

## VARIOUS SHADOWING PROPERTIES FOR PARAMETERIZED ITERATED FUNCTION SYSTEMS

Mehdi Fatehi Nia<sup>1</sup>, Seyyed Alireza Ahmadi<sup>2</sup>

*In this paper we generalize the notions of limit shadowing property and exponential limit shadowing property to parameterized iterated function systems IFS and prove some related theorems on these notions. It is proved that every uniformly contracting and every uniformly expanding IFS has the exponential limit shadowing property. Then, as an example, we give an IFS which has the limit shadowing property, but fails to have the exponential limit shadowing property and compare this result with similar ones in original discrete dynamical systems.*

**Keywords:** Exponential limit shadowing, Limit shadowing, Iterated function systems, Uniformly contracting, Uniformly expanding.

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### 1. Introduction

Iterated function systems (**IFS**) are introduced as a unified way of generating a broad class of fractals and have found numerous applications, in particular to image compression and image processing [5]. Important notions in dynamics like attractors, minimality, transitivity, and shadowing have been extended to IFS (see [3, 4, 7, 8, 9, 11]). The shadowing property plays a key role in the study of the stability of the dynamics. This property says that, near approximate trajectories, we can find exact trajectories of the system under consideration [2, 14, 16]. This property is found in hyperbolic dynamics, and it was used successfully to prove their stability, see for example [13, 17]. In [10], the authors propose a generalization of the Shadowing Property for set-valued dynamical systems, generated by parameterized IFS, and prove that if a parameterized IFS is uniformly contracting or uniformly expanding, then it has the shadowing property.

In this paper, we consider various shadowing properties for parameterized iterated function systems. First, we recall some definitions and theorems in Section 2. In this section limit shadowing and exponential limit shadowing properties for IFS are considered. Then, in Section 3, some preliminary results are proven, showing that the uniformly expanding and uniformly contracting IFS have the limit shadowing and exponential limit shadowing properties. Theorem 3.7 shows that when functions in the IFS  $\mathcal{F}$  are hyperbolic with the same stable and unstable subspaces,  $\mathcal{F}$  has the shadowing, limit shadowing and exponential limit shadowing properties. In Section 4, we give several examples to illustrate these shadowing properties. In Example ??, this is proved that when the functions in the IFS  $\mathcal{F}$  have a common attractor fixed point,  $\mathcal{F}$  has these shadowing properties locally. In Example 3 and

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<sup>1</sup>Department of Mathematics, Yazd University, 89195-741 Yazd, Iran. e-mail: [fatehiniam@yazd.ac.ir](mailto:fatehiniam@yazd.ac.ir)

<sup>2</sup> Corresponding author: Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran. e-mail: [sa.ahmadi@math.usb.ac.ir](mailto:sa.ahmadi@math.usb.ac.ir)

4 we introduce different uniformly contracting IFS. Example 4.2 obtains an IFS on the torus  $\mathbb{T}^3$  which has the shadowing, limit shadowing and exponential limit shadowing properties. Example 4.3 is one of the main results of this paper. In this example we construct a nontrivial IFS which has the limit shadowing property but has neither the shadowing property nor the exponential limit shadowing property. Example ?? presents an IFS which is neither uniformly expanding nor uniformly contracting, but has the limit shadowing and exponential limit shadowing properties.

## 2. Definitions

In this Section, we present some terminology and results which are used throughout the paper.

Let  $(X, d)$  be a complete metric space. A parameterized iterated function system (IFS)  $\mathcal{F}$  is the space  $X$  together with a family of continuous functions  $f_\lambda : X \rightarrow X$ ,  $\lambda \in \Lambda$ , where  $\Lambda$  is an arbitrary nonempty set and is denoted by

$$\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}.$$

A typical element of  $\Lambda^{\mathbb{Z}^+}$  can be denoted as  $\sigma = \{\lambda_0, \lambda_1, \dots\}$  and we use the notation

$$\mathcal{F}_{\sigma_n} = f_{\lambda_0} \circ f_{\lambda_1} \circ \dots \circ f_{\lambda_n}.$$

A sequence  $\{x_i\}_{i \geq 0}$  in  $X$  is called a  $\delta$ -pseudo orbit of the IFS  $\mathcal{F}$ , if there exists a sequence  $\sigma = \{\lambda_0, \lambda_1, \dots\} \in \Lambda^{\mathbb{Z}^+}$  such that  $d(f_{\lambda_i}(x_i), x_{i+1}) < \delta$ , for all  $i \geq 0$ . A sequence  $\{x_i\}_{i \geq 0}$  in  $X$  is called an orbit of  $\mathcal{F}$  if there exists a sequence  $\sigma = \{\lambda_0, \lambda_1, \dots\} \in \Lambda^{\mathbb{Z}^+}$  such that  $f_{\lambda_i}(x_i) = x_{i+1}$ , for all  $i \geq 0$ .

One says that the IFS  $\mathcal{F}$  has the Shadowing Property (on  $\mathbb{Z}^+$ ) if, given  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any  $\delta$ -pseudo orbit  $\{x_i\}_{i \geq 0}$  can be found an orbit  $\{y_i\}_{i \geq 0}$  for which the inequality  $d(x_i, y_i) < \epsilon$  holds for all  $i \geq 0$  [10].

**Definition 2.1.** A sequence  $\{x_n\}_{n \geq 0}$  in  $X$  is called an asymptotic pseudo-orbit of  $\mathcal{F}$  if there exists  $\sigma = \{\lambda_0, \lambda_1, \dots\} \in \Lambda^{\mathbb{Z}^+}$  such that

$$\lim_{n \rightarrow \infty} d(f_{\lambda_n}(x_n), x_{n+1}) \rightarrow 0.$$

One says that the IFS  $\mathcal{F}$  has the limit shadowing property if for any asymptotic pseudo orbit  $\{x_n\}_{n \geq 0}$  there exists an orbit  $\{y_n\}_{n \geq 0}$  so that

$$\lim_{n \rightarrow \infty} d(x_n, y_n) \rightarrow 0.$$

We introduce another kind of shadowing for which one-step errors tend to zero with exponential rate.

**Definition 2.2.** We say that a sequence  $\{t_n\}_{n \geq 0}$  of real numbers converges to zero with rate  $\theta \in (0, 1)$ , and we write  $t_n \xrightarrow{\theta} 0$ , if there exists a constant  $L > 0$  such that  $|t_n| \leq L\theta^n$  for  $n \geq 0$ .

**Definition 2.3.** Given  $\theta \in (0, 1)$ , the sequence  $\xi = \{x_n\}_{n \geq 0}$  in  $X$  is called a  $\theta$ -exponentially asymptotic pseudo-orbit of  $\mathcal{F}$  if there exists

$\sigma = \{\lambda_0, \lambda_1, \dots\} \in \Lambda^{\mathbb{Z}^+}$  such that

$$\lim_{n \rightarrow \infty} d(f_{\lambda_n}(x_n), x_{n+1}) \xrightarrow{\theta} 0.$$

One says that the IFS  $\mathcal{F}$  has the exponential limit shadowing property with exponent  $\xi$  if

there exists  $\theta_0 \in (0, 1)$  so that for any  $\theta$ -exponentially asymptotic pseudo orbit  $\{x_n\}_{n \in T}$  with  $\theta \in (\theta_0, 1)$ , there is an orbit  $\{y_n\}_{n \geq 0}$  such that

$$\lim_{n \rightarrow \infty} d(x_n, y_n) \xrightarrow{\theta} 0.$$

In the case  $\xi = 1$  we say that  $\mathcal{F}$  has the exponential limit shadowing.

Recall that the parameterized IFS  $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$  is uniformly contracting if

$$\alpha := \sup_{\lambda \in \Lambda} \sup_{x \neq y} \frac{d(f_\lambda(x), f_\lambda(y))}{d(x, y)},$$

called the *contracting ratio*, exists and is less than one. We say that  $\mathcal{F}$  is uniformly expanding if

$$\beta := \inf_{\lambda \in \Lambda} \inf_{x \neq y} \frac{d(f_\lambda(x), f_\lambda(y))}{d(x, y)},$$

called the *expanding ratio*, exists and is greater than one [10].

In this paper, we extend Theorems 2.1 and 2.2 of [10] which are related to shadowing property, to the context of limit shadowing and exponential limit shadowing properties. Here we recall the theorems.

**Theorem 2.4.** [10] *If a parameterized IFS  $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$  is uniformly contracting, then it has the shadowing property on  $\mathbb{Z}_+$ .*

**Theorem 2.5.** [10] *If a parameterized IFS  $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$  is uniformly expanding, and if each function  $f_\lambda (\lambda \in \Lambda)$  is surjective, then the IFS has the Shadowing Property on  $\mathbb{Z}_+$ .*

Finally, we present a definition which will be used in classification of parameterized IFS.

**Definition 2.6.** Suppose  $(X, d)$  and  $(Y, d')$  are compact metric spaces and  $\Lambda$  is a finite set. Let  $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$  and  $\mathcal{G} = \{Y; g_\lambda | \lambda \in \Lambda\}$  be two IFS for which the functions  $f_\lambda : X \rightarrow X$  and  $g_\lambda : Y \rightarrow Y$  are continuous for all  $\lambda \in \Lambda$ . We say that  $\mathcal{F}$  is topologically conjugate to  $\mathcal{G}$  if there is a homeomorphism  $h : X \rightarrow Y$  such that  $g_\lambda = h \circ f_\lambda \circ h^{-1}$ , for all  $\lambda \in \Lambda$ .

### 3. Results

We begin this section with a proposition in which we prove that the shadowing and limit shadowing properties are invariant under conjugacy.

**Proposition 3.1.** Suppose  $(X, d_X)$  and  $(Y, d_Y)$  are compact metric spaces and  $\Lambda$  is a finite set. Let  $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$  and  $\mathcal{G} = \{Y; g_\lambda | \lambda \in \Lambda\}$  be two conjugated IFS, then:

- (a)  $\mathcal{F}$  has the shadowing property if and only if so does  $\mathcal{G}$ .
- (b)  $\mathcal{F}$  has the limit shadowing property if and only if so does  $\mathcal{G}$ .

*Proof.* We prove only part (b). The proof of Part (a) is similar and will be omitted.

Suppose that  $h : X \rightarrow Y$  is a homeomorphism such that  $g_\lambda = h \circ f_\lambda \circ h^{-1}$  for all  $\lambda \in \Lambda$ . Let  $\mathcal{F}$  have the limit shadowing property and  $\{x_n\}_{n \geq 0}$  be an asymptotic pseudo orbit of  $\mathcal{G}$ , i.e. there exists  $\sigma = \{\lambda_0, \lambda_1, \dots\} \in \Lambda^{\mathbb{Z}_+}$  such that  $\lim_{n \rightarrow \infty} d(x_{n+1}, g_{\lambda_n}(x_n)) = 0$ . Since  $h^{-1}$  is uniformly continuous we have  $\lim_{n \rightarrow \infty} d(h^{-1}(x_{n+1}), h^{-1} \circ g_{\lambda_n}(x_n)) = 0$ . So

$$\lim_{n \rightarrow \infty} d(h^{-1}(x_{n+1}), f_{\lambda_n} \circ h^{-1}(x_n)) = 0.$$

This means that  $y_n = h^{-1}(x_n)$  is an asymptotic pseudo orbit of  $\mathcal{F}$ . Therefore there exists an orbit  $\{z_n\}_{n \geq 0}$  of  $\mathcal{F}$  so that  $d(y_n, z_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Hence the orbit  $\{w_n = h(z_n)\}_{n \geq 0}$  of  $\mathcal{G}$  satisfies  $\lim_{n \rightarrow \infty} d(x_n, w_n) = 0$ .  $\square$

In the following theorems we investigate limit shadowing and exponential limit shadowing properties in uniformly contracting and uniformly expanding IFS.

**Theorem 3.2.** *If a parameterized IFS  $\mathcal{F} = \{X : f_\lambda | \lambda \in \Lambda\}$  is uniformly contracting, then:*

- (a)  $\mathcal{F}$  has the limit shadowing Property on  $\mathbb{Z}_+$ .
- (b)  $\mathcal{F}$  has the exponential limit shadowing property on  $\mathbb{Z}_+$

*Proof.* Assume that the IFS  $\mathcal{F}$  is uniformly contracting with the contracting ratio  $\alpha$ .

(a) Suppose  $\{x_n\}_{n \geq 0}$  is an asymptotic pseudo orbit for  $\mathcal{F}$ . So there there exist  $\sigma = \{\lambda_0, \lambda_1, \lambda_2, \dots\} \in \Lambda^{\mathbb{Z}_+}$  such that  $\lim_{n \rightarrow \infty} d(f_{\lambda_n}(x_n), x_{n+1}) = 0$ . Put  $\tau_n = d(f_{\lambda_n}(x_n), x_{n+1})$ , for all  $n \geq 0$ . Consider an orbit  $\{y_n\}_{n \geq 0}$  such that  $x_0 = y_0$  and  $y_{n+1} = f_{\lambda_n}(y_n)$ , for all  $n \geq 0$ .

Now we show that  $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ .

Suppose  $\epsilon$  is an arbitrary positive number and  $M = \sup\{\tau_n\}_{n \geq 0}$ . We can find  $k \in \mathbb{N}$  such that  $M \frac{\alpha^k}{1-\alpha} < \frac{\epsilon}{2}$  and  $\tau_i < \epsilon \frac{(1-\alpha)}{2}$  for all  $i \geq k$ . Obviously,

$$d(x_1, y_1) \leq d(x_1, f_{\lambda_0}(x_0)) + d(f_{\lambda_0}(x_0), f_{\lambda_0}(y_0)) \leq \tau_0.$$

Similarly

$$\begin{aligned} d(x_2, y_2) &\leq d(x_2, f_{\lambda_1}(x_1)) + d(f_{\lambda_1}(x_1), f_{\lambda_1}(y_1)) \\ &\leq \tau_1 + \alpha d(x_1, y_1) \\ &\leq \tau_1 + \alpha \tau_0. \end{aligned}$$

And

$$\begin{aligned} d(x_3, y_3) &\leq d(x_3, f_{\lambda_2}(x_2)) + d(f_{\lambda_2}(x_2), f_{\lambda_2}(y_2)) \\ &\leq \tau_2 + \alpha d(x_2, y_2) \\ &\leq \tau_2 + \alpha(\tau_1 + \alpha d(x_1, y_1)) \\ &\leq \tau_2 + \alpha(\tau_1 + \alpha \tau_0) \\ &= \tau_2 + \alpha \tau_1 + \alpha^2 \tau_0. \end{aligned}$$

By induction, one can prove that for each  $n > 2$

$$d(x_n, y_n) \leq \tau_{n-1} + \alpha \tau_{n-2} + \dots + \alpha^{n-1} \tau_0.$$

This implies that

$$\begin{aligned} d(y_n, x_n) &\leq \tau_{n-1} + \alpha \tau_{n-2} + \dots + \alpha^{k-1} \tau_{n-k} \\ &\quad + \alpha^k \tau_{n-(k+1)} + \alpha^{k+1} \tau_{n-(k+2)} + \dots + \alpha^{n-1} \tau_0 \\ &\leq \epsilon \frac{(1-\alpha)}{2} (1 + \alpha + \dots + \alpha^{k-1}) + M \alpha^k (1 + \alpha + \dots + \alpha^{n-k-1}) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all  $n \geq k$ . Therefore  $\lim_{n \rightarrow \infty} d(y_n, x_n) \leq \epsilon$ . Since  $\epsilon > 0$  is arbitrary then  $\lim_{n \rightarrow \infty} d(y_n, x_n) = 0$ . This shows that  $\mathcal{F}$  has the limit shadowing property.

(b) We choose  $\theta_0 \in (\alpha, 1)$  and show that  $\mathcal{F}$  has the exponential limit shadowing property

with respect to this  $\theta_0$ . Let  $\{x_n\}_{n \geq 0}$  be an  $\theta$ -exponentially asymptotic pseudo orbit of  $\mathcal{F}$  with  $\theta \in (\theta_0, 1)$ , i.e. there exists  $\sigma = \{\lambda_0, \lambda_1, \dots\} \in \Lambda_{+}^{\mathbb{Z}}$  such that

$$d(f_{\lambda_n}(x_n), x_{n+1}) \xrightarrow{\theta} 0, \quad n \rightarrow \infty.$$

So there exists  $L > 0$  such that  $d(f_{\lambda_n}(x_n), x_{n+1}) \leq L\theta^n$  for  $n \geq 0$ . Consider the following orbit:

$$y_0 = x_0, \quad y_{n+1} = f_{\lambda_n}(y_n), \quad n \geq 0$$

Hence we have

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, f_{\lambda_{n-1}}(x_{n-1})) + d(f_{\lambda_{n-1}}(x_{n-1}), f_{\lambda_{n-1}}(y_{n-1})) \\ &\leq L\theta^{n-1} + \alpha d(x_{n-1}, y_{n-1}) \\ &\leq \dots \\ &\leq L(1 + \alpha\theta^{-1} + \alpha^2\theta^{-2} + \dots + \alpha^{n-1}\theta^{-n+1})\theta^{n-1} \\ &\leq \left(\frac{L}{\theta-\alpha}\right)\theta^n. \end{aligned}$$

Thus  $\mathcal{F}$  has the exponential limit shadowing property.  $\square$

**Corollary 3.3.** *Suppose  $\Lambda$  is a finite set and for every  $\lambda \in \Lambda$ ,  $f_{\lambda} : \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable function. Assume that  $p \in \mathbb{R}$  is an attractor fixed point ( $f_{\lambda}(p) = p$  and  $|f'_{\lambda}(p)| < 1$ , for all  $\lambda \in \Lambda$ ). There exist  $W \subset \mathbb{R}$  containing  $p$  such that  $f_{\lambda}(W) \subset W$ , for all  $\lambda \in \Lambda$  and  $\mathcal{F} = \{W; f_{\lambda} | \lambda \in \Lambda\}$  has the (exponential) limit shadowing property.*

*Proof.* By Proposition 4.4 of [6], for each  $\lambda \in \Lambda$  there is an open interval  $W_{\lambda}$  around  $p$  such that if  $x \in W_{\lambda}$ , then  $f_{\lambda}^n(x) \in W_{\lambda}$ , for all  $n > 0$  and  $\lim_{n \rightarrow \infty} f_{\lambda}^n(x) = p$ . Hence we can find an interval  $W \subset \bigcap_{\lambda \in \Lambda} W_{\lambda}$  and  $\varepsilon > 0$  such that if  $x \in W$ , then  $|f'_{\lambda}(x)| < 1 - \varepsilon$ , for all  $\lambda \in \Lambda$ . This implies that for all  $x, y \in W$ , we have  $\frac{|f_{\lambda}(x) - f_{\lambda}(y)|}{|x - y|} < 1 - \varepsilon$ . So  $\mathcal{F} = \{W; f_{\lambda} | \lambda \in \Lambda\}$  is a uniformly contracting IFS and has the (exponential) limit shadowing property.  $\square$

**Theorem 3.4.** *If a parameterized IFS  $\mathcal{F} = \{X : f_{\lambda} | \lambda \in \Lambda\}$  is uniformly expanding and each  $f_{\lambda}$  is surjective, then*

- (a)  $\mathcal{F}$  has the exponential limit shadowing property on  $\mathbb{Z}_{+}$ .
- (b)  $\mathcal{F}$  has the limit shadowing property on  $\mathbb{Z}_{+}$ .

*Proof.* Assume that the IFS  $\mathcal{F}$  is uniformly expanding with the expanding ratio  $\beta$ .

(a) Let  $\theta \in (0, 1)$  and let  $\{x_n\}_{n \geq 0}$  be an exponentially asymptotic pseudo orbit of  $\mathcal{F}$  with exponent  $\theta$ , i.e. there exists  $\sigma = \{\lambda_0, \lambda_1, \dots\} \in \Lambda^{\mathbb{Z}_{+}}$  such that

$$d(f_{\lambda_n}(x_n), x_{n+1}) \xrightarrow{\theta} 0, \quad n \rightarrow \infty. \quad (1)$$

We define the sequence  $\{y_n\}_{n \geq 0}$  in  $X$  as follows:

$$y_0 = x_0, \quad y_n = f_{\lambda_0}^{-1} \circ f_{\lambda_1}^{-1} \circ \dots \circ f_{\lambda_{n-1}}^{-1}(x_n), \quad n \geq 1.$$

**Claim:** The sequence  $\{y_n\}_{n \geq 0}$  is a Cauchy sequence.

Therefore the sequence  $\{y_n\}_{n \geq 0}$  is convergent to some point  $z \in X$ . Now we consider the following orbit:

$$z_0 = z, \quad z_{n+1} = f_{\lambda_n}(z_n); \quad n \geq 1.$$

For each  $n \geq 1$  and  $0 \leq k \leq n - 1$ , we define

$$y_n^{(k)} = f_{\lambda_k} \circ f_{\lambda_{k-1}} \circ \dots \circ f_{\lambda_0}(y_n).$$

So, for each  $k \geq 0$  we have  $\lim_{n \rightarrow \infty} y_n^{(k)} = z_{k+1}$ , hence we obtain

$$\begin{aligned} d(x_n, z_n) &= d(f_{\lambda_{n-1}} \circ f_{\lambda_{n-2}} \circ \cdots \circ f_{\lambda_0}(y_n), f_{\lambda_{n-1}} \circ f_{\lambda_{n-2}} \circ \cdots \circ f_{\lambda_0}(z_0)) \\ &\leq \beta^n d(y_n, z_0) \\ &\leq \frac{1}{\beta-1} \theta^n. \end{aligned}$$

Thus the IFS  $\mathcal{F}$  has the exponential limit shadowing property.

**Proof of Claim:** Given  $\lambda \in \Lambda$ , we consider the function

$$\rho_\lambda(x, y) = \begin{cases} \frac{d(f_\lambda(x), f_\lambda(y))}{d(x, y)} & x \neq y \\ \beta & x = y, \end{cases}$$

we have

$$d(x, y) = \frac{d(f_\lambda(x), f_\lambda(y))}{\rho_\lambda(x, y)} \quad x, y \in X, \quad \lambda \in \Lambda.$$

Given  $n \geq 1$  and  $0 \leq k \leq n-1$ , we denote

$$y_n^{(k)} = f_{\lambda_k} \circ f_{\lambda_{k-1}} \circ \cdots \circ f_{\lambda_0}(y_n)$$

For every  $n \geq 1$  and  $p \geq 1$ , we have

$$\begin{aligned} d(y_n, y_{n+p}) &= \frac{d(f_{\lambda_0}(y_n), f_{\lambda_0}(y_{n+p}))}{\rho_{\lambda_0}(y_n, y_{n+p})} \\ &= \frac{d(y_n^{(0)}, y_{n+p}^{(0)})}{\rho_{\lambda_0}(y_n, y_{n+p})} \\ &= \frac{d(y_n^{(1)}, y_{n+p}^{(1)})}{\rho_{\lambda_0}(y_n, y_{n+p}) \rho_{\lambda_1}(y_n^{(0)}, y_{n+p}^{(0)})} \\ &= \dots \\ &= \frac{d(x_n, y_{n+p}^{(n-1)})}{\rho_{\lambda_0}(y_n, y_{n+p}) \prod_{i=1}^{n-1} \rho_{\lambda_i}(y_n^{(i-1)}, y_{n+p}^{(i-1)})} \end{aligned}$$

We show that the following inequality holds uniformly with respect to  $n \geq 1$ :

$$d(x_n, y_{n+p}^{(n-1)}) \leq \theta^n \sum_{k=1}^p \beta^{-k}, \quad p \geq 1. \quad (2)$$

We prove this inequality by induction on  $p \geq 1$ .

For  $p = 1$ , the inequality follows from (1) and the definition of  $y_{n+1}$ .

$$\begin{aligned} d(x_n, y_{n+1}^{(n-1)}) &= \frac{d(f_{\lambda_n}(x_n), f_{\lambda_n}(y_{n+1}^{(n-1)}))}{\rho_{\lambda_n}(x_n, y_{n+1}^{(n-1)})} \\ &= \frac{d(f_{\lambda_n}(x_n), x_{n+1})}{\rho_{\lambda_n}(x_n, y_{n+1}^{(n-1)})} \\ &\leq \frac{\theta^n}{\beta}. \end{aligned}$$

Assume that the inequality (2) holds for some  $p = q \geq 1$ . We prove (2) for  $p = q + 1$ .

$$\begin{aligned} d(x_n, y_{n+q+1}^{(n-1)}) &= \frac{d(f_{\lambda_n}(x_n), f_{\lambda_n}(y_{n+q+1}^{(n-1)}))}{\rho_{\lambda_n}(x_n, y_{n+q+1}^{(n-1)})} \\ &= \frac{d(f_{\lambda_n}(x_n), y_{n+q+1}^{(n)})}{\rho_{\lambda_n}(x_n, y_{n+q+1}^{(n-1)})} \\ &\leq \frac{d(f_{\lambda_n}(x_n), x_{n+1}) + d(x_{n+1}, y_{n+q+1}^{(n)})}{\rho_{\lambda_n}(x_n, y_{n+q+1}^{(n-1)})} \\ &\leq \theta^n \sum_{k=1}^{q+1} \beta^{-k}. \end{aligned}$$

This proves (2).

Now for  $n \geq 1$  and  $p \geq 1$ , (2) gives us the following:

$$\begin{aligned} d(y_n, y_{n+p}) &\leq \frac{\theta^n \sum_{k=1}^p \beta^{-k}}{\rho_{\lambda_0}(y_n, y_{n+p}) \prod_{i=1}^{n-1} \rho_{\lambda_i}(y_n^{(i-1)}, y_{n+p}^{(i-1)})} \\ &\leq \frac{\theta^n}{(\beta-1) \rho_{\lambda_0}(y_n, y_{n+p}) \prod_{i=1}^{n-1} \rho_{\lambda_i}(y_n^{(i-1)}, y_{n+p}^{(i-1)})} \\ &\leq \frac{1}{\beta-1} \left(\frac{\theta}{\beta}\right)^n. \end{aligned}$$

The last inequality proves the claim.

(b) Let  $\{x_n\}_{n \geq 0}$  be an asymptotic pseudo orbit of  $\mathcal{F}$ . We define the sequence  $\{y_n\}_{n \geq 0}$  in  $X$  as follows:

$$y_0 = x_0, \quad y_n = f_{\lambda_0}^{-1} \circ f_{\lambda_1}^{-1} \circ \cdots \circ f_{\lambda_{n-1}}^{-1}(x_n), \quad n \geq 1.$$

Since  $\lim_{n \rightarrow \infty} d(f_{\lambda_n}(x_n), x_{n+1}) = 0$ , similar to Theorem 2.2 of [10] we can prove that  $\{y_n\}_{n > 0}$  is a convergent sequence. Let  $z$  denote its limit and consider the following sequence:

$$z_0 = z, \quad z_{n+1} = f_{\lambda_n}(z_n); \quad n \geq 1.$$

Suppose  $\epsilon$  is an arbitrary positive number. Again by use of the proof of Theorem 2.2 in [10] and the fact that there is  $n_1 > 0$  with  $d(f_{\lambda_n}(x_n), x_{n+1}) < (\beta-1)\epsilon$ , for all  $n > n_1$ , we can find  $N(\epsilon) > 0$  such that  $d(z_n, x_n) < \epsilon$ , for all  $n > N(\epsilon)$ . Then  $\lim_{n \rightarrow \infty} d(z_n, x_n) = 0$ , because  $\epsilon$  is an arbitrary positive number.  $\square$

Given complete metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , consider the product set  $X \times Y$  endowed with the metric

$$D((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

Let  $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$  and  $\mathcal{G} = \{Y; g_\gamma | \gamma \in \Gamma\}$  be two parameterized IFS. The IFS  $\mathcal{H} = \{X \times Y; \Phi_{\lambda, \gamma} | \lambda \in \Lambda, \gamma \in \Gamma\}$ , defined by  $\Phi_{\lambda, \gamma}(x, y) := (f_\lambda(x), g_\gamma(y))$  is called the product of the two IFS  $\mathcal{F}$  and  $\mathcal{G}$ . The proof of the following theorem is straightforward and omitted.

**Lemma 3.5.** *Let  $(X_i, d_i)$ ,  $i = 1, 2$ , be metric spaces and  $g_\lambda : X_1 \rightarrow X_1$  and  $h_\lambda : X_2 \rightarrow X_2$ ,  $\lambda \in \Lambda$ , be homeomorphisms. Let  $d$  be a metric on the product space  $X = X_1 \times X_2$  compatible with the uniform product structure. Let  $f_\lambda : X \rightarrow X$  be defined by  $f_\lambda(a, b) = (g_\lambda(a), h_\lambda(b))$ ,  $\lambda \in \Lambda$ . Then both  $\mathcal{G} = \{X_1; g_\lambda | \lambda \in \Lambda\}$  and  $\mathcal{H} = \{X_2 : h_\lambda | \lambda \in \Lambda\}$  have the shadowing property iff  $\mathcal{F} = \{X : f_\lambda | \lambda \in \Lambda\}$  does.*

*The same result holds for the limit and exponential shadowing properties.*

We recall a well-known result [12].

**Lemma 3.6.** *Let  $f : X \rightarrow X$  be a linear homeomorphism on a Banach space. Then  $f$  is hyperbolic if and only if the following holds. There exists Banach subspaces  $X_s, X_u \subset X$ , called stable and unstable subspaces, respectively, and a norm on  $X$  compatible with the original Banach structure such that*

$$X = X_s \oplus X_u, \quad f(X_s) = X_s, \quad f(X_u) = X_u, \quad \|f|_{X_s}\| < 1, \quad \|f^{-1}|_{X_u}\| < 1.$$

Now we extend the classical shadowing lemma in linear case for parameterized IFS.

**Theorem 3.7.** *Suppose that  $X$  is a Banach space,  $\Lambda$  is finite and  $\{f_\lambda : X \rightarrow X\}_{\lambda \in \Lambda}$  are hyperbolic linear maps with the same stable and unstable subspaces. Then the IFS  $\mathcal{F} = \{X : f_\lambda | \lambda \in \Lambda\}$  has the shadowing, limit shadowing and exponential limit shadowing properties.*

*Proof.* Take  $\mathcal{G} = \{X_s : f_\lambda | \lambda \in \Lambda\}$  and  $\mathcal{H} = \{X_u : f_\lambda | \lambda \in \Lambda\}$ . It is clear that  $\mathcal{G}$  is uniformly contracting and  $\mathcal{H}$  is uniformly expanding. So, Theorems 3.2 and 3.4 imply that  $\mathcal{G}$  and  $\mathcal{H}$  have the shadowing, limit shadowing and exponential limit shadowing properties. Then by Lemma 3.5,  $\mathcal{F}$  has the shadowing, limit shadowing and exponential limit shadowing properties.  $\square$

#### 4. Examples

In this section we show that the limit shadowing and exponential limit shadowing properties are different from other kinds of shadowing property for parameterized *IFS*. Let us recall some notions related to symbolic dynamics. Let

$$\Sigma_2 = \{(s_0s_1s_2\dots) | s_i = 0 \text{ or } 1\}.$$

We will refer to the elements of  $\Sigma_2$  as points in  $\Sigma_2$ . Let  $s = s_0s_1s_2\dots$  and  $t = t_0t_1t_2\dots$  be points in  $\Sigma_2$ . We denote the distance between  $s$  and  $t$  by  $d(s, t)$ , and define it by

$$d(s, t) = \begin{cases} 0, & s = t \\ \frac{1}{2^{k-1}}, & k = \min\{i; s_i \neq t_i\} \end{cases}$$

**Example 4.1.** Let  $f_0, f_1 : \Sigma_2 \rightarrow \Sigma_2$  be defined by  $f_0(s_0s_1s_2\dots) = 0s_0s_1s_2\dots$  and  $f_1(s_0s_1s_2\dots) = 1s_0s_1s_2\dots$  for each  $s = s_0s_1s_2\dots \in \Sigma_2$ .

It is clear that  $\mathcal{F} = \{\Sigma_2; f_0, f_1\}$  is uniformly contracting and, by Theorem 3.2 has the exponential limit shadowing property.

Note that the *IFS*  $\mathcal{F}^k$  also have the exponential limit shadowing property, for all  $k > 1$ .

For example, if  $k = 2$  then  $\mathcal{F}^2 = \{\Sigma_2; g_0, g_1, g_2, g_3\}$ , where

$$\begin{aligned} g_0(s_0s_1s_2\dots) &= f_0 \circ f_0(s_0s_1s_2\dots) = 00s_0s_1s_2\dots, \\ g_1(s_0s_1s_2\dots) &= f_1 \circ f_0(s_0s_1s_2\dots) = 10s_0s_1s_2\dots, \\ g_2(s_0s_1s_2\dots) &= f_0 \circ f_1(s_0s_1s_2\dots) = 01s_0s_1s_2\dots, \\ g_3(s_0s_1s_2\dots) &= f_1 \circ f_1(s_0s_1s_2\dots) = 11s_0s_1s_2\dots, \end{aligned}$$

for each  $s = s_0s_1s_2\dots \in \Sigma_2$ . Clearly  $\mathcal{F}^2$  is uniformly contracting and has the exponential limit shadowing property.

Consider the 3-dimensional torus

$$\mathbb{T}^3 = \mathbb{R}^3 / \mathbb{Z}^3$$

equipped with the Euclidean metric,  $d$ . In the following example we give an *IFS* on  $\mathbb{T}^3$  which has the shadowing, limit shadowing and exponential limit shadowing properties.

**Example 4.2.** Let  $f_1$  and  $f_2$  be the isomorphisms of  $\mathbb{R}^3$  which are represented, with respect to the standard basis, by the matrices:

$$A_1 = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, A_2 = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}.$$

It is clear that  $f_1$  and  $f_2$  both have the  $xy$ -plane and  $z$ -axis as the same stable and unstable subspaces, respectively. So, by Theorem 3.7 the *IFS*

$\mathcal{F} = \{\mathbb{R}^3; f_1, f_2\}$  has the shadowing, limit shadowing and exponential limit shadowing properties. Let  $\tilde{f}_1, \tilde{f}_2 : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  be Anosov diffeomorphisms of the torus  $\mathbb{T}^3$  induced by  $f_1$  and  $f_2$ . Let  $\epsilon$  be an arbitrary positive number. There exists  $\delta > 0$  such that for every points  $a, b \in \mathbb{T}^3$

with  $d(a, b) < \delta$ , we can find corresponding points  $x, y \in \mathbb{R}^3$  such that  $\|x - y\| < \epsilon$ . Similarly, for every  $\epsilon' > 0$  there exist  $\delta' > 0$  such that for every points  $x, y \in \mathbb{R}^3$ , if  $\|x - y\| < \delta'$  then  $d(\tilde{x}, \tilde{y}) < \epsilon'$ , where  $\tilde{x} = x/\mathbb{Z}^3$  and  $\tilde{y} = y/\mathbb{Z}^3$ . Then the IFS  $\mathcal{G} = \{\mathbb{T}^3; \tilde{f}_1, \tilde{f}_2\}$  has the shadowing, limit shadowing and exponential limit shadowing properties.

The following example shows that the limit shadowing property does not imply the exponential limit shadowing property.

**Example 4.3.** Consider the unit circle  $S^1$  with the coordinate  $x \in [0, 1)$ . Let  $\phi$  be a dynamical system on  $S^1$  generated by the mapping  $f : [0, 1) \rightarrow [0, 1)$  defined by  $f(x) = x - x^2(x - \frac{1}{2})(x - 1)^2$ . Ahmadi and Molaei prove that  $\phi$  has the limit shadowing property, but does not have the exponential limit shadowing property [1]. Also, suppose  $f_1 : [\frac{1}{2}, 1] \rightarrow [\frac{1}{2}, 1]$  is a map such that  $f_1(\frac{1}{2}) = \frac{1}{2}$ ,  $f_1(1) = 1$  and  $x < f_1(x) < f_1^2(x) < f(x)$  for all  $x \in (\frac{1}{2}, 1)$ . Let  $\psi_1, \psi_2$  be dynamical systems on  $S^1$  generated by the maps  $g_1, g_2 : [0, 1) \rightarrow [0, 1)$  defined by

$$g_1(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq \frac{1}{2}, \\ f_1(x) & \text{if } \frac{1}{2} \leq x < 1 \end{cases} \quad g_2(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq \frac{1}{2}, \\ f_1^2(x) & \text{if } \frac{1}{2} \leq x < 1 \end{cases}$$

Now, we prove that the IFS  $\mathcal{F} = \{S^1; \psi_1, \psi_2\}$  does not have the exponential limit shadowing property but has the limit shadowing property.

Suppose that  $\mathcal{F}$  has the exponential limit shadowing property with constants  $L > 0$  and  $\mu \in (0, 1)$ . Consider the sequence  $\{x_k\}_{k \geq n_0}$  in the previous example. Then there exist a point  $p \in S^1$ , a sequence  $\sigma = \{\lambda_0, \lambda_1, \dots\} \in \{1, 2\}^{\mathbb{Z}^+}$  and a natural number  $n_1$  such that

$$|\mathcal{F}\sigma_k(p) - x_k| \leq L\mu^{\frac{k}{2}} \text{ for } k \geq n_1.$$

Case 1.  $p \in [0, \frac{1}{2}]$ . It follows that  $\mathcal{F}\sigma_k(p) = \phi^n(p)$  for all  $n \geq 0$ . So  $d(\phi_1(p), x_k) \xrightarrow{\mu} 0$ , contradiction.

Case 2.  $p \in (\frac{1}{2}, 1)$ . Since  $x < f_1(x) < f_1^2(x) < f(x)$  for all  $x \in (\frac{1}{2}, 1)$  and  $\{x_k\}_{k \geq n_0} \subset (0, \frac{1}{2})$ , we have  $d(\phi^n(p), x_n) \leq d(\mathcal{F}\sigma_n(p), x_n)$ .

Then  $d(\phi(p), x_k) \xrightarrow{\mu} 0$ , a contradiction. So  $\mathcal{F}$  does not have the exponential limit shadowing property.

Now, we show that  $\mathcal{F}$  has the limit shadowing property.

Let  $\{x_k\}_{k \geq 0}$  be an asymptotic pseudo orbit for  $\mathcal{F}$ . So, there exists  $\sigma = \{\lambda_0, \lambda_1, \lambda_2, \dots\} \in \{1, 2\}^{\mathbb{Z}^+}$  such that  $\lim_{k \rightarrow \infty} d(\psi_{\lambda_k}(x_k), x_{k+1}) = 0$ .

Take  $J = \{i | \lambda_i = 2\}$  and

$$z_i = \begin{cases} x_i, \psi_1(x_i) & \text{if } \frac{1}{2} \leq x_i \leq 0 \text{ and } i \in J \\ x_i & \text{otherwise} \end{cases}$$

Consider the sequence  $z = z_0, z_1, z_2, \dots$ . For example, if  $0 \notin J$  and  $1 \in J$ , then  $z = x_0, x_1, \psi_1(x_1), x_2, \dots$ . Since  $d(\psi_1(x_i), \psi_1(x_i)) = 0$  and

$d(\psi_1^2(x_i), x_{i+1}) = d(f_2(x_i), x_{i+1})$ , for all  $i \in J$ , then  $z$  is an asymptotic pseudo orbit for original discrete dynamical system  $(S^1, \psi_1)$ . Theorem 3.1.2 of [15] implies that the  $(S^1, \psi_1)$  has the limit shadowing property. Therefore we can find  $p \in S^1$  such that  $\lim_{n \rightarrow \infty} d(\psi_1^n(p), y_n) = 0$ .

Take  $\gamma = \{\gamma_0, \gamma_1, \gamma_2, \dots\} \in \{1, 2\}^{\mathbb{Z}^+}$  such that

$$\gamma_i = \begin{cases} 1 & \text{if } i - 1 \in J \\ 2 & \text{otherwise} \end{cases}$$

Let  $p_0 = p$  and  $p_{i+1} = \psi_{\gamma_i}(p_i)$ , for all  $i \geq 1$ . So  $\{p_i\}_{i \geq 0}$  is an orbit of  $\mathcal{F}$  and  $\{d(p_n, x_n)\}_{n \geq 0}$  is a subsequence of  $\{d(\psi_1^n(p), y_n)\}_{n \geq 0}$ , So  $\lim_{n \rightarrow \infty} d(p_n, x_n) = 0$ .

## 5. CONCLUSION

In this paper, we generalized the notions of limit shadowing and exponential limit shadowing for parameterized iterated function systems which has been originally introduced for discrete dynamical systems. Then, we proved some results which will be needed for future studies in this connection. These results are generalizations of the previous work done by several authors [1, 15, 10]. We bring this paper to end by posing the following questions.

- 1– Does the IFS  $\mathcal{F}$  in Example 4.1 have the limit shadowing property?
- 2– Does a hyperbolic IFS (see [5], Definition 7.1) on a compact metric space have the limit shadowing property?
- 3– Does a hyperbolic IFS on a compact metric space have the exponential limit shadowing property?

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