where

$$\|\{x_i\}_{i \in J}\|_p = \left( \sum_{i \in J} \|x_i\|^p \right)^{\frac{1}{p}}.$$

Clearly,  $\oplus_p X_i$  is a Banach space and the dual space of  $\oplus_p X_i$  is  $\oplus_q X_i^*$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 < p, q < \infty$  and  $X_i^*$  is the dual space of  $X_i$  for every  $i \in J$ . Specifically, we denote  $\oplus_p X$  by  $\ell^p(X)$  and denote the dual space of  $\ell^p(X)$  by  $\ell^q(X^*)$ . Suppose that  $J = \{1, 2\}$ , then we have the following:

$$X_1 \oplus_p X_2 = \left\{ x_1 \oplus_p x_2 : x_1 \in X_1, x_2 \in X_2 \text{ with } \|x_1 \oplus_p x_2\| = (\|x_1\|^p + \|x_2\|^p)^{\frac{1}{p}} \right\}.$$

For any  $x_1^1 \oplus_p x_2^1, x_1^2 \oplus_p x_2^2 \in X_1 \oplus_p X_2, \lambda \in \mathbb{F}$ , we have

$$\begin{aligned} (x_1^1 \oplus_p x_2^1) + (x_1^2 \oplus_p x_2^2) &= (x_1^1 + x_1^2) \oplus_p (x_2^1 + x_2^2) \\ \lambda(x_1^1 \oplus_p x_2^1) &= \lambda x_1^1 \oplus_p \lambda x_2^1. \end{aligned}$$

In addition, the dual space of  $X_1 \oplus_p X_2$  is the space  $X_1^* \oplus_q X_2^*$  and  $\forall x_1 \oplus_p x_2 \in X_1 \oplus_p X_2, x_1^* \oplus_p x_2^* \in X_1^* \oplus_q X_2^*$ , we have the equation:

$$\langle x_1 \oplus_p x_2, x_1^* \oplus_q x_2^* \rangle = \langle x_1, x_1^* \rangle + \langle x_2, x_2^* \rangle.$$

Let  $T_1 \in B(X_1), T_2 \in B(X_2)$ , and we define a map  $T_1 \oplus_p T_2$  on  $X_1 \oplus_p X_2$  using the equation:

$$(T_1 \oplus_p T_2)(x_1 \oplus_p x_2) = T_1 x_1 \oplus_p T_2 x_2, \forall x_1 \oplus_p x_2 \in X_1 \oplus_p X_2.$$

We then prove that  $T_1 \oplus_p T_2 \in B(X_1 \oplus_p X_2)$  and  $T_1^* \oplus_q T_2^*$  is the adjoint of  $T_1 \oplus_p T_2$ .

A family  $\mathcal{G} = \{x_i^*\}_{i \in J} \in X^*$  is said to be a  $p$ -Bessel sequence for  $X$  if the upper frame condition in (1) holds. Equivalently, the following inequality holds:

$$\left\| \sum_{i \in J} c_i x_i^* \right\| \leq B \|\{c_i\}_{i \in J}\|, \quad \{c_i\}_{i \in J} \in \ell^q(J)$$

where  $B$  coincides with the constant in (1). The analysis operator and synthesis operator of a  $p$ -Bessel sequence  $\{x_i^*\}_{i \in J} \in X^*$  are defined as follows:

$$U_{\mathcal{G}} : X \longrightarrow \ell^p(J), \quad U_{\mathcal{G}} x = \{\langle x, x_i^* \rangle\}_{i \in J}, \quad \forall x \in X,$$

$$T_{\mathcal{G}} : \ell^q(J) \longrightarrow X^*, \quad T_{\mathcal{G}}(\{c_i\}_{i \in J}) = \sum_{i \in J} c_i x_i^*, \quad \forall \{c_i\}_{i \in J} \in \ell^q(J).$$

A family  $\Gamma = \{\Gamma_i\}_{i \in J}$  of operators in  $B(X, Y)$  is said to be a  $(p, Y)$ -operator Bessel sequence for  $X$  if the upper frame condition in (2) holds. Equivalently, the following inequality holds:

$$\left\| \sum_{i \in J} \Gamma_i^* y_i^* \right\| \leq B \|\{y_i^*\}_{i \in J}\|, \quad \forall \{y_i^*\}_{i \in J} \in \ell^q(Y^*)$$

where  $B$  coincides with the constant in (2). The analysis operator and synthesis operator are denoted by  $U_{\Gamma}, T_{\Gamma}$  respectively, where  $U_{\Gamma}$  and  $T_{\Gamma}$  are defined as follows:

$$U_{\Gamma} : X \longrightarrow \ell^p(Y), \quad U_{\Gamma}(f) = \{\Gamma_i f\}_{i \in J}, \quad \forall f \in X$$

$$T_\Gamma : \ell^q(Y^*) \longrightarrow X^*, \quad T_\Gamma(\{y_i^*\}_{i \in J}) = \sum_{i \in J} \Gamma_i^* y_i^*, \quad \forall \{y_i^*\}_{i \in J} \in \ell^q(Y^*).$$

Assuming that  $\{x_i^*\}_{i \in J} \subset X^*$  is a  $p$ -frame for  $X$ . If a  $q$ -Bessel sequence  $\{x_i\}_{i \in J} \in X$  exists, such that the following condition is satisfied:

$$x = \sum_{i \in J} \langle x, x_i^* \rangle x_i, \quad \forall x \in X; \quad x^* = \sum_{i \in J} \langle x^*, x_i \rangle x_i^*, \quad \forall x^* \in X^*, \quad (3)$$

then  $\{x_i\}_{i \in J}$  is called a dual frame of  $\{x_i^*\}_{i \in J}$  [9]. Upon checking, we verify that a dual frame of a  $p$ -frame for  $X$  is a  $q$ -frame for  $X^*$ . Therefore, if  $\mathcal{G} = \{x_i^*\}_{i \in J} \subset X^*$  is a  $p$ -Bessel sequence for  $X$  and  $\mathcal{F} = \{x_i\}_{i \in J} \subset X$  is a  $q$ -Bessel sequence for  $X^*$  satisfying (3), then  $\mathcal{G}$  and  $\mathcal{F}$  are a  $p$ -frame for  $X$  and a  $q$ -frame for  $X^*$ , respectively. In this case, we call  $(\mathcal{F}, \mathcal{G})$  a pair of dual frames. We also say that one of  $(\mathcal{F}, \mathcal{G})$  is a dual frame of the other.

Similarly, a  $(q, Y^*)$ -operator Bessel sequence  $\Lambda = \{\Lambda_i\}_{i \in J} \subset B(X^*, Y^*)$  is said to be a dual operator frame of a  $(p, Y)$ -operator Bessel sequence  $\Gamma = \{\Gamma_i\}_{i \in J} \subset B(X, Y)$  when it satisfies the following conditions:

$$x = \sum_{i \in J} \Lambda_i^* \Gamma_i x, \quad \forall x \in X, \quad x^* = \sum_{i \in J} \Gamma_i^* \Lambda_i x^*, \quad \forall x^* \in X^*. \quad (4)$$

In this case,  $\Gamma, \Lambda$  are automatically a  $(p, Y)$ -operator frame and a  $(q, Y^*)$ -operator frame, respectively.

When conditions (3) and (4) are relaxed, the concept of approximate dual frames is obtained.

**Definition 2.1.** Assume that  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\mathcal{G} = \{x_i^*\}_{i \in J} \subset X^*$  is a  $p$ -Bessel sequence for  $X$  and  $\mathcal{F} = \{x_i\}_{i \in J} \subset X$  is a  $q$ -Bessel sequence for  $X^*$ .  $(\mathcal{G}, \mathcal{F})$  is said to be a pair of approximate dual frames if  $\|I_X - T_{\mathcal{F}} U_{\mathcal{G}}\| < 1$  or  $\|I_{X^*} - T_{\mathcal{G}} U_{\mathcal{F}}\| < 1$ .

This definition is a special case of approximate dual Banach frames [17] when  $X_d = \ell^p(J)$ .

**Definition 2.2.** Assume that  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\Gamma = \{\Gamma_i\}_{i \in J} \subset B(X, Y)$  is a  $(p, Y)$ -operator Bessel sequence and  $\Lambda = \{\Lambda_i\}_{i \in J} \subset B(X^*, Y^*)$  is a  $(q, Y^*)$ -operator Bessel sequence.  $(\Gamma, \Lambda)$  is called a pair of approximate dual operator frames if  $\|I_X - T_{\Lambda} U_{\Gamma}\| < 1$  and  $\|I_{X^*} - T_{\Gamma} U_{\Lambda}\| < 1$ .

Clearly, a pair of dual frames is a pair of approximate dual frames and a pair of dual operator frames is a pair of approximate dual operator frames.

Some basic results on frames and operator frames in Banach spaces are listed in the form of lemmas, which will be used in the proofs of the main theorems in this paper.

**Lemma 2.1.** [9] If  $\mathcal{G} = \{x_i^*\}_{i \in J} \subset X^*$  is a  $p$ -frame for  $X$ , then  $X$  is reflexive.

*Proof.* By the definition of  $p$ -frames, the analysis operator  $U_{\mathcal{G}}$  of  $\mathcal{G}$  has a close range  $R(U_{\mathcal{G}})$  and is injective. This condition shows that  $X$  is isomorphic to  $R(U_{\mathcal{G}})$ . Provided that  $R(U_{\mathcal{G}})$  is a close subspace of  $\ell^p(J)$ , then  $R(U_{\mathcal{G}})$  is reflexive. Therefore,  $X$  is reflexive.  $\square$

**Lemma 2.2.** [10] Let  $\Gamma = \{\Gamma_i\}_{i \in J} \subset B(X, Y)$  be a  $(p, Y)$ -operator Bessel sequence with bound  $B_{\Gamma}$ . Then for every  $y^* \in Y^*$ , the family  $\{\Gamma_i^* y^*\}_{i \in J}$  is a  $p$ -Bessel sequence for  $X$  with bound  $B_{\Gamma} \|y^*\|$ .

The next lemma is a common result, we omit its proof.

**Lemma 2.3.** Let  $X$  be a Banach space,  $T \in B(X)$  and  $I$  be the identity operator on  $X$ . If  $\|I - T\| < 1$ , then  $T$  is invertible and  $T^{-1} = I + \sum_{n=1}^{\infty} (I - T)^n$ .

**Proposition 2.1.** *Assume that  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\Gamma = \{\Gamma_i\}_{i \in J} \subset B(X, Y)$  is a  $(p, Y)$ -operator Bessel sequence and  $\Lambda = \{\Lambda_i\}_{i \in J} \subset B(X^*, Y^*)$  is a  $(q, Y^*)$ -operator Bessel sequence. If  $(\Gamma, \Lambda)$  is a pair of approximate dual operator frames, then  $\Gamma$  and  $\Lambda$  are a  $(p, Y)$ -operator frame and a  $(q, Y^*)$ -operator frame, respectively.*

*Proof.* Since  $T_\Lambda U_\Gamma \in B(X)$ ,  $T_\Gamma U_\Lambda \in B(X^*)$ ,  $\|I - T_\Lambda U_\Gamma\| < 1$  and  $\|I - T_\Gamma U_\Lambda\| < 1$ , Lemma 2.3 implies that  $T_\Lambda U_\Gamma, T_\Gamma U_\Lambda$  are invertible. For every  $x \in X, x^* \in X^*$ , we have the following inequalities:

$$\begin{aligned} \|x\| &= \|(T_\Lambda U_\Gamma)^{-1} T_\Lambda U_\Gamma x\| \leq \|(T_\Lambda U_\Gamma)^{-1} T_\Lambda\| \|U_\Gamma x\|, \\ \|x^*\| &= \|(T_\Gamma U_\Lambda)^{-1} T_\Gamma U_\Lambda x^*\| \leq \|(T_\Gamma U_\Lambda)^{-1} T_\Gamma\| \|U_\Lambda x^*\|. \end{aligned}$$

Therefore,  $U_\Gamma, U_\Lambda$  are bounded below, and thus  $\Gamma$  is a  $(p, Y)$ -operator frame and  $\Lambda$  is a  $(q, Y^*)$ -operator frame.  $\square$

**Remark 2.1.** *For approximate dual frames, this result is also true. In this case, based on Lemma 2.1,  $X$  is a reflexive Banach space. Therefore,  $\|I_X - T_\mathcal{F} U_\mathcal{G}\| < 1$  and  $\|I_{X^*} - T_\mathcal{G} U_\mathcal{F}\| < 1$  are equivalents according to Definition 2.1, which is the reason why we use the word “or”. However,  $\|I_X - T_\Lambda U_\Gamma\| < 1$  and  $\|I_{X^*} - T_\Gamma U_\Lambda\| < 1$  are not equivalent based on Definition 2.2.*

### 3. Main results and proofs

Now, we characterize the relationship between the duality of  $p$ -frames and the duality of  $(p, Y)$ -operator frames by Theorems 3.1 and 3.2.

**Theorem 3.1.** *Let  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\mathcal{G} = \{x_i^*\}_{i \in J} \subset X^*$  be a  $p$ -Bessel sequence for  $X$  and  $\mathcal{F} = \{x_i\}_{i \in J} \subset X$  be a  $q$ -Bessel sequence for  $X^*$ . The two operator sequences are defined as follows:*

$$\begin{aligned} \Lambda_i^{\mathcal{F}} : X^* &\longrightarrow \mathbb{F}, \quad \Lambda_i^{\mathcal{F}}(x^*) = \langle x_i, x^* \rangle, \quad \forall x^* \in X^* \\ \Gamma_i^{\mathcal{G}} : X &\longrightarrow \mathbb{F}, \quad \Gamma_i^{\mathcal{G}}(x) = \langle x, x_i^* \rangle, \quad \forall x \in X, \end{aligned}$$

Therefore,  $(\mathcal{G}, \mathcal{F})$  is a pair of approximate dual frames if and only if  $(\Gamma_{\mathcal{G}}, \Lambda_{\mathcal{F}})$  is a pair of approximate dual operator frames, where  $\Gamma_{\mathcal{G}} = \{\Gamma_i^{\mathcal{G}}\}_{i \in J}$ ,  $\Lambda_{\mathcal{F}} = \{\Lambda_i^{\mathcal{F}}\}_{i \in J}$ .

*Proof.* We then show that  $\Gamma_{\mathcal{G}}$  is a  $(p, \mathbb{F})$ -operator Bessel sequence and  $\Lambda_{\mathcal{F}}$  is a  $(q, \mathbb{F})$ -operator Bessel sequence. Let  $T_{\mathcal{F}}, T_{\Lambda_{\mathcal{F}}}$  be the synthesis operators of  $\mathcal{F}, \Lambda_{\mathcal{F}}$  respectively. We also let  $U_{\mathcal{G}}, U_{\Gamma_{\mathcal{G}}}$  be the analysis operators of  $\mathcal{G}, \Gamma_{\mathcal{G}}$  respectively. For any  $x \in X$ , we obtain the following equation:

$$\begin{aligned} \|x - T_{\Lambda_{\mathcal{F}}} U_{\Gamma_{\mathcal{G}}} x\| &= \|x - \sum_{i \in J} \Lambda_i^{\mathcal{F}*} \Gamma_i^{\mathcal{G}} x\| = \|x - \sum_{i \in J} \Lambda_i^{\mathcal{F}*} \langle x, x_i^* \rangle\| \\ &= \|x - \sum_{i \in J} \langle x, x_i^* \rangle x_i\| = \|x - T_{\mathcal{F}} U_{\mathcal{G}} x\|. \end{aligned}$$

Let  $U_{\mathcal{F}}, U_{\Lambda_{\mathcal{F}}}$  be the analysis operators of  $\mathcal{F}, \Lambda_{\mathcal{F}}$  respectively and let  $T_{\mathcal{G}}, T_{\Gamma_{\mathcal{G}}}$  be the synthesis operators of  $\mathcal{G}, \Gamma_{\mathcal{G}}$  respectively. For any  $x^* \in X^*$ , we obtain the following equation:

$$\begin{aligned} \|x^* - T_{\Gamma_{\mathcal{G}}} U_{\Lambda_{\mathcal{F}}} x^*\| &= \|x^* - \sum_{i \in J} \Gamma_i^{\mathcal{G}*} \Lambda_i^{\mathcal{F}} x^*\| = \|x^* - \sum_{i \in J} \Gamma_i^{\mathcal{G}*} \langle x_i, x^* \rangle\| \\ &= \|x^* - \sum_{i \in J} \langle x^*, x_i \rangle x_i^*\| = \|x^* - T_{\mathcal{G}} U_{\mathcal{F}} x^*\|. \end{aligned}$$

Thus, by Definition 2.1 and Definition 2.2, we can prove the theorem.  $\square$

**Theorem 3.2.** Let  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\Gamma = \{\Gamma_i\}_{i \in J} \subset B(X, Y)$  be a  $(p, Y)$ -operator Bessel sequence with bound  $B_\Gamma$  and  $\Lambda = \{\Lambda_i\}_{i \in J} \subset B(X^*, Y^*)$  be a  $(q, Y^*)$ -operator Bessel sequence with bound  $B_\Lambda$  such that  $B_\Gamma B_\Lambda < \frac{1}{2}$ . If  $(\Gamma, \Lambda)$  is a pair of dual operator frames, then  $(\{\Gamma_i^* y^*\}_{i \in J}, \{\Lambda_i^* y\}_{i \in J})$  is a pair of approximate dual frames for units  $y \in Y, y^* \in Y^*$ .

*Proof.* We define an operator  $P$  as follows

$$P : \ell^p(Y) \longrightarrow \ell^p(Y), \quad P(\{y_i\}_{i \in J}) = \{\langle y_i, y^* \rangle y\}_{i \in J}.$$

Clearly,  $P$  is a linear bounded operator. Indeed, provided that  $y, y^*$  are units, we obtain the following inequality:

$$\begin{aligned} \|P(\{y_i\}_{i \in J})\|^p &= \|\{\langle y_i, y^* \rangle y\}_{i \in J}\|^p = \sum_{i \in J} \|\langle y_i, y^* \rangle y\|^p \\ &\leq \sum_{i \in J} \|y_i\|^p \|y^*\|^p \|y\|^p = \|y^*\|^p \|y\|^p \sum_{i \in J} \|y_i\|^p \leq \|\{y_i\}_{i \in J}\|^p. \end{aligned}$$

This inequality shows that  $\|P\| \leq 1$ .

In addition, Lemma 2.2 implies that  $\{\Gamma_i^* y^*\}_{i \in J}$  is a  $p$ -Bessel sequence for  $X^*$  and  $\{\Lambda_i^* y\}_{i \in J}$  is a  $q$ -Bessel sequence for  $X$ . We denote  $U_{\Gamma y^*}$  as the analysis operator of  $\{\Gamma_i^* y^*\}_{i \in J}$  and  $T_{\Lambda y}$  as the synthesis operator of  $\{\Lambda_i^* y\}_{i \in J}$ . For any  $x \in X$ , when  $(\Gamma, \Lambda)$  is a pair of dual operator frames and  $B_\Gamma B_\Lambda < \frac{1}{2}$ , we have the following:

$$\begin{aligned} \|x - T_{\Lambda y} U_{\Gamma y^*} x\| &= \left\| \sum_{i \in J} \Lambda_i^* \Gamma_i x - \sum_{i \in J} \langle x, \Gamma_i^* y^* \rangle \Lambda_i^* y \right\| \\ &= \|T_\Lambda(\{\Gamma_i x\}_{i \in J} - \{\langle x, \Gamma_i^* y^* \rangle y\}_{i \in J})\| \\ &= \|T_\Lambda(U_\Gamma x - P U_\Gamma x)\| \leq B_\Lambda \|U_\Gamma x - P U_\Gamma x\| \\ &\leq B_\Lambda \|I - P\| \cdot \|U_\Gamma x\| \leq 2B_\Lambda B_\Gamma \|x\| < \|x\|. \end{aligned}$$

Thus,  $(\{\Gamma_i^* y^*\}_{i \in J}, \{\Lambda_i^* y\}_{i \in J})$  is a pair of approximate dual frames.  $\square$

Let  $(\Gamma, \Lambda)$  be a pair of approximate dual operator frames. Then we obtain the following:

$$\|x - T_\Lambda U_\Gamma x\| \leq \|I - T_\Lambda U_\Gamma\| \cdot \|x\|, \quad \forall x \in X.$$

Assuming that  $\|I - T_\Lambda U_\Gamma\| \ll 1$ , then  $T_\Lambda U_\Gamma x$  'almost reconstructs'  $x$ . In fact, we can always find a sequence  $\{\Upsilon_n\}$  of  $(p, Y)$ -operator Bessel sequences such that  $\Upsilon_n$  becomes much closer to a dual operator frame of  $\Lambda$  as  $n \rightarrow \infty$ .

**Theorem 3.3.** Let  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\Gamma = \{\Gamma_i\}_{i \in J} \subset B(X, Y)$  be a  $(p, Y)$ -operator Bessel sequence and  $\Lambda = \{\Lambda_i\}_{i \in J} \subset B(X^*, Y^*)$  be a  $(q, Y^*)$ -operator Bessel sequence. If  $(\Gamma, \Lambda)$  is a pair of approximate dual operator frames, then  $(\Upsilon_n, \Lambda)$  is a pair of approximate dual operator frames for any  $n \in \mathbb{N}$ , where  $\Upsilon_n = \{\Upsilon_i^n\}_{i \in J}$ ,  $\Upsilon_i^n = \sum_{k=0}^n \Gamma_i(I - T_\Lambda U_\Gamma)^k$ ,  $\forall i \in J$ . Furthermore, if  $\lim_{n \rightarrow \infty} \Upsilon_n = \Upsilon = \{\Upsilon_i\}_{i \in J}$ , then  $(\Upsilon, \Lambda)$  is a pair of dual operator frames.

*Proof.* Let  $B_\Lambda$  be the Bessel bound of  $\Lambda$  and  $B_\Gamma$  be the Bessel bound of  $\Gamma$ . Given that  $(\Gamma, \Lambda)$  is a pair of approximate dual frames, then  $\|I - T_\Lambda U_\Gamma\| < 1$ . Therefore,  $T_\Lambda U_\Gamma$  is invertible and its inverse can be written via a Neumann series as  $(T_\Lambda U_\Gamma)^{-1} = \sum_{k=0}^{\infty} (I - T_\Lambda U_\Gamma)^k$ . For every  $x \in X$ , the following inequality holds:

$$\begin{aligned} \sum_{i \in J} \|\Upsilon_i^n x\|^p &= \sum_{i \in J} \left\| \sum_{k=0}^n \Gamma_i(I - T_\Lambda U_\Gamma)^k x \right\|^p \leq \sum_{i \in J} \left\| \Gamma_i \sum_{k=0}^n (I - T_\Lambda U_\Gamma)^k x \right\|^p \\ &\leq B_\Gamma^p \left\| \sum_{k=0}^n (I - T_\Lambda U_\Gamma)^k x \right\|^p \leq B_\Gamma^p \left\| \sum_{k=0}^n (I - T_\Lambda U_\Gamma)^k \right\|^p \cdot \|x\|^p. \end{aligned}$$

Thus  $\Upsilon_n = \{\Upsilon_i^n\}_{i \in J}$  is a  $(p, Y)$ -operator Bessel sequence for every  $n \in \mathbb{N}$ . Assuming that  $T_{\Upsilon_n}$  is the synthesis operator of  $\Upsilon_n$ , then we can get the following results:

$$\begin{aligned}
T_{\Upsilon_n} U_{\Lambda} x^* &= \sum_{i \in J} \Upsilon_i^{n*} \Lambda_i x^* = \sum_{i \in J} \left( \sum_{k=0}^n \Gamma_i (I - T_{\Lambda} U_{\Gamma})^k \right)^* \Lambda_i x^* \\
&= \sum_{k=0}^n (I - U_{\Gamma}^* T_{\Lambda}^*)^k \sum_{i \in J} \Gamma_i^* \Lambda_i x^* = \sum_{k=0}^n (I - T_{\Gamma} U_{\Lambda})^k T_{\Gamma} U_{\Lambda} x^* \\
&= \sum_{k=0}^n (I - T_{\Gamma} U_{\Lambda})^k [I - (I - T_{\Gamma} U_{\Lambda})] x^* \\
&= \sum_{k=0}^n (I - T_{\Gamma} U_{\Lambda})^k x^* - \sum_{k=0}^n (I - T_{\Gamma} U_{\Lambda})^{k+1} x^* \\
&= x^* - (I - T_{\Gamma} U_{\Lambda})^{n+1} x^*
\end{aligned}$$

and

$$\begin{aligned}
T_{\Lambda} U_{\Upsilon_n} x &= \sum_{i \in J} \Lambda_i^* \Upsilon_i^n x = \sum_{i \in J} \Lambda_i^* \left( \sum_{k=0}^n \Gamma_i (I - T_{\Lambda} U_{\Gamma})^k \right) x \\
&= \sum_{i \in J} \Lambda_i^* \Gamma_i \left( \sum_{k=0}^n (I - T_{\Lambda} U_{\Gamma})^k x \right) = T_{\Lambda} U_{\Gamma} \left( \sum_{k=0}^n (I - T_{\Lambda} U_{\Gamma})^k x \right) \\
&= [I - (I - T_{\Lambda} U_{\Gamma})] \left( \sum_{k=0}^n (I - T_{\Lambda} U_{\Gamma})^k x \right) \\
&= \sum_{k=0}^n (I - T_{\Lambda} U_{\Gamma})^k x - \sum_{k=0}^n (I - T_{\Lambda} U_{\Gamma})^{k+1} x \\
&= x - (I - T_{\Lambda} U_{\Gamma})^{n+1} x.
\end{aligned}$$

Hence

$$\|I - T_{\Upsilon_n} U_{\Lambda}\| = \|(I - T_{\Gamma} U_{\Lambda})^{n+1}\| \leq \|I - T_{\Gamma} U_{\Lambda}\|^{n+1} < 1$$

and

$$\|I - T_{\Lambda} U_{\Upsilon_n}\| = \|(I - T_{\Lambda} U_{\Gamma})^{n+1}\| \leq \|I - T_{\Lambda} U_{\Gamma}\|^{n+1} < 1.$$

These results show that  $(\Upsilon_n, \Lambda)$  is a pair of approximate dual operator frames for any  $n$ .

If  $\lim_{n \rightarrow \infty} \Upsilon_n = \Upsilon = \{\Upsilon_i\}_{i \in J}$ , then  $\forall \varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $\|\Upsilon_n - \Upsilon\| < \varepsilon, \forall n > N$ . Here, the norm of operator Bessel sequence was defined [10]. For every  $x^* \in X^*$ , we have the following:

$$\begin{aligned}
\|x^* - \sum_{i \in J} \Upsilon_i^* \Lambda_i x^*\| &\leq \|x^* - \sum_{i \in J} \Upsilon_i^{n*} \Lambda_i x^*\| + \left\| \sum_{i \in J} \Upsilon_i^{n*} \Lambda_i x^* - \sum_{i \in J} \Upsilon_i^* \Lambda_i x^* \right\| \\
&\leq \|x^* - T_{\Upsilon_n} U_{\Lambda} x^*\| + \left\| \sum_{i \in J} (\Upsilon_i^{n*} - \Upsilon_i^*) \Lambda_i x^* \right\| \\
&\leq \|I - T_{\Gamma} U_{\Lambda}\|^{n+1} \|x^*\| + \left( \sum_{i \in J} \|\Lambda_i x^*\|^q \right)^{\frac{1}{q}} \left( \sum_{i \in J} \|\Upsilon_i^{n*} - \Upsilon_i^*\|^p \right)^{\frac{1}{p}} \\
&= \|I - T_{\Gamma} U_{\Lambda}\|^{n+1} \|x^*\| + B_{\Lambda} \|x^*\| \cdot \|\Upsilon_n - \Upsilon\|.
\end{aligned}$$

Provided that  $\|I - T_{\Gamma} U_{\Lambda}\| < 1$ , then  $\lim_{n \rightarrow \infty} \|I - T_{\Gamma} U_{\Lambda}\|^{n+1} = 0$ . Hence as  $n \rightarrow \infty$ ,

$$\|I - T_{\Gamma} U_{\Lambda}\|^{n+1} \|x^*\| + B_{\Lambda} \|x^*\| \cdot \|\Upsilon_n - \Upsilon\| \rightarrow 0,$$

i.e.  $x^* = \sum_{i \in J} \Upsilon_i^* \Lambda_i x^*$ ,  $\forall x^* \in X^*$ . Similarly, for every  $x \in X$ , the following inequality holds:

$$\begin{aligned} \|x - \sum_{i \in J} \Lambda_i^* \Upsilon_i x\| &\leq \|x - \sum_{i \in J} \Lambda_i^* \Upsilon_i^n x\| + \|\sum_{i \in J} \Lambda_i^* \Upsilon_i^n x - \sum_{i \in J} \Lambda_i^* \Upsilon_i x\| \\ &\leq \|x - T_\Lambda U_{\Upsilon_n} x\| + \|\sum_{i \in J} \Lambda_i^* (\Upsilon_i^n - \Upsilon_i) x\| \\ &\leq \|I - T_\Lambda U_\Gamma\|^{n+1} \|x\| + B_\Lambda \|\{(\Upsilon_i^n - \Upsilon_i) x\}_{i \in J}\| \\ &= \|I - T_\Lambda U_\Gamma\|^{n+1} \|x\| + B_\Lambda \|\Upsilon_n - \Upsilon\| \cdot \|x\|. \end{aligned}$$

Provided that  $\|I - T_\Lambda U_\Gamma\| < 1$ , then  $\lim_{n \rightarrow \infty} \|I - T_\Lambda U_\Gamma\|^{n+1} = 0$ . Hence as  $n \rightarrow \infty$ ,

$$\|I - T_\Lambda U_\Gamma\|^{n+1} \|x\| + B_\Lambda \|\Upsilon_n - \Upsilon\| \cdot \|x\| \rightarrow 0,$$

i.e.  $x = \sum_{i \in J} \Lambda_i^* \Upsilon_i x$ ,  $\forall x \in X$ . Therefore,  $(\Upsilon, \Lambda)$  is a pair of dual operator frames.  $\square$

The sum or direct sum (or disjointness) of frames is a complicated problem. In Hilbert space setting, Han and Larson [15] systematically investigated the sum and disjointness of frames. By means of duality or approximate duality of frames, we can get interesting results about the sum and direct sum of  $(p, Y)$ -operator frames.

**Theorem 3.4.** *Let  $\frac{1}{p} + \frac{1}{q} = 1$ . If  $\Gamma = \{\Gamma_i\}_{i \in J} \subset B(X, Y)$  and  $\Upsilon = \{\Upsilon_i\}_{i \in J} \subset B(X, Y)$  are dual operator frames of a  $(q, Y^*)$ -operator Bessel sequence  $\Lambda = \{\Lambda_i\}_{i \in J} \subset B(X^*, Y^*)$ , then for any  $\lambda, \mu > 0$ ,  $\frac{\lambda\Gamma + \mu\Upsilon}{\lambda + \mu}$  is also a dual operator frame of  $\Lambda$ .*

*Proof.* If  $\Gamma$  and  $\Upsilon$  are dual operator frames of  $\Lambda$ , then we obtain the following:

$$\begin{aligned} x &= \sum_{i \in J} \Lambda_i^* \Gamma_i x, \quad x^* = \sum_{i \in J} \Gamma_i^* \Lambda_i x^*, \quad \forall x \in X, x^* \in X^*, \\ x &= \sum_{i \in J} \Lambda_i^* \Upsilon_i x, \quad x^* = \sum_{i \in J} \Upsilon_i^* \Lambda_i x^*, \quad \forall x \in X, x^* \in X^*. \end{aligned}$$

Hence, for any  $\lambda, \mu > 0$ , we have the following:

$$\begin{aligned} x &= \frac{\lambda \sum_{i \in J} \Lambda_i^* \Gamma_i x + \mu \sum_{i \in J} \Lambda_i^* \Upsilon_i x}{\lambda + \mu} = \sum_{i \in J} \Lambda_i^* \frac{\lambda \Gamma_i + \mu \Upsilon_i}{\lambda + \mu} x, \quad \forall x \in X, \\ x^* &= \frac{\lambda \sum_{i \in J} \Gamma_i^* \Lambda_i x^* + \mu \sum_{i \in J} \Upsilon_i^* \Lambda_i x^*}{\lambda + \mu} = \sum_{i \in J} \frac{\lambda \Gamma_i^* + \mu \Upsilon_i^*}{\lambda + \mu} \Lambda_i x^* \\ &= \sum_{i \in J} \left( \frac{\lambda \Gamma_i + \mu \Upsilon_i}{\lambda + \mu} \right)^* \Lambda_i x^*, \quad \forall x^* \in X^*. \end{aligned}$$

This result shows that  $(\frac{\lambda\Gamma + \mu\Upsilon}{\lambda + \mu}, \Lambda)$  is a pair of dual operator frames.  $\square$

For the direct sum of  $(p, Y)$ -operator frames, the result is natural. Indeed, when we let  $\Gamma^k = \{\Gamma_i^k\}_{i \in J} \subset B(X_k, Y_k)$  be a  $(p, Y_k)$ -operator frames for  $X_k$  with upper bound and lower bound  $B_k, A_k$ , with  $k = 1, 2$ , for any  $x_1 \oplus x_2 \in X_1 \oplus X_2$ , we can obtain the following:

$$\sum_{i \in J} \|(\Gamma_i^1 \oplus \Gamma_i^2)(x_1 \oplus x_2)\|^p = \sum_{i \in J} \|\Gamma_i^1 x_1 \oplus \Gamma_i^2 x_2\|^p = \sum_{i \in J} \|\Gamma_i^1 x_1\|^p + \sum_{i \in J} \|\Gamma_i^2 x_2\|^p.$$

On the one hand,

$$\sum_{i \in J} \|(\Gamma_i^1 \oplus \Gamma_i^2)(x_1 \oplus x_2)\|^p \geq \min\{A_1, A_2\} (\|x_1\|^p + \|x_2\|^p) = \min\{A_1, A_2\} (\|x_1 \oplus x_2\|^p).$$

On the other hand,

$$\sum_{i \in J} \|(\Gamma_i^1 \oplus \Gamma_i^2)(x_1 \oplus x_2)\|^p \leq \max\{B_1, B_2\}(\|x_1\|^p + \|x_2\|^p) = \max\{B_1, B_2\}(\|x_1 \oplus x_2\|^p).$$

These two inequalities imply that  $\Gamma^1 \oplus \Gamma^2$  is a  $(p, Y_1 \oplus Y_2)$ -operator frame for  $X_1 \oplus X_2$ . Similarly, this argument is true for  $n$ -tuples. Surprisingly, such a direct sum can preserve the duality and approximate duality of  $(p, Y)$ -operator frames.

**Theorem 3.5.** *Let  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\Gamma^k = \{\Gamma_i^k\}_{i \in J} \subset B(X_k, Y_k)$  be a  $(p, Y_k)$ -operator Bessel sequences for  $X_k$  and  $\Lambda^k = \{\Lambda_i^k\}_{i \in J} \subset B(X_k^*, Y_k^*)$  be a  $(q, Y_k^*)$ -operator Bessel sequences for  $X_k^*$ , with  $k = 1, 2$ . If for every  $k = 1, 2$ ,  $(\Gamma^k, \Lambda^k)$  is a pair of dual operator frames, then  $(\Gamma^1 \oplus \Gamma^2 = \{\Gamma_i^1 \oplus \Gamma_i^2\}_{i \in J}, \Lambda^1 \oplus \Lambda^2 = \{\Lambda_i^1 \oplus \Lambda_i^2\}_{i \in J})$  is a pair of dual operator frames.*

*Proof.* Clearly,  $\Gamma^1 \oplus \Gamma^2$  is a  $(p, Y_1 \oplus Y_2)$ -operator Bessel sequence for  $X_1 \oplus X_2$  and  $\Lambda^1 \oplus \Lambda^2$  is a  $(q, Y_1^* \oplus Y_2^*)$ -operator Bessel sequence for  $X_1^* \oplus X_2^*$ . For every  $x_1 \oplus x_2 \in X_1 \oplus X_2$ ,  $x_1^* \oplus x_2^* \in X_1^* \oplus X_2^*$ , we have the following:

$$\begin{aligned} x_1 \oplus x_2 &= \sum_{i \in J} \Lambda_i^{1*} \Gamma_i^1 x_1 \oplus \sum_{i \in J} \Lambda_i^{2*} \Gamma_i^2 x_2 = \sum_{i \in J} \Lambda_i^{1*} \Gamma_i^1 x_1 \oplus \Lambda_i^{2*} \Gamma_i^2 x_2 \\ &= \sum_{i \in J} (\Lambda_i^1 \oplus \Lambda_i^2)^* (\Gamma_i^1 \oplus \Gamma_i^2) (x_1 \oplus x_2), \\ x_1^* \oplus x_2^* &= \sum_{i \in J} \Gamma_i^{1*} \Lambda_i^1 x_1^* \oplus \sum_{i \in J} \Gamma_i^{2*} \Lambda_i^2 x_2^* = \sum_{i \in J} \Gamma_i^{1*} \Lambda_i^1 x_1^* \oplus \Gamma_i^{2*} \Lambda_i^2 x_2^* \\ &= \sum_{i \in J} (\Gamma_i^1 \oplus \Gamma_i^2)^* (\Lambda_i^1 \oplus \Lambda_i^2) (x_1^* \oplus x_2^*). \end{aligned}$$

Hence,  $(\Gamma^1 \oplus \Gamma^2, \Lambda^1 \oplus \Lambda^2)$  is a pair of dual operator frames.  $\square$

**Theorem 3.6.** *Let  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\Gamma^k = \{\Gamma_i^k\}_{i \in J} \subset B(X_k, Y_k)$  be a  $(p, Y_k)$ -operator Bessel sequence for  $X_k$  and  $\Lambda^k = \{\Lambda_i^k\}_{i \in J} \subset B(X_k^*, Y_k^*)$  be a  $(q, Y_k^*)$ -operator Bessel sequence for  $X_k^*$ , with  $k = 1, 2$ . If for every  $k = 1, 2$ ,  $(\Gamma^k, \Lambda^k)$  is a pair of approximate dual frames, then  $(\Gamma^1 \oplus \Gamma^2 = \{\Gamma_i^1 \oplus \Gamma_i^2\}_{i \in J}, \Lambda^1 \oplus \Lambda^2 = \{\Lambda_i^1 \oplus \Lambda_i^2\}_{i \in J})$  is a pair of approximate dual operator frames.*

*Proof.* From the definition of approximate dual operator frames,  $\|I - T_{\Lambda^1} U_{\Gamma^1}\| < 1$  and  $\|I - T_{\Lambda^2} U_{\Gamma^2}\| < 1$  hold. For every  $x_1 \oplus x_2 \in X_1 \oplus X_2$ , then

$$\begin{aligned} &\|x_1 \oplus x_2 - T_{\Lambda^1 \oplus \Lambda^2} U_{\Gamma^1 \oplus \Gamma^2}(x_1 \oplus x_2)\|^p \\ &= \|x_1 \oplus x_2 - \sum_{i \in J} (\Lambda_i^1 \oplus \Lambda_i^2)^* (\Gamma_i^1 \oplus \Gamma_i^2) (x_1 \oplus x_2)\|^p \\ &= \|x_1 \oplus x_2 - \sum_{i \in J} \Lambda_i^{1*} \Gamma_i^1 x_1 \oplus \sum_{i \in J} \Lambda_i^{2*} \Gamma_i^2 x_2\|^p \\ &= \|(x_1 - \sum_{i \in J} \Lambda_i^{1*} \Gamma_i^1 x_1) \oplus (x_2 - \sum_{i \in J} \Lambda_i^{2*} \Gamma_i^2 x_2)\|^p \\ &= \|x_1 - \sum_{i \in J} \Lambda_i^{1*} \Gamma_i^1 x_1\|^p + \|x_2 - \sum_{i \in J} \Lambda_i^{2*} \Gamma_i^2 x_2\|^p \\ &= \|x_1 - T_{\Lambda^1} U_{\Gamma^1} x_1\|^p + \|x_2 - T_{\Lambda^2} U_{\Gamma^2} x_2\|^p \\ &= \|I - T_{\Lambda^1} U_{\Gamma^1}\|^p \|x_1\|^p + \|I - T_{\Lambda^2} U_{\Gamma^2}\|^p \|x_2\|^p \\ &< \|x_1\|^p + \|x_2\|^p = \|x_1 \oplus x_2\|^p, \end{aligned}$$

i.e.  $\|I - T_{\Lambda^1 \oplus \Lambda^2} U_{\Gamma^1 \oplus \Gamma^2}\| < 1$ . Similarly,  $\|I - T_{\Gamma^1 \oplus \Gamma^2} U_{\Lambda^1 \oplus \Lambda^2}\| < 1$ . Hence,  $(\Gamma^1 \oplus \Gamma^2, \Lambda^1 \oplus \Lambda^2)$  is a pair of approximate dual operator frames.  $\square$

In the last part of this paper, we discuss the perturbations of dual operator frames and approximate dual operator frames.

**Theorem 3.7.** *Let  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\Gamma = \{\Gamma_i\}_{i \in J} \subset B(X, Y)$  be a  $(p, Y)$ -operator Bessel sequence and  $\Lambda = \{\Lambda_i\}_{i \in J} \subset B(X^*, Y^*)$  be a  $(q, Y^*)$ -operator Bessel sequence with bound  $B_\Lambda > 0$ . Suppose that sequence  $\Upsilon = \{\Upsilon_i\}_{i \in J} \subset B(X, Y)$  satisfies the following condition:*

$$\left\| \sum_{i \in J} (\Gamma_i^* - \Upsilon_i^*) y_i^* \right\| \leq \lambda \left\| \{y_i^*\}_{i \in J} \right\|, \quad \forall \{y_i^*\}_{i \in J} \in \ell^q(Y^*), \quad (5)$$

then the following statements hold.

(1) When  $\|I - T_\Gamma U_\Lambda\| < 1 - \lambda B_\Lambda$  and  $\|I - T_\Lambda U_\Gamma\| < 1 - \lambda B_\Lambda$ ,  $(\Upsilon, \Lambda)$  is a pair of approximate dual operator frames.

(2) When  $(\Gamma, \Lambda)$  is a pair of dual operator frames and  $\lambda B_\Lambda < 1$ ,  $(\Upsilon, \Lambda)$  is also a pair of approximate dual operator frames.

*Proof.* If  $\Gamma = \{\Gamma_i\}_{i \in J} \subset B(X, Y)$  is a  $(p, Y)$ -operator Bessel sequence, (5) implies that  $\Upsilon$  is a  $(p, Y)$ -operator Bessel sequence. For every  $x^* \in X^*$ , we obtain the following:

$$\begin{aligned} \|x^* - T_\Upsilon U_\Lambda x^*\| &\leq \|x^* - T_\Gamma U_\Lambda x^*\| + \|T_\Gamma U_\Lambda x^* - T_\Upsilon U_\Lambda x^*\| \\ &\leq \|I - T_\Gamma U_\Lambda\| \cdot \|x^*\| + \lambda \|U_\Lambda x^*\| \\ &\leq (\|I - T_\Gamma U_\Lambda\| + \lambda B_\Lambda) \|x^*\|. \end{aligned}$$

Provided that (5) implies that  $\Gamma - \Upsilon$  is a  $(p, Y)$ -operator Bessel sequence with bound  $\lambda$ , then the following inequality holds:

$$\left\| \{(\Gamma_i - \Upsilon_i)x\}_{i \in J} \right\| = \left( \sum_{i \in J} \|(\Gamma_i - \Upsilon_i)x\|^p \right)^{\frac{1}{p}} \leq \lambda \|x\|, \quad \forall x \in X \quad (6)$$

For every  $x \in X$ , by (6), we get the following:

$$\begin{aligned} \|x - T_\Lambda U_\Upsilon x\| &\leq \|x - T_\Lambda U_\Gamma x\| + \|T_\Lambda U_\Gamma x - T_\Lambda U_\Upsilon x\| \\ &\leq \|I - T_\Lambda U_\Gamma\| \cdot \|x\| + \|T_\Lambda(U_\Gamma - U_\Upsilon)x\| \\ &\leq \|I - T_\Lambda U_\Gamma\| \cdot \|x\| + B_\Lambda \|(U_\Gamma - U_\Upsilon)x\| \\ &\leq (\|I - T_\Lambda U_\Gamma\| + \lambda B_\Lambda) \|x\| \end{aligned}$$

Clearly, if  $\|I - T_\Gamma U_\Lambda\| < 1 - \lambda B_\Lambda$  and  $\|I - T_\Lambda U_\Gamma\| < 1 - \lambda B_\Lambda$ , then  $(\Upsilon, \Lambda)$  is a pair of approximate dual operator frames. More specifically, if  $(\Gamma, \Lambda)$  is a pair of dual operator frames and  $\lambda B_\Lambda < 1$ , then  $(\Upsilon, \Lambda)$  is a pair of approximate dual operator frames.  $\square$

**Theorem 3.8.** *Suppose that  $X, Y$  are two reflexive Banach spaces. Let  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\Gamma = \{\Gamma_i\}_{i \in J} \subset B(X, Y)$  be a  $(p, Y)$ -operator Bessel sequence with bound  $B_\Gamma > 0$ ,  $\Lambda = \{\Lambda_i\}_{i \in J} \subset B(X^*, Y^*)$  be a  $(q, Y^*)$ -operator Bessel sequence with bound  $B_\Lambda > 0$  and  $(\Gamma, \Lambda)$  be a pair of approximate dual operator frames. Suppose that the constants  $0 < \lambda_1, \lambda_2, \mu < 1$  exist, and a sequence  $\Upsilon = \{\Upsilon_i\}_{i \in J} \subset B(X, Y)$  exist, such that  $\|I - T_\Gamma U_\Lambda\| < 1 - \frac{(\lambda_1 + \lambda_2)B_\Gamma + \mu}{1 - \lambda_2} B_\Lambda$  and  $\forall J \in \mathcal{F}(J), \{y_i^*\}_{i \in J} \in \ell^q(Y^*)$ , the following condition holds:*

$$\left\| \sum_{i \in J} (\Upsilon_i^* - \Gamma_i^*) y_i^* \right\| \leq \lambda_1 \left\| \sum_{i \in J} \Gamma_i^* y_i^* \right\| + \lambda_2 \left\| \sum_{i \in J} \Upsilon_i^* y_i^* \right\| + \mu \left( \sum_{i \in J} \|y_i^*\|^q \right)^{\frac{1}{q}}, \quad (7)$$

where  $\mathcal{F}(J)$  is the set of all finite subsets of  $J$ . Then  $(\Upsilon, \Lambda)$  is a pair of approximate dual operator frames.

*Proof.* First, we need to prove that  $\Upsilon$  is a  $(p, Y)$ -operator Bessel sequence. In fact, using the inequality (7), we can get the following:

$$\begin{aligned} \left\| \sum_{i \in J} \Upsilon_i^* y_i^* \right\| &\leq \left\| \sum_{i \in J} (\Upsilon_i^* - \Gamma_i^*) y_i^* + \sum_{i \in J} \Gamma_i^* y_i^* \right\| \\ &\leq \lambda_1 \left\| \sum_{i \in J} \Gamma_i^* y_i^* \right\| + \lambda_2 \left\| \sum_{i \in J} \Upsilon_i^* y_i^* \right\| + \mu \left( \sum_{i \in J} \|y_i^*\|^q \right)^{\frac{1}{q}} + \left\| \sum_{i \in J} \Gamma_i^* y_i^* \right\| \\ &= (\lambda_1 + 1) \left\| \sum_{i \in J} \Gamma_i^* y_i^* \right\| + \lambda_2 \left\| \sum_{i \in J} \Upsilon_i^* y_i^* \right\| + \mu \left( \sum_{i \in J} \|y_i^*\|^q \right)^{\frac{1}{q}}, \end{aligned}$$

that is

$$\left\| \sum_{i \in J} \Upsilon_i^* y_i^* \right\| \leq \frac{1 + \lambda_1}{1 - \lambda_2} \left\| \sum_{i \in J} \Gamma_i^* y_i^* \right\| + \frac{\mu}{1 - \lambda_2} \left( \sum_{i \in J} \|y_i^*\|^q \right)^{\frac{1}{q}}, \quad \forall \{y_i^*\}_{i \in J} \in \ell^q(Y^*).$$

Given that  $\Gamma$  is a  $(p, Y)$ -operator Bessel sequence and  $\{y_i^*\}_{i \in J} \in \ell^q(Y^*)$ , then  $\sum_{i \in J} \Gamma_i^* y_i^*$  exists.

Thus for any  $\varepsilon > 0$ , a finite set  $J_0 \subset J$  exists, such that the following is obtained:

$$\left\| \sum_{i \in J} \Gamma_i^* y_i^* \right\| < \frac{1 - \lambda_2}{2 + 2\lambda_1} \varepsilon, \quad \left( \sum_{i \in J} \|y_i^*\|^q \right)^{\frac{1}{q}} < \frac{1 - \lambda_2}{2\mu} \varepsilon, \quad \forall J \in \mathcal{F}(J), J_0 \cap J = \emptyset.$$

Thus

$$\left\| \sum_{i \in J} \Upsilon_i^* y_i^* \right\| \leq \frac{1 + \lambda_1}{1 - \lambda_2} \left\| \sum_{i \in J} \Gamma_i^* y_i^* \right\| + \frac{\mu}{1 - \lambda_2} \left( \sum_{i \in J} \|y_i^*\|^q \right)^{\frac{1}{q}} < \varepsilon, \quad \forall \{y_i^*\}_{i \in J} \in \ell^q(Y^*).$$

Therefore,  $\sum_{i \in J} \Upsilon_i^* y_i^*$  converges and thus leads to the following result:

$$\begin{aligned} \left\| \sum_{i \in J} \Upsilon_i^* y_i^* \right\| &\leq \frac{1 + \lambda_1}{1 - \lambda_2} \left\| \sum_{i \in J} \Gamma_i^* y_i^* \right\| + \frac{\mu}{1 - \lambda_2} \left( \sum_{i \in J} \|y_i^*\|^q \right)^{\frac{1}{q}} \\ &\leq \left[ \frac{1 + \lambda_1}{1 - \lambda_2} B_\Gamma + \frac{\mu}{1 - \lambda_2} \right] \cdot \left\| \{y_i^*\}_{i \in J} \right\|. \end{aligned}$$

As  $J \rightarrow J$ , (7) implies that

$$\left\| \sum_{i \in J} (\Upsilon_i^* - \Gamma_i^*) y_i^* \right\| \leq \lambda_1 \left\| \sum_{i \in J} \Gamma_i^* y_i^* \right\| + \lambda_2 \left\| \sum_{i \in J} \Upsilon_i^* y_i^* \right\| + \mu \left( \sum_{i \in J} \|y_i^*\|^q \right)^{\frac{1}{q}}.$$

For any  $x^* \in X^*$ , using these inequalities, we obtain the following:

$$\begin{aligned} &\|x^* - T_\Upsilon U_\Lambda x^*\| = \|x^* - \sum_{i \in J} \Upsilon_i^* \Lambda_i x^*\| \\ &\leq \|x^* - \sum_{i \in J} \Gamma_i^* \Lambda_i x^*\| + \left\| \sum_{i \in J} (\Upsilon_i^* - \Gamma_i^*) \Lambda_i x^* \right\| \\ &\leq \|I - T_\Gamma U_\Lambda\| \|x^*\| + \lambda_1 \left\| \sum_{i \in J} \Gamma_i^* \Lambda_i x^* \right\| + \lambda_2 \left\| \sum_{i \in J} \Upsilon_i^* \Lambda_i x^* \right\| + \mu \left\| \{\Lambda_i x^*\}_{i \in J} \right\| \\ &< \|I - T_\Gamma U_\Lambda\| \|x^*\| + \lambda_1 B_\Gamma B_\Lambda \|x^*\| + \lambda_2 \left[ \frac{1 + \lambda_1}{1 - \lambda_2} B_\Gamma + \frac{\mu}{1 - \lambda_2} \right] B_\Lambda \|x^*\| + \mu B_\Lambda \|x^*\| \\ &= \left[ \|I - T_\Gamma U_\Lambda\| + \frac{(\lambda_1 + \lambda_2) B_\Gamma + \mu}{1 - \lambda_2} B_\Lambda \right] \|x^*\| < \|x^*\|. \end{aligned}$$

i.e.  $\|I - T_\Upsilon U_\Lambda\| < 1$ . As  $X, Y$  are reflexive, then  $\|I - T_\Lambda U_\Upsilon\| < 1$ . Hence,  $(\Upsilon, \Lambda)$  is a pair of approximate dual operator frames.  $\square$

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