

BOUNDARY VALUE PROBLEMS OF A HIGHER ORDER NONLINEAR DIFFERENCE EQUATION

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We study a higher order nonlinear difference equation. By making use of the critical point theory, some sufficient conditions for the existence of the solution to the boundary value problems are obtained.

Keywords: Existence, boundary value problems, higher order, Saddle Point Theorem, critical point theory

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1. Introduction

Recently, the theory of nonlinear difference equations have widely occurred as the mathematical models describing real life situations in computer science, electrical circuit analysis, economics, neural networks, ecology, cybernetics, biological systems, matrix theory, combinatorial analysis, optimal control, probability theory, and population dynamics. These studies cover many of the branches of nonlinear difference equations, such as stability, attractivity, periodicity, homoclinic solutions, oscillation, and boundary value problems, see [2],[4],[5], [7],[8],[9],[10],[11], [12],[18],[19],[20],[21],[22],[23],[24],[25],[25],[26], [30],[31],[32] and the references therein. For the general background of difference equations, one can refer to the monographs [1],[3].

Below \mathbf{N} , \mathbf{Z} and \mathbf{R} denote the sets of all natural numbers, integers and real numbers respectively. T is a positive integer. For any $a, b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \dots\}$, $\mathbf{Z}(a, b) = \{a, a+1, \dots, b\}$ when $a \leq b$. Besides, $*$ denotes the transpose of a vector.

The present paper considers the following higher order nonlinear difference equation

$$\sum_{i=0}^n r_i(x_{k-i} + x_{k+i}) = f(k, x_{k+M}, \dots, x_k, \dots, x_{k-M}), n \in \mathbf{N}, k \in \mathbf{Z}(1, T), \quad (1)$$

with boundary value conditions

$$x_{1-m} = x_{2-m} = \dots = x_0 = 0, x_{T+1} = x_{T+2} = \dots = x_{T+m} = 0, \quad (2)$$

where r_i is a real number, M is a given nonnegative integer, $m = \max\{n, M\}$, $f \in C(\mathbf{R}^{2M+2}, \mathbf{R})$.

When $n = 1, r_0 = -1, r_1 = 1$ and $M = 1$, (1) can be reduced to the following second order difference equation

$$\Delta^2 x_{k-1} = f(k, x_{k+1}, x_k, x_{k-1}), k \in \mathbf{Z}(1, T). \quad (3)$$

Equation (3) can be seen as an analogue discrete form of the following second order differential equation

$$\frac{d^2 x}{dt^2} = f(t, x(t+1), x(t), x(t-1)), t \in [1, T]. \quad (4)$$

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Equations similar in structure to (4) arise in the study of the periodic solutions, homoclinic orbits of functional differential equations [13],[14],[15],[16],[17],[27],[29].

In 1994, Ahlbrandt and Peterson [2] gave a formulation of generalized zeros and (n, n) -disconjugacy for even order formally self-adjoint scalar difference equation

$$\sum_{i=0}^n \Delta^i (\gamma_i(k-i) \Delta^i u(k-i)) = 0, \quad (5)$$

and Peil, Peterson [28] studied the asymptotic behavior of solutions of (5) with $\gamma_i(k) \equiv 0$ for $1 \leq i \leq n-1$. Anderson [4] in 1998 considered (5) for $k \in \mathbf{Z}(a)$ and obtained a formulation of generalized zeros and (n, n) -disconjugacy for (5).

In 2014, Deng [10] established some sufficient conditions for the boundary value problem of the following equation

$$\Delta^2 (p_{k-1} \Delta^2 u_{k-2}) - \Delta (q_k (\Delta u_{k-1})^\delta) + r_k u_k^\delta = f(k, u_k), \quad (6)$$

and gave some new results by using the critical point theory.

By Lyapunov-Schmidt reduction methods and computations of critical groups, Hu [18] in 2014 proved that the equation

$$\sum_{i=0}^n r_i (x_{k-i} + x_{k+i}) + f(k, x_k) = 0. \quad (7)$$

has four periodic solutions.

Recently, Liu, Zhang and Shi [23] considered the boundary value problem of the following difference equation

$$\Delta^2 (p_{n-1} \Delta^2 u_{n-2}) - \Delta (q_n (\Delta u_{n-1})^\delta) + r_n u_n^\delta = f(n, u_n), \quad (8)$$

by using Mountain Pass Lemma.

To the best of our knowledge, the results on boundary value problems of high order nonlinear difference equations are very scarce in the literature. What's more, (1) is a kind of difference equation containing both many advances and retardations. It has important analog in the continuous case of higher order functional differential equation for which the evolution of the function depends on its current state, its history, and its future as well. And the traditional ways of establishing the functional in [5],[18] are inapplicable to our case.

In this paper, we shall study the existence of solutions of the boundary value problem (1) with (2). First, we shall construct a suitable functional J such that solutions of the boundary value problem (1) with (2) correspond to critical points of J . Then, by using the Saddle Point Theorem, we obtain the existence of critical points of J . The motivation for the present work stems from the recent papers [10],[23].

Throughout this paper, we assume that the symbol $F'_i(u_1, \dots, u_i, \dots, u_T)$ defines the partial derivative of a function F on the i variable.

Our main results are the following theorems.

Theorem 1.1. Assume that r and F satisfy the following assumptions:

$$(r_1) \quad r_0 + \sum_{s=1}^n |r_s| < 0;$$

(F_1) there exists a function $F(t, u_M, \dots, u_0)$ which is continuously differentiable in the variable from u_M to u_0 for every $t \in \mathbf{Z}$ and satisfies

$$\sum_{i=-M}^0 F'_{2+M+i}(t+i, u_{M+i}, \dots, u_i) = f(t, u_M, \dots, u_0, \dots, u_{-M});$$

(F_2) there exist constants $c_1 > 0$, $c_2 > 0$ and $1 < \tau < 2$ such that

$$F(t, u_M, \dots, u_0) \leq c_1 \sum_{j=0}^M u_j^\tau + c_2, \forall (t, u_M, \dots, u_0) \in \mathbf{R}^{M+2}.$$

Then the boundary value problem (1) with (2) possesses at least one solution.

Example 1.1. In (1), let

$$F(k, x_{k+M}, \dots, x_k) = k \left(\sum_{i=0}^M x_{k+i}^2 \right)^{\frac{\alpha}{2}}$$

where $1 < \alpha < 2$. Since

$$\sum_{i=-M}^0 F'_{2+M+i}(k+i, x_{M+i}, \dots, x_i) = \alpha x_k \sum_{j=0}^M (k-j) \left(\sum_{i=0}^M x_{k+i-j}^2 \right)^{\frac{\alpha}{2}-1}.$$

It is easy to verify that assumptions (r_1), (F_1), (F_2) are satisfied. By 1.1, the boundary value problem (1) with (2) possesses at least one solution.

Corollary 1.1. Assume that r and F satisfy (r_1), (F_1) and the following assumption:
(F_3) there exist a constant $K_1 > 0$ such that for any $(t, y_M, \dots, y_0) \in \mathbf{R}^{M+2}$,

$$|F'_i(t, u_M, \dots, u_i, \dots, u_0)| \leq K_1, \quad i = 2, 3, \dots, M+2.$$

Then the boundary value problem (1) with (2) possesses at least one solution.

Remark 1.1. Assumption (F_3) implies that there exists a constant $K_2 > 0$ such that

$$(F'_3) \quad |F(t, u_M, \dots, u_0)| \leq K_2 + K_1 \sum_{j=0}^M |u_j|, \forall (t, u_M, \dots, u_0) \in \mathbf{R}^{M+2}.$$

Theorem 1.2. Assume that r and F satisfy (F_1), (F_3) and the following assumptions:

$$(r_2) \quad \sum_{s=0}^n r_s = 0;$$

$$(F_4) \quad \text{for every } t \in \mathbf{R}, F(t, u_M, \dots, u_0) \rightarrow +\infty \text{ as } \left(\sum_{i=0}^M u_i^2 \right)^{\frac{1}{2}} \rightarrow +\infty.$$

Then the boundary value problem (1) with (2) possesses at least one solution.

For basic knowledge of variational methods, the reader is referred to [27],[29].

2. Variational structure

Our main tool is the critical point theory. We shall establish suitable variational structure to study the existence of the boundary value problem (1) with (2). At first, we shall state some basic notations which will be used in the proofs of our main results.

Let \mathbf{R}^T be the real Euclidean space with dimension T . On one hand, we define the inner product on \mathbf{R}^T as follows:

$$\langle x, y \rangle = \sum_{j=1}^T x_j y_j, \forall x, y \in \mathbf{R}^T, \quad (9)$$

by which the norm $\|\cdot\|$ can be induced by

$$\|x\| = \left(\sum_{j=1}^T x_j^2 \right)^{\frac{1}{2}}, \forall x \in \mathbf{R}^T. \quad (10)$$

On the other hand, we define the norm $\|\cdot\|_s$ on \mathbf{R}^T as follows:

$$\|x\|_s = \left(\sum_{j=1}^T |x_j|^s \right)^{\frac{1}{s}},$$

for all $x \in \mathbf{R}^T$ and $s > 1$.

Since $\|x\|_s$ and $\|x\|_2$ are equivalent, there exist constants k_1, k_2 such that $k_2 \geq k_1 > 0$, and

$$k_1 \|x\|_2 \leq \|x\|_s \leq k_2 \|x\|_2, \forall x \in \mathbf{R}^T. \quad (11)$$

For the boundary value problem (1) with (2), consider the functional J defined on \mathbf{R}^T as follows:

$$J(x) := \frac{1}{2} \sum_{k=1}^T \sum_{i=0}^n r_i (x_{k-i} + x_{k+i}) x_k - \sum_{k=1}^T F(k, x_{k+M}, \dots, x_k), \quad (12)$$

where

$$\sum_{i=-M}^0 F'_{2+M+i}(k+i, u_{M+i}, \dots, u_i) = f(k, u_M, \dots, u_0, \dots, u_{-M}),$$

and

$$x_{1-n} = x_{2-M} = \dots = x_0 = 0, \quad x_{T+1} = x_{T+2} = \dots = x_{T+M} = 0.$$

It is easy to see that $J \in C^1(\mathbf{R}^T, \mathbf{R})$ and for any $x = \{x_k\}_{k=1}^T = (x_1, x_2, \dots, x_T)^*$, by using $x_{1-M} = x_{2-M} = \dots = x_0 = 0$, $x_{T+1} = x_{T+2} = \dots = x_{T+M} = 0$, we can compute the partial derivative as

$$\frac{\partial J}{\partial x_k} = \sum_{i=0}^n r_i (x_{k-i} + x_{k+i}) - f(k, x_{k+M}, \dots, x_k, \dots, x_{k-M}), \forall k \in \mathbf{Z}(1, T).$$

Thus, x is a critical point of J on \mathbf{R}^T if and only if

$$\sum_{i=0}^n r_i (x_{k-i} + x_{k+i}) = f(k, x_{k+M}, \dots, x_k, \dots, x_{k-M}), \forall k \in \mathbf{Z}(1, T).$$

We reduce the existence of the boundary value problem (1) with (2) to the existence of critical points of J on \mathbf{R}^T . That is, the functional J is just the variational framework of the boundary value problem (1) with (2).

For all $x \in \mathbf{R}^T$, J can be rewritten as

$$J(x) = \frac{1}{2} \langle Px, x \rangle - \sum_{k=1}^T F(k, x_{k+M}, \dots, x_k),$$

where $x = \{x_k\} \in \mathbf{R}^T$, $x_k = (x_1, x_2, \dots, x_T)^*$, $k \in \mathbf{Z}(1, T)$, and

$$-P = \begin{pmatrix} 2r_0 & r_1 & r_2 & \cdots & r_{n-1} & r_n & 0 & 0 & \cdots & 0 & r_n & r_{n-1} & \cdots & r_2 & r_1 \\ r_1 & 2r_0 & r_1 & \cdots & r_{n-2} & r_{n-1} & r_n & 0 & \cdots & 0 & 0 & r_n & \cdots & r_3 & r_2 \\ r_2 & r_1 & 2r_0 & \cdots & r_{n-3} & r_{n-2} & r_{n-1} & r_n & \cdots & 0 & 0 & 0 & \cdots & r_4 & r_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r_2 & r_3 & r_4 & \cdots & 0 & 0 & 0 & 0 & \cdots & r_n & r_{n-1} & r_{n-2} & \cdots & 2r_0 & r_1 \\ r_1 & r_2 & r_3 & \cdots & r_n & 0 & 0 & 0 & \cdots & r_{n-1} & r_{n-2} & r_{n-3} & \cdots & r_1 & 2r_0 \end{pmatrix}$$

is a $T \times T$ matrix. Assume that the eigenvalues of P are $\lambda_1, \lambda_2, \dots, \lambda_T$ respectively.

By [6], the eigenvalues of P are

$$\lambda_j = -2r_0 - \sum_{l=1}^n r_l \left\{ \exp i \frac{2j\pi}{T} \right\}^l - \sum_{l=1}^n r_l \left\{ \exp i \frac{2j\pi}{T} \right\}^{T-l} = -2 \sum_{l=0}^n r_l \cos \left(\frac{2jl\pi}{T} \right), \quad (13)$$

where $j = 1, 2, \dots, T$.

By (13), it is clear that

$$-2r_0 - 2 \sum_{l=1}^n |r_l| \leq \lambda_j \leq -2r_0 + 2 \sum_{l=1}^n |r_l|, \quad j = 1, 2, \dots, T. \quad (14)$$

Let V and W be Banach spaces, and $U \subset V$ be an open subset of V . A function $f : U \rightarrow W$ is called Fréchet-differentiable at $x \in U$ if there exists a bounded linear operator $A_x : V \rightarrow W$ such that

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - A_x(h)\|_W}{\|h\|_V} = 0.$$

We write $Df(x) = A_x$ and call it the Fréchet derivative of f at x .

For convenience, we identify $x \in \mathbf{R}^T$ with $x = (x_1, x_2, \dots, x_T)^*$.

Definition 2.1. Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E . J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\{x^{(n)}\}_{n \in \mathbf{N}} \subset E$ for which $\{J(x^{(n)})\}_{n \in \mathbf{N}}$ is bounded and $J'(x^{(n)}) \rightarrow 0$ ($n \rightarrow \infty$) possesses a convergent subsequence in E .

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

3. Main lemmas

In this section, we give two lemmas which will play important roles in the proofs of our main results.

Lemma 3.1. (*Saddle Point Theorem* [29]) Let E be a real Banach space, $E = E_1 \oplus E_2$, where $E_1 \neq \{0\}$ and is finite dimensional. Assume that $J \in C^1(E, \mathbf{R})$ satisfies the P.S. condition and

(J_1) there exist constants $\mu, \rho > 0$ such that

$$J|_{\partial B_\rho \cap E_1} \leq \mu;$$

(J_2) there exists $e \in B_\rho \cap E_1$ and a constant $\omega \geq \mu$ such that $J|_{e+E_2} \geq \omega$. Then J possesses a critical value $c \geq \omega$, where

$$c = \inf_{h \in \Gamma} \max_{x \in \bar{B}_\rho \cap E_1} J(h(x)),$$

$$\Gamma = \{h \in C(\bar{B}_\rho \cap E_1, E) \mid h|_{\partial B_\rho \cap E_1} = id\}$$

and id denotes the identity operator.

Lemma 3.2. Assume that (r_2) and $(F_1), (F_3), (F_4)$ are satisfied. Then the functional J satisfies the P.S. condition.

Proof. It is easy to see that 0 is an eigenvalue of P and $\xi = \frac{1}{\sqrt{T}}(1, 1, \dots, 1)^* \in \mathbf{R}^T$ is an eigenvector of P corresponding to 0. Let $\lambda_1, \lambda_2, \dots, \lambda_{T-1}$ be the other eigenvalues of P . Applying matrix theory, it is obvious that $\lambda_j > 0$ for all $j \in \mathbf{Z}(1, T-1)$.

Let

$$\lambda_{\max} = \max \{\lambda_j \mid j = 1, 2, \dots, T-1\}, \quad (15)$$

$$\lambda_{\min} = \min \{\lambda_j \mid j = 1, 2, \dots, T-1\}. \quad (16)$$

Set

$$Z = \ker P = \{x \in \mathbf{R}^T \mid Px = 0 \in \mathbf{R}^T\}.$$

Then

$$Z = \{x \in \mathbf{R}^T \mid x = \{c\}, c \in \mathbf{R}\}.$$

Let Y be the complement of Z . Let $\{x^{(n)}\}_{n \in \mathbf{N}} \subset \mathbf{R}^T$ be such that $\{J(x^{(n)})\}_{n \in \mathbf{N}}$ is bounded and $J'(x^{(n)}) \rightarrow 0$ as $n \rightarrow \infty$. Then there exists a positive constant D such that

$$-D \leq |J(x^{(n)})| \leq D, \forall n \in \mathbf{N}.$$

Let $x^{(n)} = y^{(n)} + z^{(n)} \in Y + Z$. First, on one hand, for n large enough, it follows from

$$-\|x\| \leq \langle -J'(x^{(n)}), x \rangle = -\langle Px^{(n)}, x \rangle + \sum_{k=1}^T f(k, x_{k+M}^{(n)}, \dots, x_k^{(n)}, \dots, x_{k-M}^{(n)}) x_k,$$

with (F_1) and (F_3) that

$$\begin{aligned} \langle Px^{(n)}, y^{(n)} \rangle &\leq \sum_{k=1}^T f(k, x_{k+M}^{(n)}, \dots, x_k^{(n)}, \dots, x_{k-M}^{(n)}) y_k^{(n)} + \|y^{(n)}\| \\ &\leq K_1 \sum_{k=1}^T |y_k^{(n)}| + \|y^{(n)}\| \\ &\leq [(K_1 + 1)\sqrt{T} + 1] \|y^{(n)}\|. \end{aligned}$$

On the other hand, it is easy to see that

$$\langle Px^{(n)}, y^{(n)} \rangle = \langle Py^{(n)}, y^{(n)} \rangle \geq \lambda_{\min} \|y^{(n)}\|^2.$$

Thus, we have

$$\lambda_{\min} \|y^{(n)}\|^2 \leq [(K_1 + 1)\sqrt{T} + 1] \|y^{(n)}\|.$$

The above inequality implies that $\{y^{(n)}\}_{n \in \mathbf{N}}$ is bounded.

Next, we shall prove that $\{z^{(n)}\}_{n \in \mathbf{N}}$ is bounded. It comes from

$$D \geq -J(x^{(n)}) = -\frac{1}{2} \langle Px^{(n)}, x^{(n)} \rangle + \sum_{k=1}^T F(k, x_{k+M}^{(n)}, \dots, x_k^{(n)})$$

$$\begin{aligned}
&= -\frac{1}{2} \langle Px^{(n)}, x^{(n)} \rangle + \sum_{k=1}^T \left[F\left(k, x_{k+M}^{(n)}, \dots, x_k^{(n)}\right) - F\left(k, z_{k+M}^{(n)}, \dots, z_k^{(n)}\right) \right] \\
&\quad + \sum_{k=1}^T F\left(k, z_{k+M}^{(n)}, \dots, z_k^{(n)}\right)
\end{aligned}$$

that

$$\begin{aligned}
&\sum_{k=1}^T F\left(k, z_{k+M}^{(n)}, \dots, z_k^{(n)}\right) \\
&\leq D + \frac{1}{2} \langle Py^{(n)}, y^{(n)} \rangle + \sum_{k=1}^T \left| F\left(k, x_{k+M}^{(n)}, \dots, x_k^{(n)}\right) - F\left(k, z_{k+M}^{(n)}, \dots, z_k^{(n)}\right) \right| \\
&\leq D + \frac{1}{2} \lambda_{\max} \|y^{(n)}\|^2 + \sum_{k=1}^T \left| \frac{\partial F\left(k, z_{k+M}^{(n)} + \alpha y_{k+M}^{(n)}, \dots, z_k^{(n)} + \alpha y_k^{(n)}\right)}{\partial u_2} \cdot y_{k+M}^{(n)} + \dots \right. \\
&\quad \left. + \frac{\partial F\left(k, z_{k+M}^{(n)} + \alpha y_{k+M}^{(n)}, \dots, z_k^{(n)} + \alpha y_k^{(n)}\right)}{\partial u_{M+1}} \cdot y_k^{(n)} \right| \\
&\leq D + \frac{1}{2} \lambda_{\max} \|y^{(n)}\|^2 + K_1 T \|y^{(n)}\|,
\end{aligned}$$

where $\alpha \in (0, 1)$. It follows that

$$\left\{ \sum_{k=1}^T F\left(k, z_{k+M}^{(n)}, \dots, z_k^{(n)}\right) \right\}$$

is bounded.

Assumption (F_4) implies that $\{z^{(n)}\}$ is bounded. Assume, for the sake of contradiction, that $\|z^{(n)}\| \rightarrow +\infty$ as $n \rightarrow \infty$. Since there exist $c^{(n)} \in \mathbf{R}^T$, $k \in \mathbf{N}$, such that $z^{(n)} = (c^{(n)}, c^{(n)}, \dots, c^{(n)})^* \in \mathbf{R}^T$, then

$$\|z^{(n)}\| = \left(\sum_{k=1}^T |z_k^{(n)}|^2 \right)^{\frac{1}{2}} = \left(\sum_{k=1}^T |c^{(n)}|^2 \right)^{\frac{1}{2}} = \sqrt{T} |c^{(n)}| \rightarrow +\infty$$

as $n \rightarrow \infty$. Since

$$F\left(k, z_{k+M}^{(n)}, \dots, z_k^{(n)}\right) = F\left(k, 0, \dots, c^{(n)}\right), 1 \leq k \leq T,$$

then $F\left(k, z_{k+M}^{(n)}, \dots, z_k^{(n)}\right) \rightarrow +\infty$ as $n \rightarrow \infty$. This contradicts the fact that

$$\left\{ \sum_{k=1}^T F\left(k, z_{k+M}^{(n)}, \dots, z_k^{(n)}\right) \right\}$$

is bounded and hence the proof of Lemma 3.2 is completed. \square

4. Proof of the main results

In this section, we shall prove that Theorems 1.1, 1.2 hold respectively via the variational methods.

Proof of Theorem 1.1. For any $x = (x_1, x_2, \dots, x_T)^* \in \mathbf{R}^T$, combining with (F_2) , it is easy to see that

$$\begin{aligned} J(x) &= \frac{1}{2} \langle Px, x \rangle - \sum_{k=1}^T F(k, x_{k+M}, \dots, x_k) \\ &\geq \frac{1}{2} \lambda_{\min} \|x\|^2 - c_1 \sum_{k=1}^T \sum_{j=0}^M |x_{k+j}|^\tau - c_2 T \\ &= \frac{1}{2} \lambda_{\min} \|x\|^2 - c_1 \sum_{j=0}^M \sum_{k=1}^T |x_{k+j}|^\tau - c_2 T \\ &\geq \frac{1}{2} \lambda_{\min} \|x\|^2 - c_1 (M+1) k_2^\tau \|x\|^\tau - c_2 T \rightarrow +\infty \end{aligned}$$

as $\|x\| \rightarrow +\infty$. By the continuity of $J(x)$, we have from the above inequality that there exist lower bounds of values of the functional $J(x)$. And this implies that $J(x)$ attains its minimal value at some point which is just the critical point of $J(x)$ with the finite norm. The proof of Theorem 1.1 is finished. \square

Remark 4.1. Due to 1.1, the conclusion of Corollary 1.1 is obviously true.

Proof of Theorem 1.2. By (r_2) , we have that the matrix P is positive definite. First, it follows from Lemma 3.2 that J satisfies the P.S. condition. Next, we shall prove the conditions (J_1) and (J_2) . For any $y \in Y$, by (F'_3) ,

$$\begin{aligned} -J(y) &= -\frac{1}{2} \langle Py, y \rangle + \sum_{k=1}^T F(k, y_{k+M}, \dots, y_k) \\ &\leq -\frac{1}{2} \lambda_{\min} \|y\|^2 + TK_2 + K_1 \sum_{k=1}^T \sum_{j=0}^M |y_{k+j}| \\ &\leq -\frac{1}{2} \lambda_{\min} \|y\|^2 + TK_2 + K_1 (M+1) T \|y\| \rightarrow -\infty \end{aligned}$$

as $\|y\| \rightarrow +\infty$. Therefore, it is easy to see that the condition (J_1) is satisfied. For any $z \in Z$, $z = (z_1, z_2, \dots, z_T)^*$, there exists $c \in \mathbf{R}$ such that $z_k = c$, for all $n \in \mathbf{Z}(1, T)$. From (F_4) , we have that there exists a constant $\rho > 0$ such that $F(k, 0, \dots, c) > 0$ for $k \in \mathbf{Z}(1, T)$ and $|c| > \rho/\sqrt{M+1}$. Let $K_3 = \min_{n \in \mathbf{Z}(1, T), |c| \leq \rho/\sqrt{M+1}} F(k, 0, \dots, c)$, $K_4 = \min\{0, K_3\}$. Then

$$F(k, 0, \dots, c) \geq K_4, \forall (k, c, \dots, c) \in \mathbf{Z}(1, T) \times \mathbf{R}^{M+1}.$$

Therefore, we have

$$-J(z) = \sum_{k=1}^T F(k, z_{k+M}, \dots, z_k) = \sum_{k=1}^T F(k, 0, \dots, c) \geq TK_4, \forall z \in Z.$$

So all the assumptions of the Saddle Point Theorem are satisfied and the proof of Theorem 1.2 is complete. \square

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