

## THE ELASTIC CONTACT OF A SPHERE WITH AN ELASTIC HALF-SPACE, A COMPARISON BETWEEN ANALYTICAL AND FINITE ELEMENTS SOLUTIONS

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*Contactul elastic al unei sfere cu un semispațiu elastic este o problemă teoretică de larg interes practic. Aceasta este primul pas în înțelegerea implicațiilor elasto-plastice ale problemei poansonului aplicat pe un astfel de mediu elastic. Cercetări mai mult sau mai puțin recente au evidențiat că formula analitică de calcul al deformației δ între două corpuri elastice este clar stabilită.*

*Lucrarea de față compară rezultatele soluției analitice cu cele date de o soluție numerică pentru un model al acestui contact elastic realizat prin metoda elementelor finite, punând în evidență posibilele erori care s-ar putea produce aplicând formula analitică.*

*The elastic contact of a sphere with an elastic half-space is a theoretical problem of high practical interest. The elastic contact is the first step in understanding the micro-indentation of elasto-plastic contact. Past and recent researches reveal that an analytical formula for the deformation  $\delta$  between two elastic bodies is clearly and well established.*

*The present paper compares the results of analytical solution with the results of a finite elements approach of this elastic contact, emphasizing possible errors which might occur by applying the analytical formula.*

**Key words:** elastic contact, Hertz theory, finite elements solution.

### 1. Introduction

In the last decades, the problem of the Hertzian contact has known extensive practical applications, being the first step in understanding the micro-indentation of an elasto-plastic contact. It is interesting to point out the model from reference [1], which allows designers to choose easily different surface topographic and material properties for rail gun technology, by using simple Hertz theory applied to the complex geometry of a fractal surface. On the other hand, some researches deal with the contact of rough surfaces of two bodies, based on statistical surface models and elasto-plastic behavior of the asperities, as shown in reference [2].

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## 2. Theoretical background

The case of elastic contact between two bodies (for instance, two spheres whose radii are  $R_1$  and  $R_2$ ) is known in technical literature [3], [4]. In this context it is important to remind some main steps from [3], this being the best known approach on this subject.

If there is no pressure between the bodies, the contact is represented by a single point  $O$  (Figure 1) and the distances  $z_1$  and  $z_2$  are given by Hertz relations. The notations are explained on figure 1:

$$z_1 + z_2 = \frac{r^2}{2R_1} + \frac{r^2}{2R_2} = \frac{r^2(R_1 + R_2)}{2R_1 R_2} = ar^2 \quad (1)$$

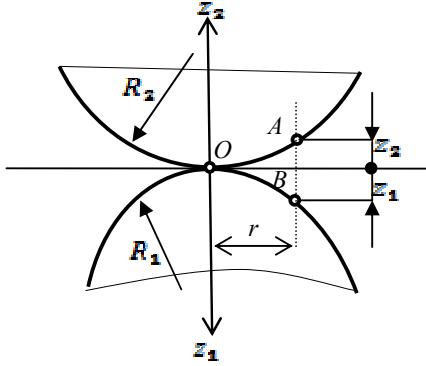


Figure 1: Contact between two spheres

Let be a vertical force  $P$ , acting on the symmetry axis ( $z$ ) of the spheres. Because of the exerted pressure, a local deformation will occur and the point of contact becomes a surface of contact. A hypothesis is introduced here: the contact surface is a small circular surface of radius  $a$ . Denoting by  $w_1$  and  $w_2$  the displacement of point  $B$  towards the point  $A$  and respectively the displacement of point  $A$  towards the point  $B$  and by  $\delta$  the distance between the new positions of  $A$  and  $B$ , the relation (1) can be rewritten:

$$z_1 + z_2 = \delta - (w_1 + w_2) = ar^2 \quad (2)$$

The next equations are proven in [3], the relevant part being represented by the following expressions:

$$\begin{aligned} w_1 + w_2 &= \frac{\pi(k_1 + k_2)p_0}{2a} \int_0^{\pi/2} (a^2 - r^2 \sin^2 \psi) d\psi = \delta - ar^2 \\ \frac{(k_1 + k_2)\pi p_0}{2a} \cdot \frac{\pi(2a^2 - r^2)}{2} &= \delta - ar^2 \end{aligned} \quad (3)$$

$$k_1 = \frac{1 - \nu_1^2}{\pi E_1}, \quad k_2 = \frac{1 - \nu_2^2}{\pi E_2}$$

where  $\nu_1, \nu_2$  are the Poisson's coefficients and  $E_1, E_2$  the Young's moduli for the two bodies. The term  $p_0$  is the maximum value of the pressure due to the applied force  $P$  over the contact surface, the distribution of this pressure  $p$  being imagined as a half-ellipsoid with axis  $a = b$  and  $c = p_0$ . It is obvious that in the middle of the contact surface, the maximum pressure will be  $p_0 = \frac{3P}{2\pi a^2}$ .

From relations (1) and (3) the next expressions are clearly determined:

$$\delta = \frac{(k_1 + k_2)(p_0 \pi^2 a)}{2}$$

$$a = (k_1 + k_2) \frac{\pi^2 p_0}{4a} = \frac{1}{2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)}$$

Finally, the equations required in applications are:

$$a = \sqrt[3]{\frac{3\pi}{4} \sqrt{\frac{P(k_1 + k_2)}{\frac{1}{R_1} + \frac{1}{R_2}}}}$$

$$\delta = \sqrt[3]{\frac{9\pi^2}{16} (k_1 + k_2)^2 P^2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \quad (4)$$

$$\eta_1 = \frac{1 - \nu_1^2}{E_1} \quad \eta_2 = \frac{1 - \nu_2^2}{E_2} \quad (5)$$

$$R_1 = \infty \quad R_2 = R \quad (6)$$

Using the notations (5) and the specific conditions (6) for the contact between a sphere with a plane, relation (4) becomes:

$$\delta = 0,8255 \sqrt[3]{\frac{P^2}{R} (\eta_1 + \eta_2)^2} \quad (7)$$

In order to have a clear image of the analytical solution, maintaining the same symbols for the above mentioned quantities (force, radius of the contact surface, distance between two points after the impact, etc) and referring directly to the contact between a sphere and a half-space, from reference [4], it is relevant to point out the next demonstration.

For a Hertzian pressure distribution, the following are true:

$$p = p_0 \left( 1 - \frac{r^2}{a^2} \right)^{\frac{1}{2}} \quad (8)$$

The resulting vertical displacement is:

$$w = \frac{\pi p_0}{(4E^*a)(2a^2 - r^2)} \quad (9)$$

where:

$$\frac{1}{E^*} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}$$

and  $\nu_1, \nu_2, E_1, E_2$  are already known to have specified significations.

The force  $P$  is:

$$P = \int_0^a p(r) 2\pi r dr = \frac{2}{3} p_0 \pi a^2 \quad (10)$$

The modeling of the mentioned contact can be seen on figure 2:

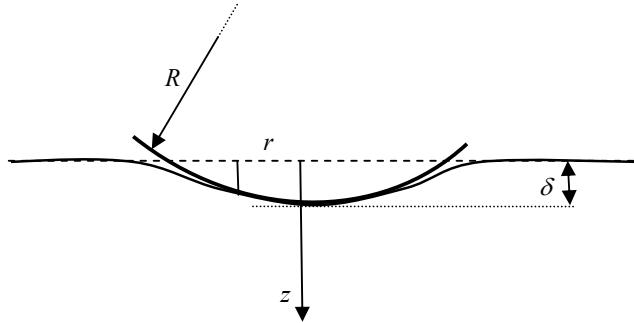


Figure 2: Contact between a sphere and a half-space

$$w = \delta - z = \delta - \frac{r^2}{2R} \quad \text{or} \quad \frac{\pi p_0}{4E^*a} (2a^2 - r^2) = \delta - \frac{r^2}{2R} \quad (11)$$

From the above relation (by identification of the terms left-right), the radius of the contact surface and the deformation fulfil the expressions:

$$a = \frac{\pi p_0 R}{2E^*} \quad \delta = \frac{\pi a p_0}{2E^*} \quad (12)$$

Using both equations, the maximum value of the contact pressure is:

$$p_0 = \frac{2E^*}{\pi} \left( \frac{\delta}{R} \right)^{\frac{1}{2}} \quad (13)$$

Relations (12) and (13) are introduced in (10):

$$P = \frac{4}{3} E^* R^{1/2} \delta^{3/2} \quad (14)$$

It is obvious, now, that the value of  $\delta$  is:

$$\delta = \sqrt[3]{\frac{9}{16} \frac{P^2}{(E^*)^2 R}} \quad (15)$$

Introducing in (7) the conditions (6), the expression (15) is equivalent to the relation (7).

### 3. The finite elements solution

The problem of elastic contact was a considerable challenge for the finite elements method until very recently. The main difficulty is to solve simultaneously both the contact surface shape and the surface area which increases with the applied force. As the area increases, the contact pressure diminishes. The equilibrium condition is achieved by an iterative process. Since the force is applied on a sphere placed on an elastic half-space, it can be logically expected that the contact surface has a circular boundary. The hypothesis that the contact surface is a circle cannot be imposed in the finite elements approach. This is one of the reasons to expect a more realistic physical solution. On the other hand, the classical analytical approach makes no clear statement about the applied force  $P$ . In practical applications, the force is applied vertically downwards (Figure 3) by other mechanical parts, which are not investigated. It is clear that a concentrated load applied on the top point of the sphere will cause a deformation affecting the spherical shape. The radius  $R$ , used in the analytical formula, will not remain valid because the shape changes. Moreover, from the point of view of finite elements method, by applying a force on a point, extreme local stresses occur, as it is known from analytical solutions. These extreme stresses can lead to numerical instabilities. One solution is to model only a quarter from the axial section of the sphere and to apply on the top surface a uniformly distributed load, having the same resultant force  $P$ . The solution is questionable, since the stress distribution over the horizontal circle passing through the sphere centre is not perfectly uniform. The solution adopted in the present paper is to apply uniform axial displacement over horizontal circular surfaces and the values of these displacements are increased in 20 steps from 0 to 0.1 mm. Two such circular surfaces were tested: one of them passing through the sphere centre and another one, very close (0.1 mm) to the upmost point of the sphere. The latter case is expected to be the closest to the real loading condition. The choice to impose displacements instead of applied force or of axial stress is necessary due to the lack of the convergence of the numerical solution if the applied force or the axial stress were used. However, the two approaches are equivalent, because the axial

force can be computed as integral of the normal (axial) stress on the surface with imposed displacements. For these reasons and using the advantage of the axial symmetry of the problem, it is recommended to investigate only half of the axial cross section through the sphere and a reasonable region is formed for the infinite half-plane.

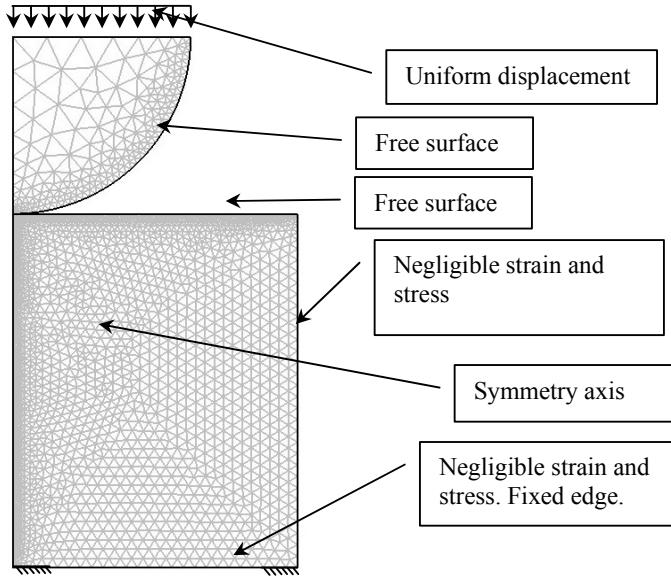


Figure 3: Geometry of the FEM model

A reasonable region is, from the finite elements point of view, a rectangle sufficiently wide, so as it has numerically negligible strains and stresses on the lowest and rightmost edges (Figure 3). This numerical requirement can be achieved by increasing the rectangle size, which represents the semi-infinite elastic domain and fixing the lowest edge. The elastic problem is solved for the two sub-domains: sphere and elastic half-space.

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr}}{r} &= 0 \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r} \sigma_{rz} &= 0 \end{aligned} \quad (16)$$

The multi-physics finite elements software package COMSOL [6] was used to determine the solution. A recent development in the available solving methods is the use of the contact elements. The software determines which

elements are in contact and in continuity for displacements and at the same time, the soft imposes normal stresses for the nodes which are in the contact domain.

A parametrical study is done for each geometrical configuration, the load increasing from zero to the imposed applied resulting force. The displacement of the point placed in the centre of the contact area can be plotted as function of the applied force.

#### 4. Numerical results

The elastic contact between a sphere of radius  $R=10\text{ mm}$  and an elastic plane is modelled by finite elements as indicated in the previous paragraph. Material data for the infinite medium are: Young's modulus  $E_1=2e11\text{ Pa}$  and Poisson's coefficient  $\nu_1=0.33$ . In the first case, the sphere material has been selected to be the same as for the infinite medium ( $E_2=2e11\text{ Pa}$ ,  $\nu_2=0.33$ ). Typical results are presented on Fig. 4, for the half-sphere model (a) and "full" sphere model (b) which is a sphere whose cap of  $0.1\text{ mm}$  is missing from the  $10\text{ mm}$  radius.

For comparison reasons, the Young's modulus of the infinite medium has been decreased in steps, such that:  $E_2/E_1=2, \dots, E_2/E_1=8$ . The same material and loading data are introduced in (7) in order to determine the deformation  $\delta$  for the same values of the applied force. The results are presented on Figure 5 as dashed line for the analytical solution and continuous line for the FEM solution.

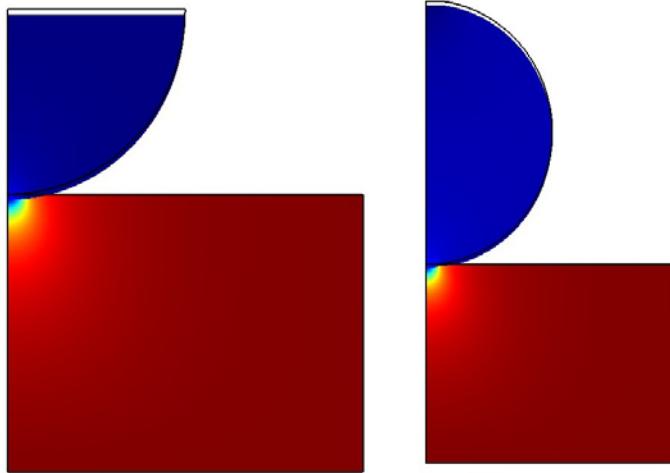


Fig. 4: Deformed shape of the contact for the two FEM models

A large difference can be seen for the case  $E_2/E_1=1$ , marked by squares. The difference between solutions decreases for increasing of the ratio  $E_2/E_1$ .

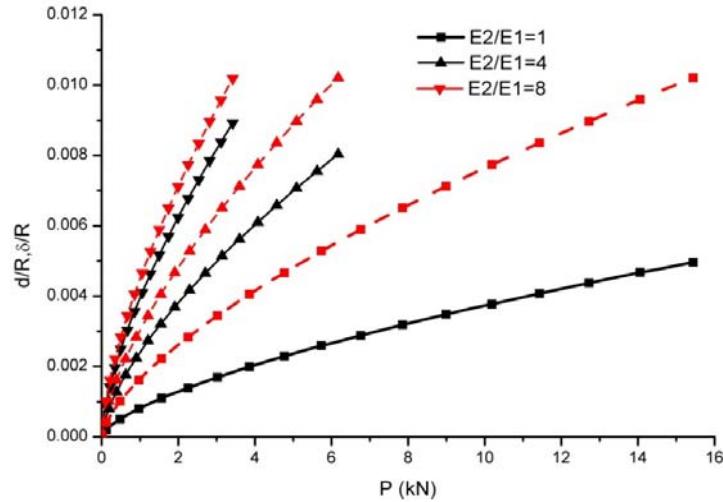


Fig. 5: Relative displacement vs. applied force for three materials. Dashed line: analytical formula, continuous line: FEM solution

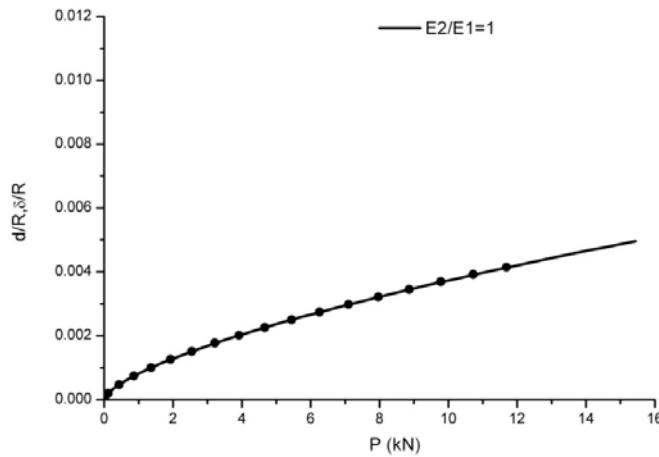


Fig. 6: FEM models comparison. Continuous line: half sphere; circles: “full” sphere.

The influence of the FEM model has been investigated and compared. The comparison of the two FEM solutions is shown on Figure 6. The model using a half-sphere (continuous line) produces practically the same results as in the case of a “full” sphere, proving that a smaller FEM model, which provides faster solutions, is a very good choice. The semi-infinite medium is modelled by cylinders. Two values were tested for the radius of the cylinder: 15 and 20 mm keeping the same height of 15 mm. The results were practically the same.

## 5. Conclusions

The investigation of the accuracy of the classical formula for the Hertzian contact between a sphere and an infinite medium has led to the following conclusions.

The theoretical approach is based on assumptions concerning the geometry of the deformed bodies. The contact surface is assumed to be a circular area, over which a parabolic load is applied.

The FEM solution provides consistently lower values for the displacement of the central contact point. The FEM solution approaches the analytical solution for increasing  $E_2/E_1$  ratio and both solutions are expected to coincide as this ratio tends to infinite.

The explanation comes from the situation that the real contact surface is not flat; it is considerably larger than the analytical solution predicts. In fact, the FEM solution accounts for the sphere deformation, the equilibrium configuration not corresponding to a spherical shape in the free area.

Practical application of the results can be used for the hardness measuring devices which need corrections and a model of a quarter of the sphere cross section is quite satisfactory.

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