

KERR-NEWMAN SOLUTIONS WITH ANALYTIC SINGULARITY AND NO CLOSED TIMELIKE CURVES

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It is shown that the Kerr-Newman solution, representing a charged and rotating stationary black hole, admits analytic extension at the singularity. This extension is obtained by using new coordinates, in which the fundamental tensor becomes smooth on the singularity ring. On the singularity, the fundamental tensor is degenerate - its determinant vanishes. The analytic extension can be naturally chosen so that the closed timelike curves normally present in the Kerr and Kerr-Newman solutions no longer exist. On the extension proposed here the electromagnetic potential is smooth, being thus able to provide non-singular models of charged spinning particles. The maximal analytic extension of this solution can be restrained to a globally hyperbolic region containing the exterior universe, having the same topology as the Minkowski spacetime. This admits a spacelike foliation in Cauchy hypersurfaces, on which the information contained in the initial data is preserved.

Keywords: Kerr-Newman black hole, singularity, closed timelike curves, information paradox.

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1. Introduction

The Kerr-Newman solutions are stationary and axisymmetric solutions of the Einstein-Maxwell equations, representing charged rotating black holes [9, 28]. The other stationary black hole solutions can be obtained as particular cases of the Kerr-Newman solutions. They are representative for all the black holes, because even the non-stationary black holes tend asymptotically in time to Kerr-Newman ones (according to the *no-hair theorem*). They have interesting properties similar to the entropy and temperature in thermodynamics, which were studied in [1, 2, 26, 25, 27, 24].

But they also have some unusual properties, which are in general considered undesirable. They, as any black hole solution, have a singularity, where some of the fields reach infinite values. The singularity is in general ring-shaped, and passing through the ring one can reach inside another universe, in which there are *closed timelike curves*, i.e. time machines (which fortunately

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don't affect the causality in the region $r > 0$). There is also another problem, the *black hole information paradox*, which refers to the loss of information inside the singularity, which, if would really happen, would cause serious problems, especially violation of unitary evolution, after the black hole evaporation [5, 6].

The fundamental tensor can be singular in two main ways which are relevant to our discussion. In the first kind of singularity, the fundamental tensor has components which diverge as approaching the singularity. The Kerr-Newman fundamental tensor is, in usual coordinates, of the first kind. The second kind is that when the fundamental tensor's components remain smooth at the singularity (and therefore finite). In the second kind, the singularity is still present, because the fundamental tensor becomes *degenerate* – i.e. its determinant becomes 0. In some cases, it is possible to change the coordinate system in which a singularity of the first kind is represented, so that in the new coordinates the singularity becomes of the second kind – it becomes degenerate¹.

The purpose of this article is to show that there are coordinates in which the singularity of the Kerr-Newman fundamental tensor becomes of degenerate type. In these coordinates, the fundamental tensor becomes smooth, and the only way the singularity manifests is that the fundamental tensor becomes degenerate (we have already developed, in [13, 14, 15], mathematical tools which allow us to make differential geometry even in this situation of degenerate fundamental tensor). In addition, we will show here that we can choose the analytic extension so that the closed timelike curves no longer exist. Moreover, we can find solutions which are globally hyperbolic and admit spacelike foliations in Cauchy hypersurfaces, ensuring therefore the conservation of information. The electromagnetic potential turns out to be smooth. New models for charged spinning particles are suggested.

The Kerr-Newman solution is usually defined in $\mathbb{R} \times \mathbb{R}^3$, where \mathbb{R} is the time coordinate, and on \mathbb{R}^3 we use spherical coordinates (r, ϕ, θ) . Let $a \geq 0$ (which characterizes the rotation), $m \geq 0$ the mass, $q \in \mathbb{R}$ the charge, and let's define the functions

$$\Sigma(r, \theta) := r^2 + a^2 \cos^2 \theta, \Delta(r) := r^2 - 2mr + a^2 + q^2.$$

Then, we define the Kerr-Newman fundamental tensor by

$$g_{tt} = -\frac{\Delta(r) - a^2 \sin^2 \theta}{\Sigma(r, \theta)}, g_{rr} = \frac{\Sigma(r, \theta)}{\Delta(r)}, g_{\theta\theta} = \Sigma(r, \theta) \quad (1)$$

¹It may happen that the fundamental tensor becomes regular after the coordinate transformation, but in this case it follows that the singularity was not genuine, it was due to the fact that the coordinates in which the regular fundamental tensor was represented are singular. This is the case of the Eddington-Finkelstein coordinates, which proved that the singularity of the event horizon is only apparent [3, 4].

$$g_{\phi\phi} = \frac{(r^2 + a^2)^2 - \Delta(r)a^2 \sin^2 \theta}{\Sigma(r, \theta)} \sin^2 \theta, \quad g_{t\phi} = g_{\phi t} = -\frac{2a \sin^2 \theta (r^2 + a^2 - \Delta(r))}{\Sigma(r, \theta)} \quad (2)$$

all other components of the fundamental tensor being equal to 0 [28].

By making $q = 0$ we obtain the Kerr solution [7, 8], while by making $a = 0$ we get the Reissner-Nordström solution [12, 10]. By making both $q = 0$ and $a = 0$ we obtain the Schwarzschild solution, which when $m = 0$ gives the empty Minkowski spacetime (see Table 1).

	$\mathbf{a} > \mathbf{0}$	$\mathbf{a} = \mathbf{0}$
$\mathbf{q} \neq \mathbf{0}$	Kerr-Newman	Reissner-Nordström
$\mathbf{q} = \mathbf{0}$	Kerr	Schwarzschild

TABLE 1. The various stationary black hole solutions, as particularizations of the Kerr-Newman solution.

2. Extending the Kerr-Newman spacetime at the singularity

Theorem 2.1. *The Kerr-Newman fundamental tensor admits an analytic extension at $r = 0$ (where the fundamental tensor is degenerate, with analytic, and not singular components).*

Proof. We will find a coordinate system in which the fundamental tensor is analytic, although degenerate. Recall that the event horizons of the black hole are given by the real solutions r_{\pm} of the equation $\Delta(r) = 0$. It is enough to make the coordinate change in a neighborhood of the singularity – in the block III, as it is usually called ([11], p. 66). This is the region $r < r_-$ if r_- is a real (and positive) number. If the equation $\Delta(r) = 0$ has no real solutions, the singularity is naked, and we can take the entire domain.

We choose the coordinates τ , ρ , and μ , so that

$$\begin{cases} t &= \tau \rho^{\mathcal{T}} \\ r &= \rho^{\mathcal{S}} \\ \phi &= \mu \rho^{\mathcal{M}}, \\ \theta &= \theta \end{cases} \quad (3)$$

with $\mathcal{S}, \mathcal{T}, \mathcal{M} \in \mathbb{N}$ are natural numbers, to be determined in order to make the fundamental tensor analytic. The expression of the fundamental tensor when passing from coordinates (x^a) to the new coordinates $(x^{a'})$ is given by

$$g_{a'b'} = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} g_{ab}, \quad (4)$$

where Einstein's summation convention is used. In our case, the Jacobian of the coordinate transformation is

$$\frac{\partial(t, r, \phi, \theta)}{\partial(\tau, \rho, \mu, \theta)} = \begin{pmatrix} \frac{\partial t}{\partial \tau} & \frac{\partial t}{\partial \rho} & \frac{\partial t}{\partial \mu} & \frac{\partial t}{\partial \theta} \\ \frac{\partial r}{\partial \tau} & \frac{\partial r}{\partial \rho} & \frac{\partial r}{\partial \mu} & \frac{\partial r}{\partial \theta} \\ \frac{\partial \phi}{\partial \tau} & \frac{\partial \phi}{\partial \rho} & \frac{\partial \phi}{\partial \mu} & \frac{\partial \phi}{\partial \theta} \\ \frac{\partial \theta}{\partial \tau} & \frac{\partial \theta}{\partial \rho} & \frac{\partial \theta}{\partial \mu} & \frac{\partial \theta}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \rho^{\mathcal{T}} & \mathcal{T}\tau\rho^{\mathcal{T}-1} & 0 & 0 \\ 0 & \mathcal{S}\rho^{\mathcal{S}-1} & 0 & 0 \\ 0 & \mathcal{M}\mu\rho^{\mathcal{M}-1} & \rho^{\mathcal{M}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5)$$

The components can be arranged as in Table 2.

	$\cdot/\partial\tau$	$\cdot/\partial\rho$	$\cdot/\partial\mu$	$\cdot/\partial\theta$
$\partial t/\cdot$	$\rho^{\mathcal{T}}$	$\mathcal{T}\tau\rho^{\mathcal{T}-1}$	0	0
$\partial r/\cdot$	0	$\mathcal{S}\rho^{\mathcal{S}-1}$	0	0
$\partial \phi/\cdot$	0	$\mathcal{M}\mu\rho^{\mathcal{M}-1}$	$\rho^{\mathcal{M}}$	0
$\partial \theta/\cdot$	0	0	0	1

TABLE 2. The Jacobian components of the coordinate change.

We want to make sure that the new expression of the fundamental tensor becomes smooth even on the ring singularity. For this, we want that all the terms in the right hand side of equation (4) are smooth. To ensure this, we have to make sure that the Jacobian components cancel the singularities of the fundamental tensor's components, even when $\cos \theta = 0$.

The least power of ρ on the ring singularity, in each of the fundamental tensor's components listed in equations (1), and (2) are respectively

$$\mathcal{O}_{\rho}(g_{tt}) = -2\mathcal{S}, \quad (6)$$

$$\mathcal{O}_{\rho}(g_{\phi\phi}) = -2\mathcal{S}, \quad (7)$$

$$\mathcal{O}_{\rho}(g_{t\phi}) = \mathcal{O}_{\rho}(g_{\phi t}) = -2\mathcal{S}, \quad (8)$$

these components being obtained by dividing polynomial expressions in ρ by $\Sigma(r, \theta)$. None of the other components can become singular on the ring singularity.

The least power of ρ in each of the Jacobians components from Table 2 are given in Table 3.

Let's take the fundamental tensor's components and see if the singularities are canceled by the components of the Jacobian.

We check each component g_{ab} of the fundamental tensor by looking up the rows labeled by $\partial x^a/\cdot$ and $\partial x^b/\cdot$ in Table 3.

For example, the term

$$\frac{\partial t}{\partial \rho} \frac{\partial t}{\partial \tau} g_{tt} \quad (9)$$

	$\cdot/\partial\rho$	$\cdot/\partial\tau$	$\cdot/\partial\mu$	$\cdot/\partial\theta$
$\partial t/\cdot$	\mathcal{T}	$\mathcal{T} - 1$	0	0
$\partial r/\cdot$	0	$\mathcal{S} - 1$	0	0
$\partial\phi/\cdot$	0	$\mathcal{M} - 1$	\mathcal{M}	0
$\partial\theta/\cdot$	0	0	0	0

TABLE 3. The least power of ρ in the Jacobian components of the coordinate change.

satisfies

$$\mathcal{O}_\rho \left(\frac{\partial t}{\partial \rho} \frac{\partial t}{\partial \tau} g_{tt} \right) = (\mathcal{T} - 1) + \mathcal{T} - 2\mathcal{S}, \quad (10)$$

hence \mathcal{T} needs to satisfy $2\mathcal{T} \geq 2\mathcal{S} + 1$.

From equations (6), (7), and (8) is easy to see that we have to do this only for the components of the fundamental tensor with indices t and ϕ . From the equation (4) we see that we are interested only in the rows $\partial t/\cdot$ and $\partial\phi/\cdot$ from the Table 3. It follows then that each of the components of the Jacobian having the form $\partial t/\cdot$ and $\partial\phi/\cdot$ has to contain ρ to at least the power \mathcal{S} , to cancel the singularities of the fundamental tensor's components. It follows that the conditions

$$\begin{cases} \mathcal{S} & \geq 1 \\ \mathcal{T} & \geq \mathcal{S} + 1 \\ \mathcal{M} & \geq \mathcal{S} + 1, \end{cases} \quad (11)$$

where $\mathcal{S}, \mathcal{T}, \mathcal{M} \in \mathbb{N}$, ensure the smoothness (and the analyticity for that matter) of the fundamental tensor on the ring singularity, in the new coordinates. In the new coordinates, none of the fundamental tensor's components become infinite at the singularity. \square

Remark 2.1. *The Kerr-Newman solution has a ring singularity, where $r = 0$ and $\cos\theta = 0$. By using Kerr-Schild coordinates, we can see that it can be analytically extended through the disk defined by $r = 0$ to another spacetime region which looks similar, but is not isometric to the region with $r > 0$, since there $r < 0$ (see Fig. 1). On the other hand it is easy to check that, if we use our coordinates with even \mathcal{S} , \mathcal{T} , and \mathcal{M} , then the analytic extension to $\rho < 0$ gives a region which is isometric to that with $\rho > 0$, with the isometry given by identifying the points $(\rho, \tau, \mu, \theta)$ and $(-\rho, \tau, \mu, \theta)$.*

Remark 2.2. *Our global solution described in the Remark 2.1 shows that, for even \mathcal{S} , \mathcal{T} , and \mathcal{M} , we can eliminate the region where $r < 0$. In this case, the closed timelike curves known to appear in the standard Kerr and Kerr-Newman solutions, are no longer present. Therefore, if these closed timelike curves were considered as violating the causality, to avoid them we just take \mathcal{S} , \mathcal{T} , and \mathcal{M} to be even and make the identification of $(\rho, \tau, \mu, \theta)$ and $(-\rho, \tau, \mu, \theta)$.*

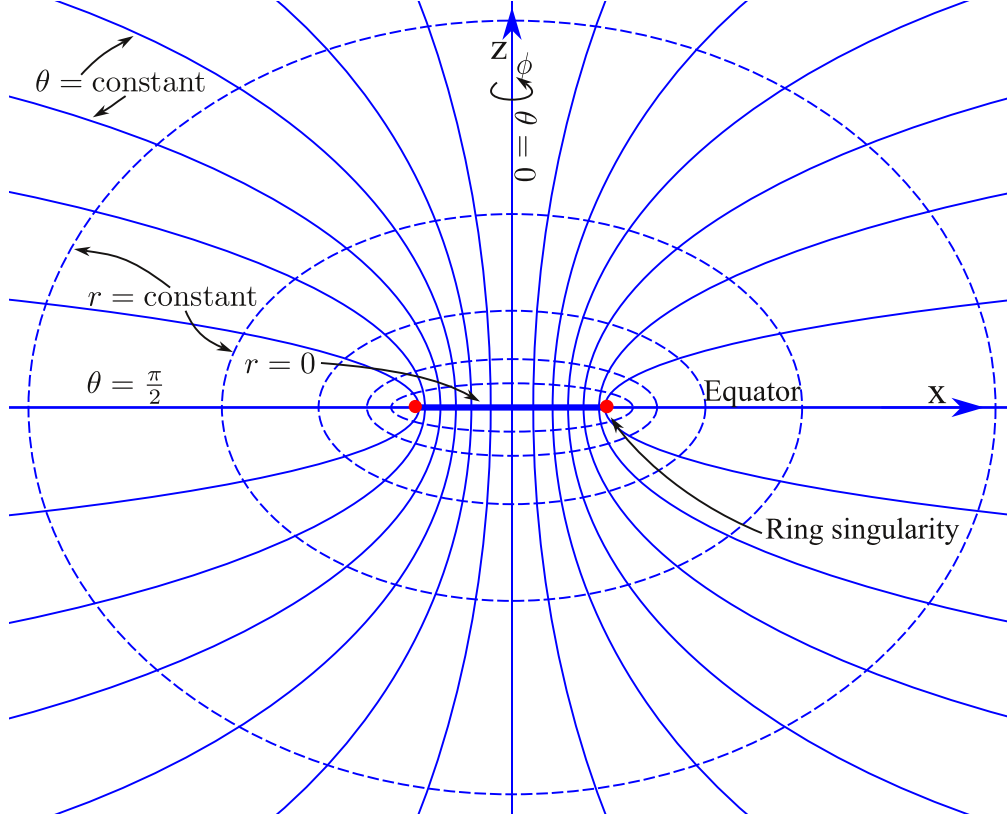


FIGURE 1. The Kerr (and Kerr-Newman) solution, in Kerr-Schild coordinates. The standard solution admits an analytic continuation beyond the disk $r = 0$, into another spacetime which contains closed timelike curves. If we take in our solution \mathcal{S} , \mathcal{T} , and \mathcal{M} to be even, we can identify isometrically the regions $\rho < 0$ and $\rho > 0$, and obtain by this a removal of the wormhole and of the closed timelike curves.

Remark 2.3. *If $a \rightarrow 0$, then we recover the Reissner-Nordström solution. The neck $r = 0$ connecting the two regions $r > 0$ and $r < 0$ converges to a point, as well as the ring singularity delimiting it. This point is the $r = 0$ singularity of the Reissner-Nordström solution, and it still can be viewed as connecting the region $r > 0$ with a region $r < 0$. This can be now put in relation with the extension through singularity of some of the Reissner-Nordström solutions developed in [17], which suggest that for odd \mathcal{S} the singularity connects the spacetime region $r > 0$ with a region $r < 0$.*

3. The electromagnetic field

One distinctive feature of our extension is that it has smooth electromagnetic potential and electromagnetic field. This may be important in particular when using the Kerr-Newman black holes to model charged particles.

The electromagnetic potential of the Kerr-Newman solution is the 1-form

$$A = -\frac{qr}{\Sigma(r, \theta)}(dt - a \sin^2 \theta d\phi), \quad (12)$$

which becomes, in our coordinates,

$$A = -\frac{q\rho^{\mathcal{S}}}{\Sigma(r, \theta)}(\rho^{\mathcal{T}}d\tau + \mathcal{T}\tau\rho^{\mathcal{T}-1}d\rho - a \sin^2 \theta \rho^{\mathcal{M}}d\mu), \quad (13)$$

because from the Table 2 it follows that

$$dt = \rho^{\mathcal{T}}d\tau + \mathcal{T}\tau\rho^{\mathcal{T}-1}d\rho \quad (14)$$

$$dr = \mathcal{S}\rho^{\mathcal{S}-1}d\rho \quad (15)$$

and

$$d\phi = \mathcal{M}\mu\rho^{\mathcal{M}-1}d\rho + \rho^{\mathcal{M}}d\mu. \quad (16)$$

The singularity of the electromagnetic potential A at $\rho = 0$ and $\cos \theta = 0$ is removed in our case, since $\mathcal{T} > \mathcal{S}$ and $\mathcal{M} > \mathcal{S}$, from the conditions (11). Similarly, since the electromagnetic field $F = dA$, we conclude that the electromagnetic field F is smooth too.

4. The global solution

The Penrose-Carter diagrams of our solution depend on the various combinations of the parameters a, q, m . For the Schwarzschild solution they were presented in [16], and for the Reissner-Nordström in [17]. In general it is admitted that the Kerr and Kerr-Newman solutions have Penrose-Carter diagrams similar to those for the Reissner-Nordström solution, although there are some differences due to the fact that the symmetry is not spherical, but axisymmetric, that the singularity is ring-shaped, and of the closed timelike curves in the region $r < 0$. Since our solution can eliminate the closed timelike curves (Remark 2.1), we expect a better similarity with the Reissner-Nordström case, and consequently similar Penrose-Carter diagrams. This would allow similar spacelike foliations of the spacetime as those presented in [17] for the Reissner-Nordström case, except that the singularity is ring-shaped (see Figure 2). The foliations are obtained exactly as in the Reissner-Nordström case [17], by using the same Schwarz-Christoffel mappings. As in that case, to obtain maximal globally hyperbolic extensions, we don't take the maximal analytic continuations of the solutions for $a^2 + q^2 \geq m^2$ beyond the Cauchy horizons. To avoid these horizons, we limit the foliations to globally hyperbolic regions containing the exterior universe.

5. The significance of the analytic extension at the singularity

The analytic extension beyond the singularity obtained here completes the series of results obtained for the Schwarzschild [16] and Reissner-Nordström [17] solutions. As in those simpler cases, it becomes clear that the singularity can coexist with the geometric and topological structures of the spacetime, in

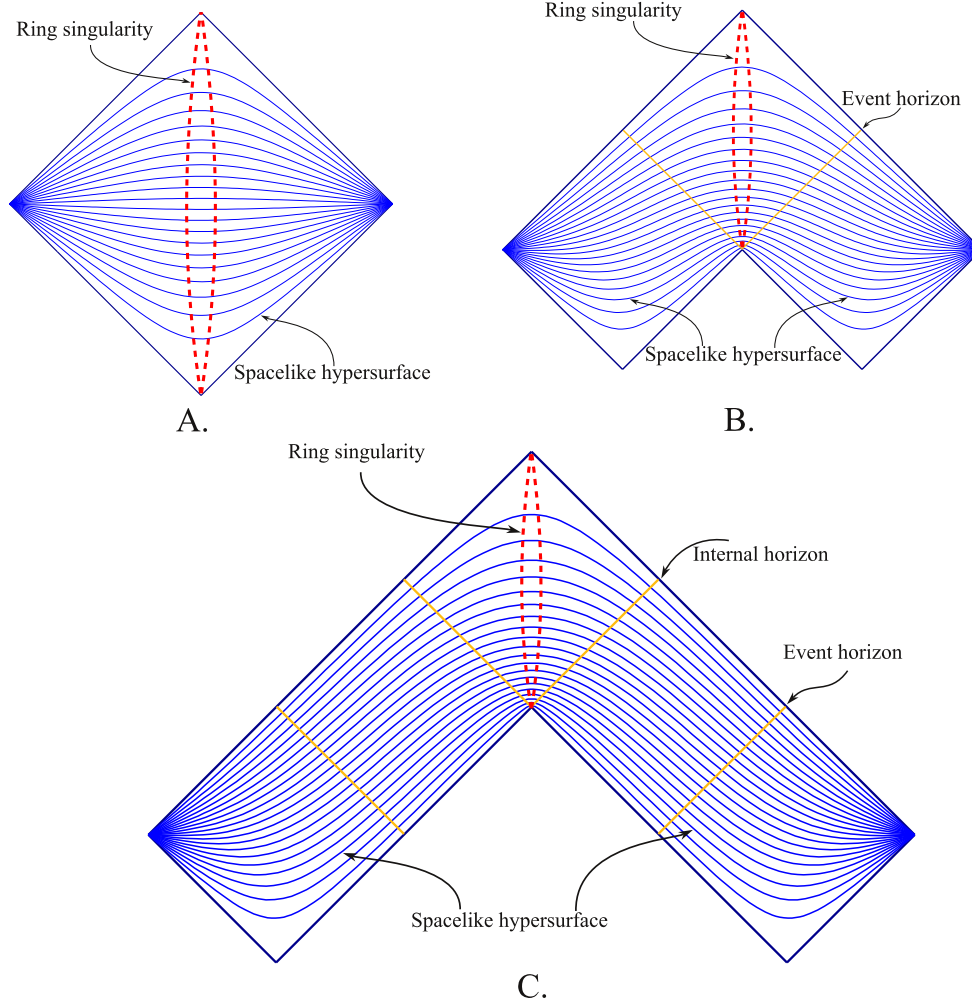


FIGURE 2. **A.** Space-like foliation of the naked Kerr-Newman solution ($a^2 + q^2 > m^2$). **B.** Space-like foliation of the extremal Kerr-Newman solution with $a^2 + q^2 = m^2$. **C.** Space-like foliation of the non-extremal Kerr-Newman solution ($a^2 + q^2 < m^2$).

a way which doesn't destroy the information contained in the fields. As in the other cases, we can extrapolate for the case when the black hole is not eternal, *e.g.* when it evaporates. This is because the Kerr-Newman solution is, according to the no-hair theorem, representative for all kinds of black holes.

The fact that the fundamental tensor is allowed to become degenerate is not a problem, because, as shown in [13, 14, 15, 18, 19, 21, 20, 22, 23], we have now the mathematical apparatus to deal with this kind of singularities.

In conclusion, despite the singularities present inside the black holes, there is no reason to consider that the Kerr-Newman black holes destroy causality, the evolution equations and the information conservation. Moreover, we

obtained charged singularities with smooth electromagnetic potential, leading to models of charged particles for which the electromagnetic potential and field are non-singular. The Kerr-Newman black holes are the most general stationary solution. The no-hair theorem makes them typical for our universe. They are typical even for the evaporating black holes, because the foliations presented here allow smooth modifications of the parameters m , q , and a , while preserving the topology. This is why we can be more optimistic about the singularities of the black holes in general.

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