

A PARABOLIC p -BIHARMONIC EQUATION WITH LOGARITHMIC NONLINEARITY

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We study a parabolic p -biharmonic equation with the logarithmic nonlinearity. Based on the difference and variation methods, we prove the existence of weak solutions for the initial boundary problem. We also discuss the large time behavior and the propagation of perturbations of solutions.

Keywords: p -biharmonic equation, Logarithmic nonlinearity, Existence, Finite speed of propagation.

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1. Introduction

In this paper, we study the parabolic p -biharmonic equation with the logarithmic nonlinearity

$$\frac{\partial u}{\partial t} + \Delta(|\Delta u|^{p-2}\Delta u) = \lambda|u|^{q-2}u \log(|u|), \quad x \in \Omega, t > 0, \quad (1)$$

where $\lambda > 0$, $p > q > \frac{p}{2} + 1$, $p > \frac{n}{2}$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with the smooth boundary.

The equation (1) is supplemented with the natural boundary value conditions

$$u = \Delta u = 0, \quad x \in \partial\Omega, t > 0, \quad (2)$$

and the initial value condition

$$u(x, 0) = u_0(x), \quad x \in \Omega. \quad (3)$$

In the past years, the p -biharmonic equation has been intensively studied. Kefi and Rădulescu [11] investigated the $p(x)$ -biharmonic equation

$$\Delta(|\Delta u|^{p(x)-2}\Delta u) + a(x)|u|^{p(x)-2}u = \lambda(V_1(x)|u|^{q(x)-2}u - V_2(x)|u|^{\alpha(x)-2}u).$$

They proved the existence of at least one nontrivial weak solution. Liu, Chen and Almualemi [13] studied

$$\Delta(|\Delta u|^{p-2}\Delta u) + V(x)|u|^{p-2}u = f(x, u).$$

They obtained the existence of the Nehari type ground state solutions. Pavel Drábek and Mitsuharu Ôtani [6] considered the following equation

$$\Delta(|\Delta u|^{p-2}\Delta u) = \lambda|u|^{p-2}u \quad (4)$$

and proved that (4) and (2) had a principal positive eigenvalue λ_1 which is simple and isolated. However, only a few papers are devoted to the parabolic p -biharmonic equation. Hao and Zhou [7] studied the nonlocal p -biharmonic parabolic equation

$$\frac{\partial u}{\partial t} + \Delta(|\Delta u|^{p-2}\Delta u) = |u|^q - \int_{\Omega} |u|^q dx.$$

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They proved the blow-up, extinction and non-extinction of the solutions to the equation. The relevant equations have also been studied in [1, 12, 16].

In this paper, we study the parabolic p -biharmonic equation with the logarithmic nonlinearity. The second order parabolic equation with the logarithmic nonlinearity is diffusely studied. Chen considered the semilinear heat equation with the logarithmic nonlinearity [3] and the semilinear pseudo-parabolic equations with the logarithmic nonlinearity [4]. Ji, Yin and Cao [10] established the existence of positive periodic solutions and discussed the instability of such solutions for the semilinear pseudo-parabolic equation with the logarithmic source. Nhana and Truong [14] discussed the equation

$$u_t - \Delta u_t - \Delta_p u = |u|^{p-2} u \log(|u|),$$

where Δ_p is the p -Laplacian. He, Gao and Wang [9] study the pseudo-parabolic p -Laplacian equation

$$u_t - \Delta u_t - \Delta_p u = |u|^{q-2} u \log(|u|).$$

In this paper, we first study the existence of weak solutions for the problem (1)-(3). Now, we introduce weak solutions in the sense as following

Definition 1.1. *A function u is said to be a weak solution of the problem (1)–(3), if the following conditions are satisfied:*

- 1) $u \in L^\infty(0, T; W_0^{2,p}(\Omega)) \cap C(0, T; L^2(\Omega))$,
 $\frac{\partial u}{\partial t} \in L^\infty(0, T; W^{-2,p'}(\Omega))$, where p' is the conjugate exponent of p ;
- 2) For any $\varphi \in C_0^\infty(Q_T)$, the following integral equality holds:

$$-\iint_{Q_T} u \frac{\partial \varphi}{\partial t} dx dt + \iint_{Q_T} |\Delta u|^{p-2} \Delta u \Delta \varphi dx dt - \lambda \iint_{Q_T} |u|^{q-2} u \log(|u|) \varphi dx dt = 0;$$

- 3) $u(x, 0) = u_0(x)$, in $L^2(\Omega)$

This paper is arranged as follows. We first discuss the existence of weak solutions by employing the variation methods in Section 2. Based on the energy techniques, Hardy inequality and Poincaré inequality, we also obtain the large time behavior and the finite speed of propagation of perturbations subsequently.

2. Existence

Let k be a nonnegative integer, $p > 1$. The family of functions

$$\{u; D^\alpha u \in L^p(\Omega), \text{ for any } \alpha \text{ with } |\alpha| \leq k\}$$

with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\int_{\Omega} \sum_{|\alpha| \leq k} |D^\alpha u|^p dx \right)^{1/p}$$

is called a Sobolev space, denoted by $W^{k,p}(\Omega)$. $W_0^{k,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

In this section, we are going to study the existence of weak solutions. Next, we state the main result.

Theorem 2.1. *Assume that $u_0 \in W_0^{2,p}(\Omega)$, $p > q > \frac{p}{2} + 1$, $p > \frac{n}{2}$. Then the problem (1)-(3) admits at least one weak solution.*

To prove the Theorem 2.1, we first consider the following elliptic problem

$$\frac{1}{h} (u_{k+1} - u_k) + \Delta (|\Delta u_{k+1}|^{p-2} \Delta u_{k+1}) - \lambda |u_{k+1}|^{q-2} u_{k+1} \log(|u_{k+1}|) = 0, \quad (5)$$

$$u_{k+1}\Big|_{\partial\Omega} = \Delta u_{k+1}\Big|_{\partial\Omega} = 0, \quad k = 0, 1, \dots, N-1, \quad (6)$$

where $h = \frac{T}{N}$, u_0 is the initial value.

Now, we state and prove a preliminary result.

Lemma 2.1. *For any fixed k , if $u_k \in L^2(\Omega)$, the problem (5)-(6) admits a weak solution $u_{k+1} \in W_0^{2,p}(\Omega)$, such that for any $\varphi \in C_0^\infty(\Omega)$, there holds*

$$\begin{aligned} \frac{1}{h} \int_{\Omega} (u_{k+1} - u_k) \varphi dx + \int_{\Omega} |\Delta u_{k+1}|^{p-2} \Delta u_{k+1} \Delta \varphi dx \\ - \lambda \int_{\Omega} |u_{k+1}|^{q-2} u_{k+1} \log(|u_{k+1}|) \varphi dx = 0. \end{aligned} \quad (7)$$

Proof. We introduce the following functionals on the space $W_0^{2,p}(\Omega)$

$$\begin{aligned} F[u] &= \frac{1}{p} \int_{\Omega} |\Delta u|^p dx, \\ G[u] &= \frac{1}{2} \int_{\Omega} |u|^2 dx, \\ E[u] &= -\frac{1}{q} \int_{\Omega} |u|^q \log(|u|) dx + \frac{1}{q^2} \int_{\Omega} |u|^q dx, \\ H[u] &= F[u] + \frac{1}{h} G[u] + \lambda E[u] - \int_{\Omega} f u dx, \end{aligned}$$

where $f \in L^2(\Omega)$ is a known function. By the Sobolev imbedding theorem, the Young inequality, the Poincaré inequality and the fact $u^{-\mu} \log u \leq (e\mu)^{-1}$, for $u \geq 1$, $\mu > 0$, we see that for $C_1 > 0$,

$$\begin{aligned} H[u] &= \frac{1}{p} \int_{\Omega} |\Delta u|^p dx + \frac{1}{2h} \int_{\Omega} |u|^2 dx - \frac{\lambda}{q} \int_{\Omega} |u|^q \log(|u|) dx + \frac{\lambda}{q^2} \int_{\Omega} |u|^q dx - \int_{\Omega} f u dx \\ &\geq \frac{1}{p} \int_{\Omega} |\Delta u|^p dx + \frac{1}{2h} \int_{\Omega} |u|^2 dx - \frac{\lambda}{q} \int_{\{x \in \Omega; |u(x)| \geq 1\}} |u|^q \log(|u|) dx \\ &\quad + \frac{\lambda}{q^2} \int_{\Omega} |u|^q dx - \int_{\Omega} f u dx \\ &\geq \frac{1}{p} \int_{\Omega} |\Delta u|^p dx + \frac{1}{2h} \int_{\Omega} |u|^2 dx - (e\mu)^{-1} \frac{\lambda}{q} \int_{\Omega} |u|^{q+\mu} dx + \frac{\lambda}{q^2} \int_{\Omega} |u|^q dx - \int_{\Omega} f u dx \\ &\geq \frac{1}{p} \int_{\Omega} |\Delta u|^p dx + \frac{1}{2h} \int_{\Omega} |u|^2 dx - \varepsilon \int_{\Omega} |\nabla u|^p dx + \frac{\lambda}{q^2} \int_{\Omega} |u|^q dx - \int_{\Omega} f u dx - C \\ &\geq \frac{1}{2p} \int_{\Omega} |\Delta u|^p dx - C_1 \int_{\Omega} |f|^2 dx - C. \end{aligned}$$

Recalling $u|_{\partial\Omega} = 0$ and the L^p theory for elliptic equation([5]), we know

$$\|u\|_{W^{2,p}} \leq C \|\Delta u\|_{L^p}.$$

Hence $H[u] \rightarrow +\infty$, if $\|u\|_{W^{2,p}} \rightarrow +\infty$, which shows that $H[u]$ satisfies the coercive condition. Furthermore, $H[u]$ is weakly lower semicontinuous on $W_0^{2,p}(\Omega)$. So, it follows from the theory in [2] that there exists $u_* \in W_0^{2,p}(\Omega)$, such that

$$H[u_*] = \inf H[u],$$

and u_* is the weak solution of the Euler equation corresponding to $H[u]$, namely

$$\frac{1}{h} u + \Delta(|\Delta u|^{p-2} \Delta u) - \lambda |u|^{q-2} u \log(|u|) = f.$$

Choosing $f = \frac{1}{h}u_k$, we obtain the desired conclusion. The proof is complete. \square

Now, we construct an approximate solution u^h of the problem (1)-(3) by defining

$$u^h(x, t) = u_k(x), \quad kh < t \leq (k+1)h, \quad k = 0, 1, \dots, N-1,$$

$$u^h(x, 0) = u_0(x).$$

The desired solution of the problem (1)-(3) will be obtained as the limit of some subsequence of $\{u^h\}$. To this purpose, we need some uniform estimates on u^h .

Lemma 2.2. *If u_k be a weak solution of the problem (5)-(6), the following estimates hold*

$$h \sum_{k=1}^N \int_{\Omega} |\Delta u_k|^p dx \leq C, \quad (8)$$

$$\sup_{0 < t < T} \int_{\Omega} |\Delta u^h(x, t)|^p dx \leq C, \quad (9)$$

where C is a constant independent of h, k .

Proof. i) Taking $\varphi = u_{k+1}$ in the integral equality (7), we deduce

$$\begin{aligned} & \frac{1}{h} \int_{\Omega} |u_{k+1}|^2 dx + \int_{\Omega} |\Delta u_{k+1}|^p dx \\ &= \frac{1}{h} \int_{\Omega} u_{k+1} u_k dx + \lambda \int_{\Omega} |u_{k+1}|^q \log(|u_{k+1}|) dx. \end{aligned} \quad (10)$$

Using the Young inequality ($ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$), we see that

$$\begin{aligned} & \frac{1}{h} \int_{\Omega} |u_{k+1}|^2 dx + \int_{\Omega} |\Delta u_{k+1}|^p dx \\ & \leq \frac{1}{2h} \int_{\Omega} |u_k|^2 dx + \frac{1}{2h} \int_{\Omega} |u_{k+1}|^2 dx + \lambda \int_{\Omega} |u_{k+1}|^q \log(|u_{k+1}|) dx, \end{aligned}$$

that is

$$\begin{aligned} & \frac{1}{2h} \int_{\Omega} |u_{k+1}|^2 dx + \int_{\Omega} |\Delta u_{k+1}|^p dx - \lambda \int_{\Omega} |u_{k+1}|^q \log(|u_{k+1}|) dx \\ & \leq \frac{1}{2h} \int_{\Omega} |u_k|^2 dx. \end{aligned} \quad (11)$$

Summing up these inequalities for k from 0 to $N-1$, we have

$$h \sum_{k=1}^N \int_{\Omega} |\Delta u_k|^p dx - h\lambda \sum_{k=1}^N \int_{\Omega} |u_k|^q \log(|u_k|) dx \leq \frac{1}{2} \int_{\Omega} |u_0|^2 dx.$$

On the other hand, by (10), we derive

$$\begin{aligned} & \frac{1}{h} \int_{\Omega} |u_{k+1}|^2 dx + \int_{\Omega} |\Delta u_{k+1}|^p dx \\ & \leq \frac{1}{2h} \int_{\Omega} |u_k|^2 dx + \frac{1}{2h} \int_{\Omega} |u_{k+1}|^2 dx + \int_{\{x \in \Omega; |u_{k+1}(x)| \geq 1\}} |u_{k+1}|^q \log(|u_{k+1}|) dx. \end{aligned}$$

The fact $u^{-\alpha} \log u \leq (e\alpha)^{-1}$, for $u \geq 1, \alpha > 0$ yields

$$\begin{aligned} & \frac{1}{2h} \int_{\Omega} |u_{k+1}|^2 dx + \int_{\Omega} |\Delta u_{k+1}|^p dx \\ & \leq \frac{1}{2h} \int_{\Omega} |u_k|^2 dx + \lambda(e\alpha)^{-1} \int_{\{x \in \Omega; |u_{k+1}(x)| \geq 1\}} |u_{k+1}|^{q+\alpha} dx \\ & \leq \frac{1}{2h} \int_{\Omega} |u_k|^2 dx + \frac{1}{2} \int_{\Omega} |\Delta u_{k+1}|^p dx. \end{aligned} \quad (12)$$

Summing (12) on k from 0 to $N-1$ and canceling the same terms on both sides, that yields

$$h \sum_{k=1}^N \int_{\Omega} |\Delta u_k|^p dx \leq 2 \int_{\Omega} |u_0|^2 dx.$$

Therefore, (8) holds.

ii) Choosing $\varphi = u_{k+1} - u_k$ in the (7) and integrating by parts, we know

$$\begin{aligned} & \frac{1}{h} \int_{\Omega} |u_{k+1} - u_k|^2 dx + \int_{\Omega} |\Delta u_{k+1}|^{p-2} \Delta u_{k+1} \Delta(u_{k+1} - u_k) dx \\ & = \lambda \int_{\Omega} |u_{k+1}|^{q-2} u_{k+1} \log(|u_{k+1}|) (u_{k+1} - u_k) dx \\ & = \lambda \int_{\{x \in \Omega; |u_{k+1}(x)| \geq 1\}} |u_{k+1}|^{q-2} u_{k+1} \log(|u_{k+1}|) (u_{k+1} - u_k) dx \\ & \quad + \lambda \int_{\{x \in \Omega; |u_{k+1}(x)| < 1\}} |u_{k+1}|^{q-2} u_{k+1} \log(|u_{k+1}|) (u_{k+1} - u_k) dx \\ & \leq \lambda \int_{\{x \in \Omega; |u_{k+1}(x)| \geq 1\}} |u_{k+1}|^{q-2} u_{k+1} \log(|u_{k+1}|) (u_{k+1} - u_k) dx \\ & \quad + \lambda C(h) \int_{\{x \in \Omega; |u_{k+1}(x)| < 1\}} |u_{k+1}|^{2q-2-2\alpha} dx + \frac{1}{2h} \int_{\Omega} |u_{k+1} - u_k|^2 dx. \end{aligned}$$

Since the first term of the left hand side of the above equality is nonnegative, it follows that

$$\begin{aligned} & \int_{\Omega} |\Delta u_{k+1}|^p dx \leq \int_{\Omega} |\Delta u_{k+1}|^{p-2} \Delta u_{k+1} \Delta u_k dx \\ & \quad + \lambda \int_{\{x \in \Omega; |u_{k+1}(x)| \geq 1\}} |u_{k+1}|^q \log(|u_{k+1}|) dx \\ & \quad - \lambda \int_{\{x \in \Omega; |u_{k+1}(x)| \geq 1\}} |u_{k+1}|^{q-2} u_{k+1} \log(|u_{k+1}|) u_k dx \\ & \quad + \lambda C(h) \int_{\{x \in \Omega; |u_{k+1}(x)| < 1\}} |u_{k+1}|^{2q-2-2\alpha} dx \\ & \leq \frac{p-1}{p} \varepsilon \int_{\Omega} |\Delta u_{k+1}|^p dx + \frac{1}{p} \varepsilon^{-(p-1)} \int_{\Omega} |\Delta u_k|^p dx \\ & \quad + \lambda \int_{\{x \in \Omega; |u_{k+1}(x)| \geq 1\}} |u_{k+1}|^q \log(|u_{k+1}|) dx \\ & \quad - \lambda \int_{\{x \in \Omega; |u_{k+1}(x)| \geq 1\}} |u_{k+1}|^{q-2} u_{k+1} \log(|u_{k+1}|) u_k dx \\ & \quad + \lambda C(h) \int_{\{x \in \Omega; |u_{k+1}(x)| < 1\}} |u_{k+1}|^{2q-2-2\beta} dx \\ & = \frac{p-1}{p} \varepsilon \int_{\Omega} |\Delta u_{k+1}|^p dx + \frac{1}{p} \varepsilon^{-(p-1)} \int_{\Omega} |\Delta u_k|^p dx + I_1 + I_2 + I_3. \end{aligned}$$

Noting that

$$I_1 \leq \int_{\{x \in \Omega; |u_{k+1}(x)| \geq 1\}} |u_{k+1}|^{q+\alpha} dx \leq \int_{\Omega} |u_{k+1}|^{q+\alpha} dx,$$

taking $\alpha = p - q$, we obtain

$$I_1 \leq \int_{\Omega} |u_{k+1}|^p dx.$$

In addition, by the Hölder inequality, we derive

$$\begin{aligned} |I_2| &\leq \lambda \frac{q-1}{q} \varepsilon_1 \int_{\{x \in \Omega; |u_{k+1}(x)| \geq 1\}} |u_{k+1}|^q \log(|u_{k+1}|) dx \\ &\quad + \lambda \frac{1}{q} \varepsilon_1^{-(q-1)} \int_{\{x \in \Omega; |u_{k+1}(x)| \geq 1\}} |u_k|^q \log(|u_{k+1}|) dx \\ &= I_a + I_b. \end{aligned}$$

Similar to the proof of the I_1 ,

$$I_a \leq \lambda \frac{q-1}{q} \varepsilon_1 \int_{\Omega} |u_{k+1}|^{q+\alpha} dx \leq \lambda \frac{q-1}{q} \varepsilon_1 \int_{\Omega} |u_{k+1}|^p dx.$$

It follows from the Young inequality,

$$\begin{aligned} I_b &\leq \lambda \frac{1}{q} \varepsilon_1^{-(q-1)} \int_{\{x \in \Omega; |u_{k+1}(x)| \geq 1\}} |u_k|^q |u_{k+1}|^\alpha dx \\ &\leq \lambda \frac{1}{q} \varepsilon_1^{-(q-1)} \frac{q}{p} \varepsilon_2 \int_{\{x \in \Omega; |u_{k+1}(x)| \geq 1\}} |u_k|^p dx \\ &\quad + \lambda \frac{1}{q} \varepsilon_1^{-(q-1)} \frac{p-q}{p} \varepsilon_2^{-\frac{q}{p-q}} \int_{\{x \in \Omega; |u_{k+1}(x)| \geq 1\}} |u_{k+1}|^p dx \\ &\leq \lambda \frac{1}{q} \varepsilon_1^{-(q-1)} \frac{q}{p} \varepsilon_2 \int_{\Omega} |u_k|^p dx \\ &\quad + \lambda \frac{1}{q} \varepsilon_1^{-(q-1)} \frac{p-q}{p} \varepsilon_2^{-\frac{q}{p-q}} \int_{\Omega} |u_{k+1}|^p dx. \end{aligned}$$

For I_3 , by $q > \frac{p}{2} + 1$, we have

$$I_3 \leq \lambda C(h, \beta) \int_{\Omega} |u_{k+1}|^p dx.$$

On the other hand, using the Poincaré inequality, we know

$$\int_{\Omega} |u|^p dx \leq C \int_{\Omega} |\Delta u|^p dx.$$

Combining the above estimates, we obtain

$$\begin{aligned} &\left(1 - \frac{p-1}{p} \varepsilon - \lambda \frac{q-1}{q} \varepsilon_1 C - \lambda C - \lambda \frac{1}{q} \varepsilon^{-(q-1)} \frac{p-q}{p} \varepsilon_2^{-\frac{q}{p-q}} C - \lambda C(h, \beta) C \right) \int_{\Omega} |\Delta u_{k+1}|^p dx \\ &\leq \left(\frac{1}{p} \varepsilon^{-(p-1)} - \lambda \frac{1}{q} \varepsilon^{-(q-1)} \frac{q}{p} \varepsilon_2 C \right) \int_{\Omega} |\Delta u_k|^p dx. \end{aligned}$$

Chosing suitable $\varepsilon, \varepsilon_1, \varepsilon_2$ and letting λ sufficiently small, we have

$$\begin{aligned} &1 - \frac{p-1}{p} \varepsilon - \lambda \frac{q-1}{q} \varepsilon_1 C - \lambda C - \lambda \frac{1}{q} \varepsilon^{-(q-1)} \frac{p-q}{p} \varepsilon_2^{-\frac{q}{p-q}} C - \lambda C(h, \beta) C \\ &= \frac{1}{p} \varepsilon^{-(p-1)} - \lambda \frac{1}{q} \varepsilon^{-(q-1)} \frac{q}{p} \varepsilon_2 C \equiv \gamma. \end{aligned}$$

For any m with $1 \leq m \leq N - 1$, summing up the above inequality for k from 0 to $m - 1$, we get

$$\int_{\Omega} |\Delta u_m|^p dx \leq \int_{\Omega} |\Delta u_0|^p dx.$$

Hence (9) holds. \square

Lemma 2.3. *Let u_{k+1} be the weak solution of the problem (5)-(6). Then the following estimate holds*

$$-Ch \leq \int_{\Omega} |u_{k+1}|^2 dx - \int_{\Omega} |u_k|^2 dx \leq 0, \quad (13)$$

where C is a constant independent of h .

Proof. To prove the first inequality, choosing $\varphi = u_k$ as a test function in (7), integrating by parts, we derive

$$\begin{aligned} \frac{1}{h} \int_{\Omega} |u_k|^2 dx &= \frac{1}{h} \int_{\Omega} u_{k+1} u_k dx + \int_{\Omega} |\Delta u_{k+1}|^{p-2} \Delta u_{k+1} \Delta u_k dx \\ &\quad + \lambda \int_{\Omega} |u_{k+1}|^{q-2} \log(|u_{k+1}|) u_{k+1} u_k dx. \end{aligned}$$

The Hölder inequality and the estimate (9) imply

$$\begin{aligned} \frac{1}{h} \int_{\Omega} |u_k|^2 dx &\leq \frac{1}{h} \int_{\Omega} u_{k+1} u_k dx + \frac{p-1}{p} \int_{\Omega} |\Delta u_{k+1}|^p + \frac{1}{p} \int_{\Omega} |\Delta u_k|^p dx \\ &\quad + C \int_{\Omega} |u_{k+1}|^p dx + C \int_{\Omega} |u_k|^p dx \\ &\leq \frac{1}{2h} \int_{\Omega} |u_{k+1}|^2 dx + \frac{1}{2h} \int_{\Omega} |u_k|^2 dx + C. \end{aligned}$$

So, we obtain

$$-Ch \leq \int_{\Omega} |u_{k+1}|^2 dx - \int_{\Omega} |u_k|^2 dx.$$

In addition, using (12) again, we see that

$$\int_{\Omega} |u_{k+1}|^2 dx - \int_{\Omega} |u_k|^2 dx \leq 0.$$

The proof is complete. \square

Proof of Theorem 2.1. First, we define the operator A^t by $A^t(\Delta u^h) = |\Delta u_k|^{p-2} \Delta u_k$, $\Delta^h u^h = u_{k+1} - u_k$, where $kh < t \leq (k+1)h$, $k = 0, 1, \dots, N-1$. By the equation (5) and (8) in Lemma 2.2, we know that

$$\frac{1}{h} \Delta^h u^h \text{ is bounded in } L^{\infty}(0, T; (W^{2,p}(\Omega))'). \quad (14)$$

Combining (7), (9), with (14) and employing the compactness results ([15]), we conclude that there exists a subsequence of $\{u^h\}$ (which we denote as the original sequence), such that

$$\begin{aligned} u^h &\xrightarrow{*} u, \quad \text{in } L^{\infty}(0, T; W^{2,p}(\Omega)), \\ u^h &\rightarrow u, \quad \text{in } C(0, T; L^2(\Omega)), \\ \frac{1}{h} (u_{k+1} - u_k) &\xrightarrow{*} \frac{\partial u}{\partial t}, \quad \text{in } L^{\infty}(0, T; (W^{2,p}(\Omega))'), \\ A^t(\Delta u^h) &\xrightarrow{*} w, \quad \text{in } L^{\infty}(0, T; L^{p'}(\Omega)), \end{aligned}$$

where p' is the conjugate exponent of p . Therefore, by (7), we know that, for any $\varphi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T} \left(\frac{1}{h} \Delta^h u^h \varphi + A^t(\Delta u^h) \Delta \varphi - \lambda |u^h|^{p-2} u^h \log |u^h| \varphi \right) dx dt = 0.$$

Sending $h \rightarrow 0$ yields

$$\frac{\partial u}{\partial t} + \Delta w - \lambda |u|^{p-2} u \log |u| = 0, \quad (15)$$

in the sense of distributions.

Now, we prove that $w = |\Delta u|^{p-2} \Delta u$ a.e. in Q_T . Set

$$f_h(t) = \frac{t - kh}{2h} \left(\int_{\Omega} |u_{k+1}|^2 dx - \int_{\Omega} |u_k|^2 dx \right) + \frac{1}{2} \int_{\Omega} |u_k|^2 dx,$$

where $kh < t \leq (k+1)h$, $k = 0, 1, \dots, N-1$. It follows from (11) that

$$\frac{1}{2} \int_{\Omega} |u_k|^2 dx - Ch \leq f_h(t) \leq \frac{1}{2} \int_{\Omega} |u_k|^2 dx,$$

and

$$-C \leq f'_h(t) \leq 0.$$

According to the Ascoli–Arzela theorem, there exists a function $f(t) \in C([0, T])$, such that

$$\lim_{h \rightarrow 0} f_h(t) = f(t) \quad \text{uniformly for } t \in [0, T].$$

Using (13), we have

$$\lim_{h \rightarrow 0} \frac{1}{2} \int_{\Omega} |u^h|^2 dx = f(t) \quad \text{uniformly for } t \in [0, T]. \quad (16)$$

It follows from (11) that

$$\frac{1}{2} \int_{\Omega} |u_N|^2 dx + \iint_{Q_T} |\Delta u^h|^p dx dt - \lambda \iint_{Q_T} |u^h|^q \log(|u^h|) dx dt \leq \frac{1}{2} \int_{\Omega} |u_0|^2 dx.$$

Passing to limits as $h \rightarrow 0$ in above inequality and using (16), we obtain

$$\begin{aligned} & \lim_{h \rightarrow 0} \left(\iint_{Q_T} |\Delta u^h|^p dx dt - \lambda \iint_{Q_T} |u^h|^q \log(|u^h|) dx dt \right) \\ & \leq f(0) - f(T) \\ & = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^{T-\varepsilon} (f(t) - f(t+\varepsilon)) dt \\ & = \lim_{\varepsilon \rightarrow 0} \lim_{h \rightarrow 0} \frac{1}{2\varepsilon} \int_0^{T-\varepsilon} \int_{\Omega} (|u^h(x, t)|^2 - |u^h(x, t+\varepsilon)|^2) dx dt. \end{aligned}$$

Consider the functional $G[u] = \frac{1}{2} \int_{\Omega} |u|^2 dx$. Obviously, $G[u]$ is convex and $\frac{\delta G[u]}{\delta u} = u$. Thus, we get

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |u^h(x, t)|^2 dx - \frac{1}{2} \int_{\Omega} |u^h(x, t+\varepsilon)|^2 dx \\ & \leq \int_{\Omega} (u^h(x, t) - u^h(x, t+\varepsilon)) u^h(x, t) dx, \end{aligned}$$

that is

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{1}{2\varepsilon} \int_0^{T-\varepsilon} \int_{\Omega} (|u^h(x, t)|^2 - |u^h(x, t + \varepsilon)|^2) dx dt \\ & \leq \frac{1}{\varepsilon} \int_0^{T-\varepsilon} \int_{\Omega} (u(x, t) - u(x, t + \varepsilon)) u dx dt. \end{aligned}$$

Therefore

$$\lim_{h \rightarrow 0} \iint_{Q_T} \left(\iint_{Q_T} |\Delta u^h|^p dx dt - \lambda \iint_{Q_T} |u^h|^q \log(|u^h|) dx dt \right) \leq - \int_0^T \langle \frac{\partial u}{\partial t}, u \rangle dt,$$

where $\langle \cdot, \cdot \rangle$ denotes inner product. From (15), we conclude that

$$\lim_{h \rightarrow 0} \iint_{Q_T} |\Delta u^h|^p dx dt \leq \int_0^T \int_{\Omega} w \Delta u dx dt. \quad (17)$$

Again by $\frac{\delta F[u]}{\delta u} = \Delta(|\Delta u|^{p-2} \Delta u)$ and the convexity of $F[u]$, for any $g \in L^\infty(0, T; W_0^{2,p}(\Omega))$, we know

$$\frac{1}{p} \iint_{Q_T} |\Delta g|^p dx dt - \frac{1}{p} \iint_{Q_T} |\Delta u^h|^p dx dt \geq \iint_{Q_T} (|\Delta u^h|^{p-2} \Delta u^h) \Delta(g - u^h) dx dt.$$

By (17) and the fact that $F(u)$ is weakly lower semicontinuous, letting $h \rightarrow 0$ in the above equality, we have

$$\begin{aligned} & \frac{1}{p} \iint_{Q_T} |\Delta g|^p dx dt - \frac{1}{p} \iint_{Q_T} |\Delta u|^p dx dt \\ & \geq - \int \int_{Q_T} w \Delta(u - g) dx dt. \end{aligned}$$

Replacing g by $\varepsilon g + u$, we see that

$$\frac{1}{\varepsilon} (F[u + \varepsilon g] - F[u]) \geq \iint_{Q_T} w \Delta g dx dt.$$

Letting $\varepsilon \rightarrow 0$, which implies that

$$\iint_{Q_T} \frac{\delta F[u]}{\delta u} g dx dt = \iint |\Delta u|^{p-2} \Delta u \Delta g dx dt \geq \iint_{Q_T} w \Delta g dx dt.$$

Due to the arbitrariness of g , we get the opposite inequality of the above inequality. Therefore

$$w = |\Delta u|^{p-2} \Delta u.$$

The strong convergence of u^h in $C(0, T; L^2(\Omega))$ and the fact that $u^h(x, 0) = u_0(x)$ implies that u satisfies the initial value condition. The proof is complete. \square

3. Large time behavior

This section is devoted to the large time behavior of solutions. To this purpose, we first show that

Theorem 3.1. *Suppose that u be a weak solution obtained in Theorem 2.1, then for any $0 \leq \rho \in C^2(\overline{\Omega})$,*

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \rho(x) |u(x, t)|^2 dx - \frac{1}{2} \int_{\Omega} \rho(x) |u_0(x)|^2 dx \\ & = \lambda \iint_{Q_t} \rho |u|^q \log(|u(x, \tau)|) dx d\tau - \iint_{Q_t} |\Delta u|^{p-2} \Delta u \Delta(\rho(x) u(x, \tau)) dx d\tau, \end{aligned} \quad (18)$$

where $Q_t = \Omega \times (0, t)$.

Proof. In the proof of Theorem 2.1, we know that

$$f(t) = \frac{1}{2} \int_{\Omega} |u(x, t)|^2 dx \in C([0, T]).$$

Similarly, we can also easily prove that for any $0 \leq \rho(x) \in C^2(\bar{\Omega})$,

$$f_{\rho}(t) = \frac{1}{2} \int_{\Omega} \rho(x) |u(x, t)|^2 dx \in C([0, T]).$$

Consider the functional

$$\Phi_{\rho}[v] = \frac{1}{2} \int_{\Omega} \rho(x) |v(x)|^2 dx.$$

It is easy to see that $\Phi_{\rho}[v]$ is a convex functional on $L^2(\Omega)$.

For any $\tau \in (0, T)$ and $h > 0$, we have

$$\Phi_{\rho}[u(\tau + h)] - \Phi_{\rho}[u(\tau)] \geq \langle u(\tau + h) - u(\tau), \rho(x)u(x, \tau) \rangle.$$

By $\frac{\delta \Phi_{\rho}[v]}{\delta v} = \rho(x)v$, for any fixed $t_1, t_2 \in [0, T], t_1 < t_2$, integrating the above inequality with respect to τ over (t_1, t_2) , we have

$$\int_{t_2}^{t_2+h} \Phi_{\rho}[u(\tau)] d\tau - \int_{t_1}^{t_1+h} \Phi_{\rho}[u(\tau)] d\tau \geq \int_{t_1}^{t_2} \langle u(\tau + h) - u(\tau), \rho(x)u \rangle d\tau.$$

Multiplying the both side of the above inequality by $\frac{1}{h}$, and letting $h \rightarrow 0$, we obtain

$$\Phi_{\rho}[u(t_2)] - \Phi_{\rho}[u(t_1)] \geq \int_{t_1}^{t_2} \langle \frac{\partial u}{\partial t}, \rho(x)u \rangle d\tau.$$

Similarly, we have

$$\Phi_{\rho}[u(\tau)] - \Phi_{\rho}[u(\tau - h)] \leq \langle (u(\tau) - u(\tau - h)), \rho(x)u \rangle.$$

Thus

$$\Phi_{\rho}[u(t_2)] - \Phi_{\rho}[u(t_1)] \leq \int_{t_1}^{t_2} \langle \frac{\partial u}{\partial t}, \rho(x)u \rangle d\tau,$$

and hence

$$\Phi_{\rho}[u(t_2)] - \Phi_{\rho}[u(t_1)] = \int_{t_1}^{t_2} \langle \frac{\partial u}{\partial t}, \rho(x)u \rangle d\tau.$$

Taking $t_1 = 0, t_2 = t$, we get from the definition of solutions that

$$\begin{aligned} & \Phi_{\rho}[u(t)] - \Phi_{\rho}[u(0)] \\ &= \int_0^t \langle -\Delta(|\Delta u|^{p-2} \Delta u) + \lambda |u|^{q-2} u \log(|u|), \rho(x)u(\tau) \rangle d\tau \\ &= - \int_0^t \langle |\Delta u|^{p-2} \Delta u, \Delta[\rho(x)u(\tau)] \rangle d\tau + \int_0^t \langle \lambda |u|^{q-2} u \log(|u|), \rho(x)u(\tau) \rangle d\tau. \end{aligned}$$

The proof is complete. \square

Theorem 3.2. *Let u be the weak solution of the problem (1)-(3), $p > 2$. Then*

$$\int_{\Omega} |u(x, t)|^2 dx \leq \frac{C_3}{(C_1 t + C_2)^{\alpha}}, \quad C_i > 0 \ (i = 1, 2, 3) \ \alpha = \frac{2}{p-2}.$$

Proof. Taking $\rho(x) = 1$ in the equality (18), we have

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |u(x, t)|^2 dx - \frac{1}{2} \int_{\Omega} |u_0(x)|^2 dx \\ &= - \int_0^t \int_{\Omega} |\Delta u|^p dx dt + \lambda \iint_{Q_t} |u|^q \log(|u|) dx dt. \end{aligned} \tag{19}$$

Let $f(t) = \frac{1}{2} \int_{\Omega} |u(x, t)|^2 dx$. By (3.2), we have

$$f'(t) = - \int_{\Omega} |\Delta u|^p dx + \lambda \int_{\Omega} |u|^q \log(|u|) dx \leq -\frac{1}{2} \int_{\Omega} |\Delta u|^p dx \leq 0.$$

Noticing that $u \in W_0^{2,p}(\Omega)$ and employing the Poincaré inequality, we see that

$$\int_{\Omega} |u(x, t)|^2 dx \leq C \int_{\Omega} |\Delta u|^2 dx \leq C \left(\int_{\Omega} |\Delta u|^p dx \right)^{2/p},$$

which implies $f(t) \leq C|f'(t)|^{2/p}$.

Again by $f'(t) \leq 0$, we get $f'(t) \leq -Cf(t)^{p/2}$, and hence

$$\int_{\Omega} |u(x, t)|^2 dx \leq \frac{1}{(C_1 t + C_2)^{\alpha}}, \quad \alpha = \frac{2}{p-2}, C_i > 0, i = 1, 2.$$

The proof is complete. \square

4. Finite speed of propagation of solutions

Theorem 4.1. *If $|\sigma_n(0)| \leq b$, and u is the weak solution of the problem (1)-(3), then for any fixed $t > 0$, we have*

$$\sigma_n(t) - \sigma_n(0) \leq C t^{\alpha} \left(\int_0^t \int_{\Omega} |\Delta u|^p dx dt \right)^{\beta},$$

where C is a constant depending on p, n, b ; $\sigma_n(t) = \sup\{z; x \in \text{supp } u(\cdot, t)\}$, $z = x_n$; $\alpha > 0, \beta > 0$ and $b > 0$ are constants independently of t .

To prove the theorem 4.1, we need the following lemma.

Lemma 4.1. ([1]) *Let $f_s(z) = \int_z^{\infty} (x-z)^s g(x) dx$, $0 \leq g(x) \in L^1(R_+)$, $k > 0$, $\alpha > 0$, $\theta > 0$, $s \geq 1$, and $0 < h \leq s < w = \frac{\theta h}{\theta-1}$. Assume $f_{s-h}(0)$ is finite and*

$$f_s(z) \leq k^{\alpha} (f_{s-h}(z))^{\theta}, \quad \forall z \geq 0.$$

Then the support of f_0 is a bounded interval $[0, l]$ and

$$l \leq (w-s+1) k^{\frac{\alpha}{(\theta-1)(w-s)}} f_0(0)^{\frac{1}{w-s}}.$$

Proof of Theorem 4.1. Without loss of generality, we assume $\sigma_n(t) > 0$. Taking $\rho(x) = (z - z_0)_+^s$, $z_0 \geq b$, $s \geq 2p$ in (18), we see that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} (z - z_0)_+^s |u(x, t)|^2 dx \\ &= - \int_0^t \int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta [(z - z_0)_+^s u] dx d\tau + \lambda \iint_{Q_t} (z - z_0)_+^s |u(\tau)|^q \log(|u(\tau)|) dx d\tau \\ &\equiv I. \end{aligned}$$

A simple calculation shows that

$$\begin{aligned} I &= - \int_0^t \int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta [(z - z_0)_+^s u] dx d\tau - \lambda \iint_{Q_t} (z - z_0)_+^s |u(\tau)|^p dx d\tau \\ &= - \int_0^t \int_{\Omega} (z - z_0)_+^s |\Delta u|^p dx d\tau - 2 \int_0^t \int_{\Omega} \nabla [(z - z_0)_+^s] \nabla u |\Delta u|^{p-2} \Delta u dx d\tau \\ &\quad - \int_0^t \int_{\Omega} s(s-1)(z - z_0)_+^{s-2} u |\Delta u|^{p-2} \Delta u dx d\tau \\ &\quad + \lambda \iint_{Q_t} (z - z_0)_+^s |u(\tau)|^q \log(|u(\tau)|) dx d\tau. \end{aligned}$$

The Hölder inequality implies

$$\begin{aligned}
I &\leq - \int_0^t \int_{\Omega} (z - z_0)_+^s |\Delta u|^p dx d\tau + \frac{1}{4} \int_0^t \int_{\Omega} (z - z_0)_+^s |\Delta u|^p dx d\tau \\
&\quad + C_1 \int_0^t \int_{\Omega} (z - z_0)_+^{s-p} |\nabla u|^p dx d\tau + \frac{1}{4} \int_0^t \int_{\Omega} |\Delta u|^p (z - z_0)_+^s dx d\tau \\
&\quad + C_2 \int_0^t \int_{\Omega} (z - z_0)_+^{s-2p} |u|^p dx d\tau + C\lambda \int_0^t \int_{\Omega} (z - z_0)_+^s |u(\tau)|^p dx d\tau \\
&\leq - \frac{1}{2} \int_0^t \int_{\Omega} (z - z_0)_+^s |\Delta u|^p dx d\tau + C_1 \int_0^t \int_{\Omega} (z - z_0)_+^{s-p} |\nabla u|^p dx d\tau \\
&\quad + C_2 \int_0^t \int_{\Omega} (z - z_0)_+^{s-2p} |u|^p dx d\tau.
\end{aligned}$$

In addition, the Hardy inequality [8] shows that

$$\int_{\Omega} (z - z_0)_+^{s-2p} |u|^p dx \leq \left(\frac{p}{s-2p+1} \right)^p \int_{\Omega} (z - z_0)_+^{s-p} |D_z u|^p dx.$$

Therefore, we get

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega} (z - z_0)_+^s |u|^2 dx + \frac{1}{2} \int_0^t \int_{\Omega} (z - z_0)_+^s |\Delta u|^p dx d\tau \\
&\leq C_3 \int_0^t \int_{\Omega} (z - z_0)_+^{s-p} |\nabla u|^p dx d\tau + C_4 \int_0^t \int_{\Omega} (z - z_0)_+^{s-p} |D_z u|^p dx d\tau \\
&\leq C \int_0^t \int_{\Omega} (z - z_0)_+^{s-p} |\nabla u|^p dx d\tau,
\end{aligned} \tag{20}$$

which implies

$$\sup_{0 < \tau \leq t} \int_{\Omega} (z - z_0)_+^s |u|^2 dx \leq C \iint_{Q_t} (z - z_0)_+^{s-p} |\nabla u|^p dx d\tau \tag{21}$$

and

$$\iint_{Q_t} (z - z_0)_+^s |\Delta u|^p dx d\tau \leq C \iint_{Q_t} (z - z_0)_+^{s-p} |\nabla u|^p dx d\tau. \tag{22}$$

Combining the (21) with the Hardy inequality, we obtain

$$\sup_{0 < \tau \leq t} \int_{\Omega} (z - z_0)_+^s |u|^2 dx \leq C \iint_{Q_t} (z - z_0)_+^s |\Delta u|^p dx d\tau. \tag{23}$$

Set

$$E_s(z_0) = \iint_{Q_t} (z - z_0)_+^s |\Delta u|^p dx d\tau, \quad E_0(z_0) = \int_0^t \int_{\Omega} |\Delta u|^p dx d\tau.$$

Using (22) and the weighted Nirenberg inequality, we see that

$$\begin{aligned}
E_{2p+1}(z_0) &\leq C_1 \iint_{Q_t} (z - z_0)_+^{p+1} |\nabla u|^p dx d\tau \\
&\leq C \int_0^t \left(\int_{\Omega} (z - z_0)_+^{p+1} |\Delta u|^p dx \right)^a \left(\int_{\Omega} (z - z_0)_+^{p+1} |u|^2 dx \right)^{(1-a)p/2} d\tau,
\end{aligned}$$

where $\frac{1}{p} = \frac{1}{p+2} + a\left(\frac{1}{p} - \frac{2}{p+2}\right) + (1-a)\frac{1}{2}$, therefore

$$a = \frac{\frac{1}{p} - \frac{1}{p+2} - \frac{1}{2}}{\frac{1}{p} - \frac{2}{p+2} - \frac{1}{2}} < 1.$$

It follows from (23), that

$$\begin{aligned}
E_{2p+1}(z_0) &\leq C \left(\iint_{Q_t} (z - z_0)_+^{p+1} |\Delta u|^p dx d\tau \right)^{(1-a)p/2} \int_0^t \int_{\Omega} ((z - z_0)_+^{p+1} |\Delta u|^p dx)^a d\tau \\
&\leq C [E_{p+1}(z_0)]^{(1-a)p/2} \left(\iint_{Q_t} (z - z_0)_+^{p+1} |\Delta u|^p dx d\tau \right)^a t^{1-a} \\
&\leq C E_{p+1}(z_0)^{(1-a)p/2+a} t^{1-a}.
\end{aligned}$$

We are going to obtain from the above inequality that $\Delta u = 0$ a. e. for $z_0 > b$ and $0 < \tau < t$. By (23), we know that $u = 0$ a. e. on the same set. By Lemma 4.1, we obtain Theorem 4.1. The proof is complete. \square

5. Conclusions

The parabolic equation with the logarithmic nonlinearity is important. The second order parabolic equation with the logarithmic nonlinearity is diffusely studied. In this paper, we study the higher order parabolic equation with the logarithmic nonlinearity. We study the existence of weak solutions. The main difficulties for treating the problem are caused by the nonlinearity of the principal part and the logarithmic nonlinearity. The method used for treating the second order parabolic equation with the logarithmic nonlinearity seems not applicable to the present situation. Our method is based on the variation methods. Using the energy techniques, Hardy inequality and Poincaré inequality, we also obtain the large time behavior and the finite speed of propagation of perturbations subsequently.

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REFERENCES

- [1] F. Bernis, *Qualitative properties for some nonlinear higher order degenerate parabolic equations*, Houston J. Math. 14(3)(1988), 319-352.
- [2] K. Chang, *Critical point theory and its applications*, Shanghai Sci. Tech. Press, Shanghai, 1986.
- [3] H. Chen, P. Luo, G. Liu *Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity*, J. Math. Anal. Appl., 422(2015), 84-98.
- [4] H. Chen, S. Tian, *Initial boundary value problem for a class of semilinear pseudo-parabolic equations with logarithmic nonlinearity*, J. Differential Equations, 258(2015), 4424-4442.
- [5] Y. Chen and L. Wu, *Second order elliptic equations and elliptic systems*, Science Press, Beijing, 1991.
- [6] Pavel Drábek and Mitsuharu Ôtani, *Global bifurcation result for the p -biharmonic operator*, Electronic Journal of Differential Equations, (48)(2001), 1-19.
- [7] A. Hao, J. Zhou, *Blowup, extinction and non-extinction for a nonlocal p -biharmonic parabolic equation*, Applied Mathematics Letters, 64(2017) 198-204.
- [8] G. H. Hardy, J. E. Littlewood and G. P'olya, *Inequalities*, Cambridge University press, Cambridge, 1934.
- [9] Y. He, H. Gao, H. Wang, *Blow-up and decay for a class of pseudo-parabolic p -Laplacian equation with logarithmic nonlinearity*, Computers and Mathematics with Applications, 75(2018), 459-469.
- [10] S. Ji, J. Yin, Y. Cao, *Instability of positive periodic solutions for semilinear pseudo-parabolic equations with logarithmic nonlinearity*, J. Differential Equations, 261(2016), 5446-5464.

- [11] K. Kefi and V. D. Rădulescu, *On a $p(x)$ -biharmonic problem with singular weights*, Z. Angew. Math. Phys., (2017) 68:80
- [12] C. Liu and J. Guo, *Weak solutions for a fourth order degenerate parabolic equation*, Bulletin of the Polish Academy of Sciences, Mathematics, 54(1)(2006), 27-39.
- [13] X. Liu, H. Chen, B. Almualemi, *Ground state solutions for p -biharmonic equations*, Electronic Journal of Differential Equations, 2017(45)(2017), 1-9.
- [14] L. C. Nhana, L. X. Truong, *Global solution and blow-up for a class of pseudo p -Laplacian evolution equations with logarithmic nonlinearity*, Computers and Mathematics with Applications, 73(2017), 2076-2091.
- [15] J. Simon, Compact sets in the space $L^p(0, T; B)$, Ann. Math. Pure Appl., 146(1987), 65-96.
- [16] L. Wang, C. Zhang and S. Zhou, *Existence and uniqueness of weak solutions for a 2D low-curvature equation*, J. Math. Anal. Appl., 452(2017), 297-311.