

## GENERALIZED RATIONNAL EFFICIENCY IN MULTIOBJECTIVE PROGRAMMING

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*A problem that sometimes occurs in multiobjective optimization is the existence of a large set of Pareto-optimal solutions. Hence the decision making based on selecting a unique preferred solution becomes difficult. Considering models with rational B-efficiency relieves some of the burden from the decision maker by shrinking the solution set. This paper focuses on solving multiobjective optimization problems by introducing the concept of rational B-efficiency. In this paper, first some theoretical and practical aspects of rationally B-efficient solutions are discussed. Then an algorithm to generate a subset of Pareto-optimal solutions is presented which aims to offer a limited number of representative solutions to the decision maker.*

**Keywords:** Pareto, Nondominated, Efficiency, Multiobjective programming.

**MSC2010:** 90C29, 91B08, 90B50.

### 1. Introduction

The field of multiobjective programming has grown considerably in different directions in the setting of optimality conditions and duality theory since the 1980s. It has been enriched by the applications of various types of generalizations of convexity theory and many authors have worked in this field and contributed to the results and literature available. Pitea and Postolache, for example, recently introduced the concept of quasiinvexity in a geometric framework, see [12-15].

Several multiobjective optimization approaches exist that generate finite sets of Pareto optimal solutions, which can be overwhelming to the decision maker in the task of selecting the most appropriate solution to implement. The classical preference based methods are categorized as a priori methods, a posteriori methods, and interactive methods. A priori methods use decision maker preferences to bias the search of optimal solutions towards a preferred region,

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for example by changing the definition of dominance [10, 21], by weighting differently the objectives [19], by assigning reference values (goals) and priority levels to the objectives [20], by assuming a utility function describing the decision maker behaviour and interest in the alternative solutions [9]. A posteriori methods aim to generate a representative set of Pareto optimal solutions and the decision maker chooses the best one among them ([1], [11], [18], [4], [22]). Interactive methods allow the decision maker to guide the search by alternating optimization and preference articulation iteratively ([2], [7], [17], [9], [16]). In this paper, we focus on a priori techniques for attaining the decision maker preferences.

The paper is organized as follows. In Section 2, terminology is introduced and basic concepts are defined. The concept of rational  $B$ -efficiency is introduced in Section 3. Then, to make it practical, rational  $B$ -efficiency is defined in terms of vector inequalities. In Section 4, an algorithm is presented to generate a subset of Pareto-optimal solutions. Next, two numerical examples are provided to confirm the method. Finally, Section 5 concludes the paper.

## 2. Terminology

Throughout this article the following notations are used. Let  $\mathbb{R}^m$  be the Euclidean vector space and  $y', y'' \in \mathbb{R}^m$ .  $y' \leq y''$  denotes  $y'_i \leq y''_i$  for all  $i = 1, 2, \dots, m$ .  $y' < y''$  denotes  $y'_i < y''_i$  for all  $i = 1, 2, \dots, m$ .  $y' \leq y''$  denotes  $y' \leq y''$  but  $y' \neq y''$ .

Consider a decision problem defined as an optimization problem with  $m$  objective functions. For simplification we assume, without loss of generality, that the objective functions are to be minimized. The problem can be formulated as follows:

$$\begin{aligned} & \min (f_1(x), f_2(x), \dots, f_m(x)), \\ & \text{subject to } x \in X \end{aligned} \tag{1}$$

where  $x$  denotes a vector of decision variables selected from the feasible set  $X$  and  $f(x) = (f_1(x), f_2(x), \dots, f_m(x))$  is a vector function that maps the feasible set  $X$  into the objective (criterion) space  $\mathbb{R}^m$ . We refer to the elements of the objective space as outcome vectors. An outcome vector  $y$  is attainable if it expresses outcomes of a feasible solution, i.e.,  $y = f(x)$  for some  $x \in X$ . The set of all attainable outcome vectors will be denoted by  $Y = f(X)$ .

In single objective minimization problems, we compare the objective values at different feasible decisions to select the best decision. Decisions are ranked according to the objective values at those decisions and the decision resulting in the least smallest objective value is the most preferred decision. Similarly, to make the multiobjective optimization model operational, one needs to assume some solution concept specifying what it means to minimize multiobjective functions. The solution concepts are defined by the properties of the corresponding preference model. We assume that solution concepts depend

only on the evaluation of the outcome vectors while not taking into account any other solution properties not represented within the outcome vectors. Thus, we can limit our considerations to the preference model in the objective space  $Y$ . In the following, some basic concepts and definitions of preference relations are reviewed from [8].

**Definition 2.1.** Let  $y', y'' \in \mathbb{R}^m$  and let  $\preceq$  be a relation of weak preference defined on  $\mathbb{R}^m \times \mathbb{R}^m$ . The corresponding relations of strict preference  $\prec$  and indifference  $\simeq$  are defined as follows:

$$y' \prec y'' \Leftrightarrow (y' \preceq y'' \text{ and not } y'' \preceq y'), \quad (2)$$

$$y' \simeq y'' \Leftrightarrow (y' \preceq y'' \text{ and } y'' \preceq y'). \quad (3)$$

**Definition 2.2.** Preference relations satisfying the following axioms are called rational preference relations:

1. Reflexivity: for all  $y \in \mathbb{R}^m$ :  $y \preceq y$ .
2. Transitivity: for all  $y', y'', y''' \in \mathbb{R}^m$ :  $y' \preceq y''$  and  $y'' \preceq y''' \Rightarrow y' \preceq y'''$ .
3. Strict monotonicity: for all  $y \in \mathbb{R}^m$ :  $y - \epsilon e_i \prec y$  for  $\epsilon > 0$  where  $e_i$  denotes the  $i^{\text{th}}$  unit vector in  $\mathbb{R}^m$ .

The rational preference relations allow us to formalize the Pareto-optimal solution concept with the following definitions.

**Definition 2.3.** The outcome vector  $y' \in Y$  rationally dominates  $y'' \in Y$  iff  $y' \prec y''$  for all rational preference relations  $\preceq$ .

An outcome vector  $y$  is rationally nondominated if and only if there does not exist another outcome vector  $y'$  such that  $y'$  rationally dominates  $y$ . Analogously, a feasible solution  $x \in X$  is an efficient or Pareto-optimal solution of the multiobjective problem (1) if and only if  $y = f(x)$  is rationally nondominated.

### 3. Rational $B$ -efficiency

In this section, we will introduce a rational  $B$ -dominance relation to generate rationally  $B$ -efficient solutions. The following definitions are necessary notion for the solution concepts of interest in this paper.

**Definition 3.1.** Let  $A = \{1, 2, \dots, m\}$  be the index set of objective functions

$$f = (f_1, f_2, \dots, f_m),$$

and  $n$  be a positive integer such that  $n \leq m$ . A collection  $B = \{B_k \subset A : k = 1, 2, \dots, n\}$  is called a partition of  $A$  if  $\bigcup_{k=1}^n B_k = A$  and  $B_i \cap B_j = \emptyset$  for all  $i \neq j$ , where  $i, j \in \{1, 2, \dots, n\}$  and  $B_k$  is index set of objective functions in class  $k$ .

Let  $y \in Y$  and  $B = \{B_1, B_2, \dots, B_n\}$  be a partition of the set  $\{1, 2, \dots, m\}$ , we define

$$S_B(y) = \left( \sum_{j \in B_1} y_j, \sum_{j \in B_2} y_j, \dots, \sum_{j \in B_n} y_j \right).$$

**Definition 3.2.** Suppose that  $y', y'' \in Y$  are two outcome vectors, we say that  $y'$  rationally  $B$ -dominates  $y''$ , iff  $S_B(y') \prec S_B(y'')$  for all rational preference relations  $\preceq$ , and that denoted by  $y' \prec_{Br} y''$ .

**Definition 3.3.** Suppose that  $y', y'' \in Y$  are two outcome vectors, we say that  $y'$  is rationally  $B$ -nondominated, iff there does not exist  $y''$  such that  $y''$  rationally  $B$ -dominates  $y'$ .

**Definition 3.4.** We say that feasible solution  $x \in X$  is a rationally  $B$ -efficient solution of the multiobjective problem (1), iff  $y = f(x)$  is rationally  $B$ -nondominated.

Similar to the relation of rational  $B$ -dominance, we can define the relation of rational  $B$ -indifference,  $\simeq_{Br}$ , and the relation of rational weak  $B$ -dominance,  $\preceq_{Br}$ . We say that  $y' \simeq_{Br} y''$  iff  $S_B(y') \simeq S_B(y'')$  for all rational preference relations  $\preceq$ , and also  $y' \preceq_{Br} y''$  iff  $S_B(y') \preceq S_B(y'')$  for all rational preference relations  $\preceq$ . The relations  $\prec_{Br}$ ,  $\simeq_{Br}$  and  $\preceq_{Br}$  satisfy conditions (2-3). Moreover, relation  $\preceq_{Br}$  holds the properties of reflexivity, transitivity and strict monotonicity, thus it is a rational preference relation.

To make it practical, rational  $B$ -efficiency is defined in terms of vector inequalities. In order to do that, we define a certain preference relation.

**Definition 3.5.** Suppose that  $y', y'' \in Y$  are two outcome vectors. We define the relation  $\leq_{Bir}$  as follows:

$$y' \leq_{Bir} y'' \Leftrightarrow S_B(y') \leq S_B(y''). \quad (4)$$

Also, we can define the relations  $<_{Bir}$  and  $=_{Bir}$  as follows:

$$\begin{aligned} y' <_{Bir} y'' &\Leftrightarrow (y' \leq_{Bir} y'' \text{ and not } y'' \leq_{Bir} y'), \\ y' =_{Bir} y'' &\Leftrightarrow (y' \leq_{Bir} y'' \text{ and } y'' \leq_{Bir} y'). \end{aligned}$$

It is clear that the preference relation  $\leq_{Bir}$ , satisfies reflexivity, transitivity and monotonicity. This means that, the relation (4) is a rational preference relation. Note that when  $B_k = \{k\}$  for all  $k = 1, 2, \dots, m$ , the relation  $\leq_{Bir}$  becomes the relation Pareto.

In the following, we will discuss the relationship between two preference  $\preceq_{Br}$  and  $\leq_{Bir}$ .

**Theorem 3.1.** Suppose that  $y', y'' \in Y$  are two outcome vectors. We have

$$\begin{aligned} y' \preceq_{Br} y'' &\Leftrightarrow y' \leq_{Bir} y'', \\ y' \prec_{Br} y'' &\Leftrightarrow y' <_{Bir} y''. \end{aligned}$$

**Proof.** We only prove the first statement, the other statement is proved similarly. Obviously, the relation  $\preceq_{Br}$  implies  $\leq_{Bir}$ , because the relation  $\leq_{Br}$  is a rational preference relation. Conversely, Suppose that  $y' \leq_{Bir} y''$ . So

$$\sum_{j \in B_k} y'_j \leq \sum_{j \in B_k} y''_j,$$

for  $k = 1, 2, \dots, n$ . If  $\epsilon_j = \sum_{j \in B_k} y''_j - \sum_{j \in B_k} y'_j$ , due to property of strictly monotonic, we have

$$\begin{aligned} S_B(y') &= \left( \sum_{j \in B_1} y''_j - \epsilon_1, \sum_{j \in B_2} y''_j - \epsilon_2, \dots, \sum_{j \in B_n} y''_j - \epsilon_n \right) \\ &\preceq \left( \sum_{j \in B_1} y''_j, \sum_{j \in B_2} y''_j, \dots, \sum_{j \in B_n} y''_j \right) = S_B(y'') \end{aligned} \quad (5)$$

for any rational preference relation  $\preceq$ . Thus  $y' \preceq_{Br} y''$ .  $\square$

By applying Theorem 3.1 and Definition 3.5, we have the following statement.

**Corollary 3.1.** *Suppose that  $y', y'' \in Y$  are two outcome vectors. We have*

$$\begin{aligned} y' \preceq_{Br} y'' &\Leftrightarrow S_B(y') \leq S_B(y''), \\ y' \prec_{Br} y'' &\Leftrightarrow S_B(y') < S_B(y''). \end{aligned}$$

**Remark 3.1.** *If  $B_k = \{k\}$  for all  $k = 1, 2, \dots, m$ , we have Proposition 1.1 from [8].*

Note that Corollary 3.1 permits one to express rational  $B$ -efficiency for problem (1) in terms of the standard efficiency for the multiobjective problem with objectives  $S_B(f(x))$ ,

$$\min\{S_B(f(x)) : x \in X\}. \quad (6)$$

Engau and Wiecek [3] are proposed Pareto-optimal solutions of the multiobjective problem (6) to coordinate efficient solutions of subproblems.

**Corollary 3.2.** *Feasible solution  $x \in X$  is a rationally  $B$ -efficient solution of the multiobjective problem (1), if and only if it is an efficient solution of the multiobjective problem (6).*

**Proof.** Let  $x$  be a rationally  $B$ -efficient solution of the multiobjective problem (1) and suppose that  $x$  is not an efficient solution of the multiobjective problem (6). Then a feasible vector  $x'$  must exist such that  $S_B(f(x')) \leq S_B(f(x))$  or  $f(x') \prec_{Br} f(x)$  due to corollary 3.1. Hence,  $f(x')$  rationally  $B$ -dominates  $f(x)$ , which contradicts the rational  $B$ -efficiency of  $x$ . By the same method the after is trivial.  $\square$

Note that if  $x$  is a rationally  $B$ -efficient solution of multiobjective problem (1), then it is also Pareto-optimal solution for this problem. Therefore, to reduce Pareto-optimal solutions, we can use rationally  $B$ -efficient solutions.

#### 4. Reduce rationally $B$ -efficient solutions

In this section with integration classes, we introduce an algorithm to generate a subset of rationally efficient solutions which aims to provide a limited number of representative solutions to the decision maker.

**Definition 4.1.** Let  $E = \{E_1, E_2, \dots, E_n\}$  be a partition of  $\{1, 2, \dots, m\}$  and  $I = \{I_1, I_2, \dots, I_t\}$  be a partition of  $\{1, 2, \dots, n\}$ . The generated partition  $B = \{B_1, B_2, \dots, B_t\}$  by  $E$  and  $I$  of  $\{1, 2, \dots, m\}$  is defined by

$$B_k = \bigcup_{j \in I_k} E_j \quad (k = 1, 2, \dots, t),$$

where  $I_k$  is index set classes in partition  $E$  should be integrated for class  $k$  in partition  $B$ .

**Example 4.1.** Suppose that  $E_1 = \{1, 2\}$ ,  $E_2 = \{3\}$ ,  $E_3 = \{4\}$ ,  $I_1 = \{1, 2\}$  and  $I_2 = \{3\}$ , The generated partition by  $E$  and  $I$  is  $B_1 = E_1 \cup E_2 = \{1, 2, 3\}$  and  $B_2 = E_3 = \{4\}$ .

In order to compare outcome vectors with preference relation  $\leq_{Bir}$ , we can use the linear cumulative map  $S_E$  with preference relation  $\leq_{Iir}$ .

**Theorem 4.1.** Let  $E, I$  and  $B$  be as Definition 4.1. For any two outcome vectors  $y', y'' \in Y$ , we have

$$y' <_{Bir} y'' \Leftrightarrow S_E(y') <_{Iir} S_E(y'').$$

**Proof.** Since  $B_k = \bigcup_{j \in I_k} E_j$ , for all  $k = 1, 2, \dots, t$ , we have

$$\begin{aligned} y' <_{Bir} y'' &\Leftrightarrow \sum_{i \in B_k} y'_i \leq \sum_{i \in B_k} y''_i, \quad (k = 1, 2, \dots, t) \\ &\Leftrightarrow \sum_{i \in \bigcup_{j \in I_k} E_j} y'_i \leq \sum_{i \in \bigcup_{j \in I_k} E_j} y''_i \\ &\Leftrightarrow \sum_{i \in I_k} \sum_{j \in E_i} y'_j \leq \sum_{i \in I_k} \sum_{j \in E_i} y''_j \\ &\Leftrightarrow S_E(y') <_{Iir} S_E(y''). \end{aligned}$$

□

In the following, we will investigate the relationship between preference relations  $\leq_{Eir}$  and  $\leq_{Bir}$ .

**Theorem 4.2.** Let  $E, I$  and  $B$  be as Definition 4.1. For any two outcome vectors  $y', y'' \in Y$ , if  $y' <_{Eir} y''$  then  $y' <_{Bir} y''$ .

The condition  $B_k = \bigcup_{j \in I_k} E_j$  is necessary to the above theorem. Since for example, if  $E_1 = \{1, 2\}$ ,  $E_2 = \{3\}$ ,  $B_1 = \{1\}$ ,  $B_2 = \{2, 3\}$ ,  $y' = (5.5, 4.5, 4)$  and  $y'' = (5, 5, 4.5)$ . We have  $S_E(y') = (10, 4)$ ,  $S_E(y'') = (10, 4.5)$ ,  $S_B(y') = (5.5, 8.5)$  and  $S_B(y'') = (5, 9.5)$ . Hence  $y' <_{Eir} y''$  but  $y' \not<_{Bir} y''$ .

**Remark 4.1.** If  $E_j = \{j\}$  for all  $j = 1, 2, \dots, m$  then for each partition  $B$

$$y' \leq_p y'' \Rightarrow y' <_{Bir} y'',$$

for any two outcome vectors  $y', y'' \in Y$ .

Following example shows that the converse of the above remark is not true in general.

**Example 4.2.** Let  $B_1 = \{1, 2\}$ ,  $B_2 = \{3, 4\}$ ,  $y' = (4, 1, 3, 2)$  and  $y'' = (5, 1, 2, 3)$ . We have  $S_B(y') = (5, 5)$  and  $S_B(y'') = (6, 5)$ , hence  $y' <_{Bir} y''$  But  $y' \not\leq_p y''$ .

Below, we will investigate the relationship between rationally  $B$ -efficient solutions and rationally  $E$ -efficient solutions.

**Theorem 4.3.** Let  $E, I$  and  $B$  be as Definition 4.1, also let  $y \in Y$  be a outcome vector. If  $y$  is rationally  $B$ -nondominated, then it is rationally  $E$ -nondominated.

**Proof.** Suppose that  $y$  is not rationally  $E$ -nondominated. Then there exists a vector  $y' \in Y$  such that  $y' \prec_{Er} y$ , so  $y' <_{Eir} y$  due to Theorem 3.1. By applying Theorem 4.2, we deduce that  $y' <_{Bir} y$ . This means that  $y' \prec_{Br} y$ , by Theorem 3.1.  $\square$

**Corollary 4.1.** Let  $E, I$  and  $B$  be as Definition 4.1 and let  $x \in X$  be a feasible solution. If  $x$  is a rationally  $B$ -efficient solution of multiobjective problem (1), then it is a rationally  $E$ -efficient solution of problem (1).

This result suggests that the set of rationally  $B$ -efficient solutions is contained within the set of rationally  $E$ -efficient solutions. In particular if  $E_j = \{j\}$  for  $j = 1, 2, \dots, m$ , then the set of rationally  $B$ -efficient solutions is contained within the set of rationally efficient solutions. By using Corollary 3.2, we can rewrite Corollary 4.1 as follows:

**Corollary 4.2.** Let  $E, I$  and  $B$  be as Definition 4.1. Also, let  $x$  be an efficient solution of the multiobjective problem (6), then it is an efficient solution of the multiobjective problem

$$\min\{S_E(f(x)) : x \in X\}. \quad (7)$$

According to Corollary 4.1 and Corollary 3.2, an algorithm is offered to generate rationally  $B$ -efficient solutions, whereby is reduced rationally  $B$ -efficient solutions of the multiobjective problem (1).

**Algorithm 4.1.**

Step 1: Determine a partition  $E = \{E_1, E_2, \dots, E_n\}$  of  $\{1, 2, \dots, m\}$  according to the decision maker.

Step 2: put  $t = 1$ .

Step 3: Consider the partition  $I$ , where  $I_1 = \{1, 2, \dots, t\}$ ,  $I_2 = \{t + 1\}$ ,  $\dots$ ,  $I_{n-t+1} = \{n\}$ .

*Step 4: Calculate the partition  $B$ , where  $B_k = \bigcup_{j \in I_k} E_j$  for  $k = 1, 2, \dots, n - t + 1$ .*

*Step 5: Solve the multiobjective problem (6).*

*Step 6: If the decision maker chooses the desired solution, stop.*

*Step 7: Otherwise put  $t = t + 1$ , if  $t > n$  stop, the model does not answer.*

*Step 8: Otherwise, go to Step 3.*

In the first iteration of Algorithm 4.1 rationally  $E$ -efficient solutions are computed, then these solutions are gradually reduced by rationally  $B$ -efficient solutions, in the next iterations. In particular, if  $E_j = \{j\}$  for all  $j = 1, 2, \dots, m$ , then Pareto-optimal solutions are computed in the first iteration of this algorithm.

In the following example, we investigate the effectiveness of rational  $B$ -dominance relation to generate the rationally  $B$ -efficient solutions which is subset of Pareto-optimal solutions. For this purpose, a large number of random solutions are generated for scalable test function. From this large set of solutions, efficient solutions or equivalently, the nondominated solutions with respect to Pareto and rational  $B$ -dominance are calculated.

**Example 4.3.** *The test problem considered is the  $FI[5]$ ,*

$$\begin{aligned} \min_{x \in R^2} \quad & y = \{f_1(x), f_2(x), f_3(x), f_4(x), f_5(x), f_6(x)\} \\ & f_1(x) = x_1^2 + (x_2 + 1)^2 \\ & f_2(x) = (x_1 - 0.5)^2 + (x_2 + 0.5)^2 \\ & f_3(x) = (x_1 - 1)^2 + x_2^2 \\ & f_4(x) = (x_1 + 1)^2 + x_2^2 \\ & f_5(x) = (x_1 - 0.5)^2 + (x_2 - 0.5)^2 \\ & f_6(x) = x_1^2 + (x_2 - 1)^2 \\ & x_1, x_2 \in [-1, 1]. \end{aligned}$$

In Figure 1 from 5000 random solutions, 2873 solutions (blue point) are rationally nondominated. 2537 solutions (green pentagram) are rationally  $B$ -nondominated, which are obtained by assuming  $B_1 = \{1, 2\}$ ,  $B_2 = \{3\}$ ,  $B_3 = \{4\}$ ,  $B_4 = \{5\}$  and  $B_5 = \{6\}$ , in the first iteration of the algorithm. 1887 solutions (yellow square) are rationally  $B$ -nondominated, which are obtained by assuming  $B_1 = \{1, 2, 3\}$ ,  $B_2 = \{4\}$ ,  $B_3 = \{5\}$  and  $B_4 = \{6\}$ , in the second iteration of the algorithm. In the third iteration of the algorithm, 623 solutions (red diamond) are rationally  $B$ -nondominated, which are obtained by assuming  $B_1 = \{1, 2, 3, 4\}$ ,  $B_2 = \{5\}$  and  $B_3 = \{6\}$ . 167 solutions (cyan circle) are rationally  $B$ -nondominated, which in the fourth iteration of the algorithm are obtained by assuming  $B_1 = \{1, 2, 3, 4, 5\}$  and  $B_2 = \{6\}$  and one solution (black star) is rationally  $B$ -nondominated, which is obtained by  $B_1 = \{1, 2, 3, 4, 5, 6\}$ , in the fifth iteration of the algorithm.



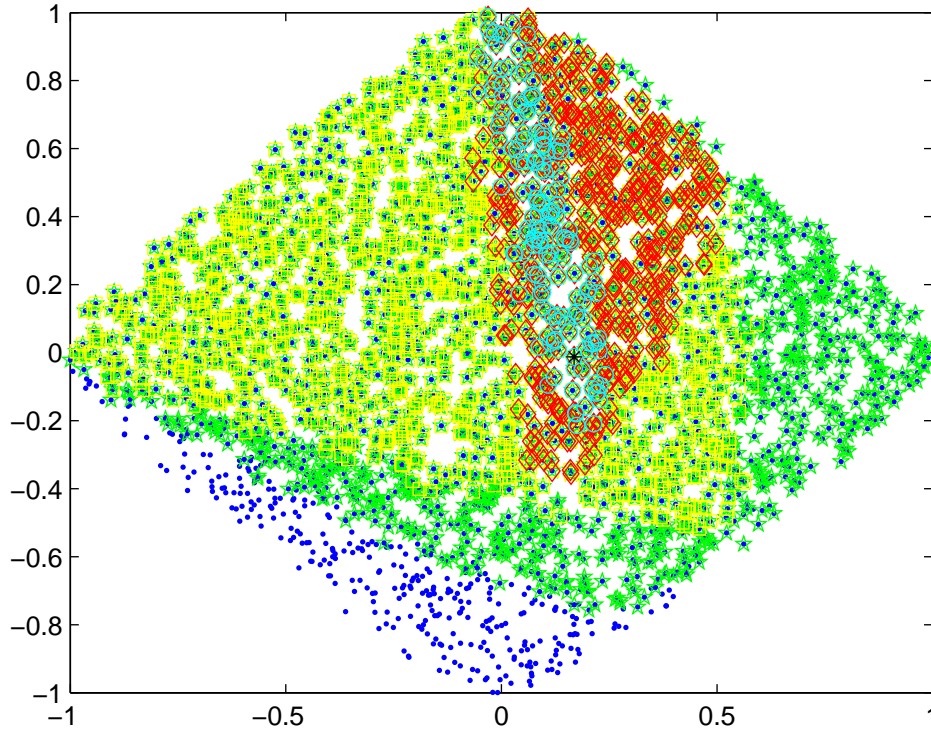


FIGURE 1. The Pareto and rational  $B$ -dominance fronts (objective space) for the  $F1$  problem (2 variables and 6 objectives) with  $E_1 = \{1, 2\}$ ,  $E_2 = \{3\}$ ,  $E_3 = \{4\}$ ,  $E_4 = \{5\}$  and  $E_5 = \{6\}$ .

To illustrate the importance of choosing  $E$ , example 4.3 is solved again with  $E_1 = \{1, 2\}$ ,  $E_2 = \{3, 4\}$  and  $E_3 = \{5, 6\}$ , which it is shown in Figure 2.

In Figure 2 from 5000 random solutions, 2923 solutions (blue point) are rationally nondominated. 447 solutions (green pentagram) are rationally  $B$ -nondominated, which are obtained by assuming  $B_1 = \{1, 2\}$ ,  $B_2 = \{3, 4\}$ , and  $B_3 = \{5, 6\}$ , in the first iteration of the algorithm. 151 solutions (yellow star) are rationally  $B$ -nondominated, which are obtained by assuming  $B_1 = \{1, 2, 3, 4\}$  and  $B_2 = \{5, 6\}$ , in the second iteration of the algorithm. Finally, one solution (red diamond) is rationally  $B$ -nondominated, which is obtained by assuming  $B_1 = \{1, 2, 3, 4, 5, 6\}$  in the third iteration of the algorithm.

## 5. Conclusion

In this paper, we introduced a new multiobjective optimization problem and obtained rationally  $B$ -efficient solutions of the original problem by seeking efficient solutions of this new problem. The concept of rational  $B$ -efficiency is obtained by rational preference relations on a certain cumulative vector. We

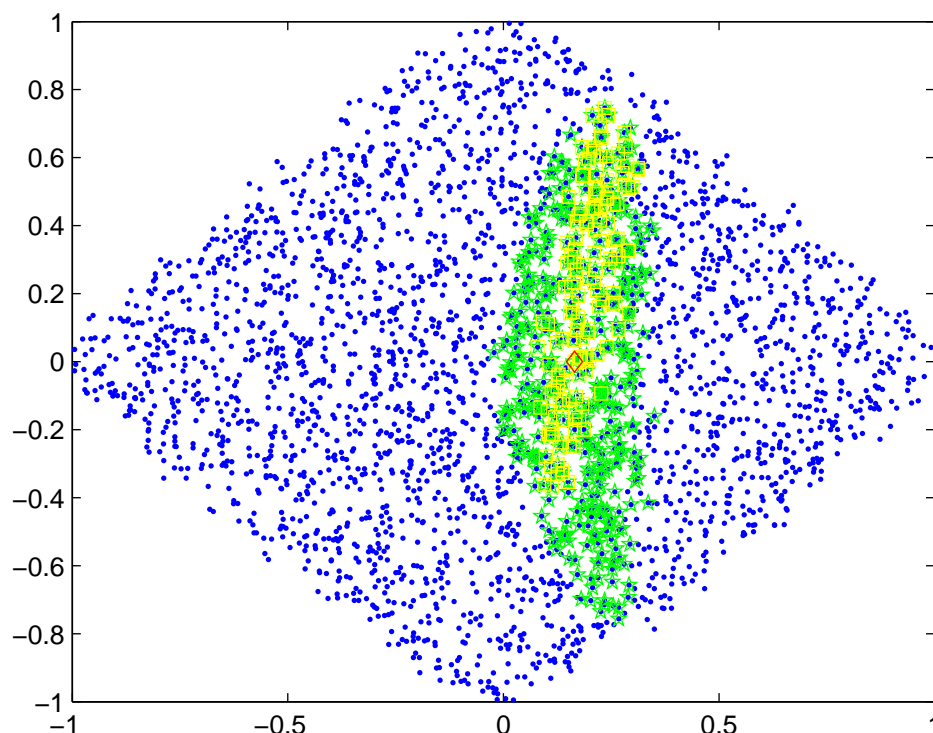


FIGURE 2. The Pareto and rational  $B$ -dominance fronts (objective space) for the  $F1$  problem (2 variables and 6 objectives) with  $E_1 = \{1, 2\}$ ,  $E_2 = \{3, 4\}$  and  $E_3 = \{5, 6\}$ .

examined the relationship between rationally  $B$ -efficient solutions and rationally  $E$ -efficient solutions, and we also proved that rationally  $B$ -nondominated points can be found as rationally nondominated points. Moreover, two experiments were carried out on randomly generated solutions in order to better compare the rational dominance and rational  $B$ -dominance. These experiments indicated that the size of the rational  $B$ -nondominated solution set is considerably smaller than the size of the rational nondominated solution set.

## REFERENCES

- [1] *P. Armand and C. Malivert*, Determination of the efficient set in multiobjective linear programming, *J. Optimiz. Theory App.*, **70**(1991), 467-489.
- [2] *K. I. Cho and S. H. Kim*, An improved Interactive hybrid method for the linear multi-objective knapsack problem, *Comput. Oper. Res.*, **24**(1997), No. 11, 991-1003.

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- [3] *A. Engau and M. M. Wiecek*, Interactive coordination of objective decompositions in multiobjective programming, *Manage. Sci.*, **54**(2008), 1350-1363.
  - [4] *J. P. Evans and R. E. Steuer*, A revised simplex method for multiple objective programs, *Math. Program.*, **5**(1973), 54-72.
  - [5] *M. Farina and P. Amato*, On the optimal solution definition for many criteria optimization problems, In *Proceedings of the NAFIPS-FLINT International Conference (2002)*, 233-238.
  - [6] *J. Horn*, *Multicriteria Decision Making and Evolutionary Computation*, Technical report, 1996.
  - [7] *H. Katagiri, M. Sakawa, K. Kato and I. Nishizaki*, Interactive multiobjective fuzzy random linear programming: Maximization of possibility and probability, *Eur. J. Oper. Res.*, **188**(2008), 530-539.
  - [8] *M. M. Kostreva and W. Ogryczak*, Linear optimization with multiple equitable criteria, *RAIRO Oper. Res.*, **33**(1999), No. 3, 275-297.
  - [9] *B. Malakooti*, A decision support system and a heuristic interactive approach for solving discrete multiple criteria problems, *IEEE T. Syst. Man. Cyb.*, **18**(1988), 273-284.
  - [10] *J. Molina, L. V. Santana, A. G. Hernandez-Diaz, C. A. Coello Coello and R. Caballero*, g-Dominance: Reference point based dominance for multiobjective metaheuristics, *Eur. J. Oper. Res.*, **197**(2009), 685-692.
  - [11] *N. Padhye, S. Kalia and K. Deb*, Multi-Objective Optimization and Multi-criteria Decision Making for FDM Using Evolutionary Approaches, *KanGAL Report 2009007*, December 2009.
  - [12] *A. Pitea*, On efficiency conditions for new constrained minimum problem, *U. Politeh. Buch. Ser A*, **71**(2009), No. 3, 61-68.
  - [13] *A. Pitea and M. Postolache*, Duality theorems for a new class of multitime multiobjective variational problems, *J. Glob. Optim.*, **54**(2012), No. 1, 47-58.
  - [14] *A. Pitea and M. Postolache*, Minimization of vectors of curvilinear functionals on the second order jet bundle. Necessary conditions, *Optim. Lett.*, **6**(2012), No. 3, 459-470.
  - [15] *A. Pitea and M. Postolache*, Minimization of vectors of curvilinear functionals on the second order jet bundle. Sufficient efficiency conditions, *Optim. Lett.*, **6**(2012), No. 8, 1657-1669.
  - [16] *B. Roy and D. Bouyssou*, Comparison of two decision-aid models applied to a nuclear power plant siting example, *Eur. J. Oper. Res.*, **25**(1986), 200-215.

- [17] *Steuer RE*, Multiple Criteria Optimization: Theory, Computation, and Application, Krieger Publishing: Malabar, 1986.
- [18] *V. Venkat, S. Jacobson and J. Stori*, A post-optimality analysis algorithm for multiobjective optimization, *Comput. Optim. Appl.*, **28**(2004), 357-372.
- [19] *J. B. Yang*, Multiple Criteria Decision Making Methods and Applications, Hunan Publishing House, Changsha PR China, 1996.
- [20] *J. B. Yang*, Minimax reference point approach and its application for multiobjective optimisation, *Eur. J. Oper. Res.*, **126**(2000), 541-556.
- [21] *E. Zio, P. Baraldi and N. Pedroni*, Optimal power system generation scheduling by multi- objective genetic algorithms with preferences, *Reliab. Eng. Syst. Safe.*, **94**(2009), 432-444.
- [22] *E. Zio and R. Bazzo*, A clustering procedure for reducing the number of representative solutions in the Pareto front of multiobjective optimization problems, *Eur. J. Oper. Res.*, **210**(2011), No. 3, 624-634.