

## ESSENTIAL CHARACTER CONTRACTIBILITY OF BANACH ALGEBRAS

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*In the present paper we introduce and study the concept of essential left  $\varphi$ -contractibility of a Banach algebra  $A$ , where  $\varphi \in \Delta(A)$ , the character space of  $A$ . An example is given to show that the class of essentially left  $\varphi$ -contractible Banach algebras is larger than that of left  $\varphi$ -contractible Banach algebras. We show that under certain conditions, left  $\varphi$ -contractibility and essential left  $\varphi$ -contractibility are equivalent. Moreover, we prove some hereditary properties and investigate left  $\varphi$ -contractibility for certain class of Banach algebras.*

**Keywords:** Essential left  $\varphi$ -contractibility, essential left character contractibility, left  $\varphi$ -contractibility, Banach algebra.

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### 1. Introduction

Let  $A$  be a Banach algebra and let  $X$  be a Banach  $A$ -bimodule. A derivation is a linear map  $D: A \rightarrow X$  such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in A).$$

For every  $x \in X$ , define  $ad_x$  by  $ad_x(a) = a \cdot x - x \cdot a$  ( $a \in A$ ). It is easily seen that  $ad_x$  is a derivation. Derivations of this form are called inner.

The notion of amenability for Banach algebras was introduced and studied by Johnson [8]. Ghahramani and Loy [4] introduced the notion of essential amenable Banach algebras and investigated the essential amenability of certain Banach algebras such as symmetric Segal algebras. The Banach algebra  $A$  is called essentially amenable if continuous derivations from  $A$  into the duals of neo-unital Banach  $A$ -bimodules are inner.

Let  $A$  be a Banach algebra and  $\varphi \in \Delta(A)$ . Kaniuth et al. [9] have recently introduced and studied the interesting notion of  $\varphi$ -amenability. In fact a Banach algebra  $A$  is called  $\varphi$ -amenable if for every Banach  $A$ -bimodule  $X$  with the left module action  $a \cdot x = \varphi(a)x$  ( $x \in X$ ), ( $a \in A$ ), every continuous derivation from  $A$  into  $X^*$  is inner. Furthermore,  $A$  is called character amenable if it has a bounded right approximate identity and it is  $\varphi$ -amenable for all  $\varphi \in \Delta(A)$  (see [13]).

In [14], Nasre-Isfahani and Nemati have introduced and studied the concept of essential  $\varphi$ -amenability and essential character amenability of Banach algebras.

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They showed that for certain Banach algebras,  $\varphi$ -amenability and essential  $\varphi$ -amenability are equivalent.

A Banach algebra  $A$  is called left  $\varphi$ -contractible if for every Banach  $A$ -bimodule  $X$  with the right module action  $a \cdot x = \varphi(a)x$  ( $a \in A, x \in X$ ), every continuous derivation from  $A$  into  $X$  is inner; in addition,  $A$  is called left character contractible if it is left  $\varphi$ -contractible for all  $\varphi \in \Delta(A) \cup \{0\}$ . These notions were recently introduced and studied by Hu, Monfared and Traynor [7] as right  $\varphi$ -contractibility and right character contractibility respectively; see also [12, 15].

The aim of the present work is to introduce and study the concept of essential left  $\varphi$ -contractibility and essential left character contractibility of Banach algebras. This paper is organized as follows:

In section 2 first by an example we show that essential left  $\varphi$ -contractibility is weaker than left  $\varphi$ -contractibility. We then prove that for certain Banach algebras, the two notions are equivalent.

Hereditary properties of essential left  $\varphi$ -contractibility are investigated in section 3. Moreover, in this section we characterize the essential left character contractibility of various Banach algebras related to locally compact groups.

In section 4, we investigate left character contractibility for certain class of Banach algebras consisting of Lau product  $A \times_{\theta} B$ , where  $\theta \in \Delta(B)$  and projective tensor product  $A \bar{\otimes} B$  for two Banach algebras  $A$  and  $B$ . Moreover, for a Banach algebra  $A$  and a Banach  $A$ -bimodule  $X$ , we study the essential left character contractibility of module extension  $A \oplus_1 X$ .

## 2. Essential left character contractibility

We start this section with the following:

Let  $A$  be a Banach algebra. An  $A$ -bimodule  $X$  is neo-unital if  $X = A \cdot X = X \cdot A$ , where  $A \cdot X = \{a \cdot x \mid a \in A, x \in X\}$  and  $X \cdot A = \{x \cdot a \mid a \in A, x \in X\}$ .

**Definition 2.1.** Let  $A$  be a Banach algebra and let  $\varphi \in \Delta(A)$ . We say that  $A$  is essentially left  $\varphi$ -contractible if for every neo-unital Banach  $A$ -bimodule  $X$  with the right module action  $x \cdot a = \varphi(a)x$  ( $a \in A, x \in X$ ), every continuous derivation from  $A$  into  $X$  is inner. We also say that  $A$  is essentially left 0-contractible if for every Banach  $A$ -bimodule  $X$  with the zero right action such that  $A \cdot X = X$  every continuous derivation from  $A$  into  $X$  is inner.

**Definition 2.2.** A Banach algebra  $A$  is called essentially left character contractible if it is essentially left  $\varphi$ -contractible for all  $\varphi \in \Delta(A) \cup \{0\}$ .

Similar definitions hold for essentially right  $\varphi$ -contractible and right character contractible Banach algebras.  $A$  is called character contractible if it is both left and right character contractible.

Clearly every left  $\varphi$ -contractible Banach algebra is essentially left  $\varphi$ -contractible but the converse is not true. The following example shows that there are essentially left  $\varphi$ -contractible Banach algebras which are not left  $\varphi$ -contractible.

**Example 2.1.** Let  $A$  be a nonzero Banach space. Define a product on  $A$  by  $ab=0$  ( $a, b \in A$ ). With this product  $A$  is a Banach algebra. Since  $A$  has no right identity, by Proposition 3.4 of [15],  $A$  is not left 0-contractible. We now show that  $A$  is essential left 0-contractible. To this end, suppose that  $X$  is a neo-unital Banach  $A$ -bimodule with the right module action  $x \cdot a = 0$  ( $a \in A, x \in X$ ) and  $D: A \rightarrow X$  is a continuous derivation. Let  $a \in A$ . Since  $X$  is a neo-unital Banach  $A$ -bimodule, there exist  $a' \in A$  and  $x' \in X$  such that  $D(a) = a' \cdot x'$ . Also there exist  $a'' \in A$  and  $x'' \in X$  such that  $x' = a'' \cdot x''$ . Hence,

$$D(a) = a' \cdot x' = a' \cdot (a'' \cdot x'') = (a' a'') \cdot x'' = 0.$$

So  $D=0$ . Therefore  $A$  is essential left 0-contractible.

Recall from [4] that a point derivation  $d$  at a character  $\varphi$  of an algebra  $A$  is a linear functional satisfying  $d(ab) = d(a)\varphi(b) + \varphi(a)d(b)$  ( $a, b \in A$ ). That is,  $d$  is a derivation into the  $A$ -bimodule  $\square$  with the module actions

$$a \cdot \lambda = \lambda \cdot a = \lambda \varphi(a) \quad (a \in A, \lambda \in \square). \quad (1)$$

**Proposition 2.1.** *Let  $A$  be a Banach algebra and let  $\varphi \in \Delta(A)$ . If  $A$  is essentially left  $\varphi$ -contractible, then any bounded point derivation at  $\varphi$  is trivial.*

*Proof.* Let  $A$  be essentially left  $\varphi$ -contractible and  $d: A \rightarrow \square$  be a point derivation at  $\varphi$ . Since  $d$  is a continuous derivation and  $\square$  is a neo-unital Banach  $A$ -bimodule by actions defined as (1), it follows that  $d=0$ .  $\square$

**Corollary 2.1.** *Suppose that  $A$  admits a non-zero continuous point derivation. Then  $A$  is not essentially left character contractible.*

To prove our next results we need to quote the following definition and theorem.

**Definition 2.3.** [7] Let  $A$  be a Banach algebra and  $\varphi \in \Delta(A)$ . A left [right]  $\varphi$ -diagonal for  $A$  is an element  $m$  of  $A \hat{\otimes} A$  such that

$$(i) \ m \cdot a = \varphi(a)m \ [a \cdot m = \varphi(a)m] \ \langle \varphi \otimes \varphi, m \rangle = \varphi(\pi(m)) = 1. \quad (i) \ (i; (a \in A))$$

If  $m$  is both left and right  $\varphi$ -diagonal, it is called a  $\varphi$ -diagonal.

**Theorem 2.1.** [2,5] (*Cohen-Hewitt factorization theorem*) *Let  $X$  be a left module over a Banach algebra  $A$  with a left approximate identity  $(e_\alpha)_\alpha$ . Then an element  $x \in X$  can be factorized as a product  $x = ax'$  (for some  $a \in A$  and  $x' \in X$ ) whenever  $\lim_\alpha e_\alpha x = x$ .*

**Proposition 2.3.** *Let  $A$  be a Banach algebra with a left approximate identity and let  $\varphi \in \Delta(A)$ . Then  $A$  is left  $\varphi$ -contractible if and only if  $A$  is essentially left  $\varphi$ -contractible.*

*Proof.* Clearly every  $\varphi$ -contractible Banach algebra is essentially  $\varphi$ -contractible. Conversely, suppose that  $A$  is essentially left  $\varphi$ -contractible. Consider the Banach  $A$ -bimodule  $A \bar{\otimes} A$  (the projective tensor product of  $A$ ) with the module actions given by  $a \cdot (b \otimes c) = ab \otimes c$  and  $(b \otimes c) \cdot a = \varphi(a)b \otimes c$  ( $a, b, c \in A$ ). Let  $(e_\alpha)_\alpha$  be a left approximate identity for  $A$ . For every  $a, b \in A$ , we have

$$a \otimes b = (\lim_\alpha e_\alpha a) \otimes b = \lim_\alpha e_\alpha \cdot (a \otimes b).$$

So, by Theorem 2.1,  $A \bar{\otimes} A$  is a neo-unital Banach  $A$ -bimodule. Let  $m_0 \in A \bar{\otimes} A$  be such that  $\langle \varphi \otimes \varphi, m_0 \rangle = 1$ , and  $D: A \rightarrow A \bar{\otimes} A$  define by  $D(a) = a \cdot m_0 - m_0 \cdot a$  ( $a \in A$ ). Clearly,  $\ker(\varphi \otimes \varphi)$  is a subset of  $\ker D$  is a continuous derivation and the image of  $\ker(\varphi \otimes \varphi)$ . From the essentially left  $\varphi$ -contractibility of  $A$ , it follows that there exists  $m_1 \in \ker(\varphi \otimes \varphi)$  such that  $D = ad_{m_1}$ . Let  $m = m_0 - m_1$ . Hence  $a \cdot m = \varphi(a)m$  and  $\langle \varphi \otimes \varphi, m \rangle = 1$  ( $a \in A$ ). Then  $m$  is a right  $\varphi$ -diagonal for  $A$ . So Theorem 6.3 of [7], implies that  $A$  is left  $\varphi$ -contractible (as right  $\varphi$ -contractible introduced in [7]).  $\square$

**Proposition 2.3.** *Let  $A$  be a Banach algebra with a left approximate identity. Then  $A$  is left 0-contractible if and only if  $A$  is essentially left 0-contractible.*

*Proof.* Suppose that  $A$  is essentially left 0-contractible. Let  $X = A \oplus_1 A$  (the  $l_1$ -direct sum of  $A$ ) be the Banach  $A$ -bimodule with the module actions given by

$$a \cdot (b, c) = (ab, ac) \text{ and } (b, c) \cdot a = (0, 0) \quad (a, b, c \in A).$$

Let  $(e_\alpha)_\alpha$  be a left approximate identity for  $A$ . Since for every  $a, b \in A$ ,  $(a, b) = \lim_\alpha e_\alpha \cdot (a, b)$ , by Theorem 2.1,  $X$  is a neo-unital Banach  $A$ -bimodule. By our assumption, the continuous derivation  $D: A \rightarrow X$ ,  $D(a) = (a, a)$ , is inner and hence  $D = ad_{(a_0, b_0)}$  for some  $(a_0, b_0) \in X$ . Thus  $a_0$  is a right identity for  $A$ . From Proposition 3.4 of [15], it follows that  $A$  is left 0-contractible. The converse is obvious.  $\square$

As a consequence of Propositions 2.2 and 2.3, we have the following theorem.

**Theorem 2.2.** *Let  $A$  be a Banach algebra with a left approximate identity. Then  $A$  is left character contractible if and only if  $A$  is essentially left character contractible.*

### 3. Hereditary properties of essential left $\varphi$ -contractibility

In this section we discuss hereditary properties of essential left  $\varphi$ -contractibility.

**Proposition 3.1.** *Let  $A$  and  $B$  be Banach algebras and suppose that  $h: A \rightarrow B$  is a continuous epimorphism. If  $\psi \in \Delta(B) \cup \{0\}$  and  $A$  is essentially left  $\psi \circ h$ -contractible, then  $B$  is essentially left  $\psi$ -contractible.*

*Proof.* Assume that  $A$  is essentially left  $\psi \circ h$ -contractible. Let  $D: B \rightarrow X$  be a continuous derivation for a neo-unital Banach  $B$ -bimodule  $X$  with the right action  $x \cdot b = \psi(b)x$  ( $b \in B, x \in X$ ). By the actions

$$a \bullet x = h(a) \cdot x, \quad x \bullet a = x \cdot h(a) \quad (a \in A, x \in X),$$

$X$  is a neo-unital Banach  $A$ -bimodule such that  $x \bullet a = x \cdot h(a) = \psi(h(a))x$  ( $a \in A, x \in X$ ). Let  $\bar{D} := D \circ h: A \rightarrow X$ . Clearly,  $\bar{D}$  is a continuous derivation and so there exists  $x \in X$  such that  $\bar{D}(a) = a \bullet x - x \bullet a$  ( $a \in A$ ). This yields that  $D$  is inner.

Therefore  $B$  is essentially left  $\psi$ -contractible.  $\square$

**Corollary 3.1.** *Let  $A$  and  $B$  be Banach algebras and suppose that  $h: A \rightarrow B$  is a continuous epimorphism. If  $A$  is essentially left character contractible, then  $B$  is essentially left character contractible.*

It is well known that  $A \bar{\otimes} B$ , the projective tensor product of  $A$  and  $B$  is a Banach algebra with respect to the canonical multiplication defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2 \otimes b_1 b_2) \quad (a_1, a_2 \in A, b_1, b_2 \in B).$$

For  $f \in A^*$  and  $g \in B^*$ , let  $f \otimes g$  denote the element of  $(A \bar{\otimes} B)^*$  satisfying  $f \otimes g(a \otimes b) = f(a)g(b)$  for all  $a \in A$  and  $b \in B$ . Then, with this notion,

$$\Delta(A \bar{\otimes} B) = \{\varphi \otimes \psi : \varphi \in \Delta(A), \psi \in \Delta(B)\}.$$

**Proposition 3.2.** *Let  $A$  be a Banach algebra,  $\varphi \in \Delta(A)$  and let  $I$  be a closed two-sided ideal of  $A$  with a left approximate identity which is also a left approximate identity for  $A$  such that  $I \not\subset \ker(\varphi)$ . Then  $A$  is essentially left  $\varphi$ -contractible if and only if  $I$  is essentially left  $\varphi|_I$ -contractible.*

*Proof.* Suppose that  $A$  is essentially left  $\varphi$ -contractible. Applying Proposition 3.8 of [15] and Proposition 2.2, we conclude that  $I$  is left  $\varphi|_I$ -contractible. Therefore  $I$  is essentially left  $\varphi|_I$ -contractible.

Conversely, assume that  $I$  is essentially left  $\varphi|_I$ -contractible and Consider the Banach  $I$ -bimodule  $A \bar{\otimes} A$  with the module actions given by

$$a \cdot (b \otimes c) = ab \otimes c \quad \text{and} \quad (b \otimes c) \cdot a = \varphi(a)b \otimes c \quad (a \in I, b, c \in A).$$

Let  $(e_\alpha)_\alpha$  be a left approximate identity for  $I$  which is also a bounded approximate identity for  $A$ . For every  $a, b \in A$ , we have

$$a \otimes b = (\lim_\alpha e_\alpha a) \otimes b = \lim_\alpha e_\alpha \cdot (a \otimes b).$$

So, by Theorem 2.1,  $A \bar{\otimes} A$  is a neo-unital Banach  $I$ -bimodule. Choose  $m_0 \in A \bar{\otimes} A$  such that  $\langle \varphi \otimes \varphi, m_0 \rangle = 1$ , and define the inner derivation  $D: I \rightarrow A \bar{\otimes} A$  by  $D(a) = a \cdot m_0 - m_0 \cdot a$  ( $a \in I$ ). By a method similar to the proof of Proposition 2.2, one can show that there exists  $m \in A \bar{\otimes} A$  such that  $a \cdot m = \varphi(a)m$  ( $a \in I$ ) and  $\langle \varphi \otimes \varphi, m \rangle = 1$ . So for every  $a \in A$ ,

$$a \cdot m = (\lim_{\alpha} e_{\alpha} a) \cdot m = \lim_{\alpha} e_{\alpha} a \cdot m = \lim_{\alpha} \varphi(e_{\alpha} a) m = \varphi(a) m.$$

It follows that  $m$  is a right  $\varphi$ -diagonal for  $A$ . Therefore Theorem 6.3 of [7], implies that  $A$  is left  $\varphi$ -contractible. Now Proposition 2.2 shows that  $A$  is essentially left  $\varphi$ -contractible.  $\square$

Let  $G$  be a locally compact group with the left Haar measure  $m_G$  and let  $L^1(G)$  be the group algebra,  $M(G)$  be the measure algebra and  $LUC(G)$  be the space of all left uniformly continuous functions on  $G$  (for more details see [6]).

**Theorem 3.1.** *Let  $G$  be a locally compact group. Then the following assertions are equivalent:*

- (i)  $L^1(G)^{**}$  is essentially left character contractible;
- (ii)  $LUC(G)^*$  is essentially left character contractible;
- (iii)  $M(G)$  is essentially left character contractible;
- (iv)  $L^1(G)$  is essentially left character contractible;
- (v)  $G$  is finite.

*Proof.* Since the restriction maps from  $L^1(G)^{**}$  onto  $LUC(G)^*$  and from  $LUC(G)^*$  onto  $M(G)$  are continuous epimorphisms, the equivalences (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii) follow from Corollary 3.1.

Also since  $M(G)$  and  $L^1(G)$  have bounded approximate identity, the equivalences (iii)  $\Leftrightarrow$  (iv)  $\Leftrightarrow$  (v) follows from Corollary 6.2 of [15] and Theorem 2.2.

(v)  $\Rightarrow$  (i). Assume that  $G$  is finite. By Corollary 6.5 of [15],  $L^1(G)^{**}$  is left character contractible and so it is essentially left character contractible.  $\square$

A linear subspace of  $L^1(G)$  is said to be a Segal algebra, and denoted by  $S(G)$ , if it satisfies the following conditions:

- (i)  $S(G)$  is dense in  $L^1(G)$ ;
- (ii)  $S(G)$  is a Banach space under some norm  $\|\cdot\|_S$  and  $\|f\| \leq C \|f\|_S$  for all  $f \in S(G)$  and for some constant  $C > 0$ ;
- (iii)  $S(G)$  is left translation invariant,  $\|\delta_x * f\|_S = \|f\|_S$  ( $f \in S(G)$ ,  $x \in G$ ), and the map  $x \mapsto \delta_x * f$  from  $G$  into  $S(G)$  is continuous for all  $f \in S(G)$ .

For more information about the Segal algebras see [1].

Let  $\hat{G}$  denote the dual group of  $G$  consisting of all continuous homomorphism from  $G$  into the circle group  $\mathbb{T}$ . For  $\rho \in \hat{G}$ , define  $\varphi_{\rho} \in \Delta(L^1(G))$  by  $\varphi_{\rho}(f) = \int_G \overline{\rho(g)} f(g) dm_G(g)$  ( $f \in L^1(G)$ ). It is well-known that there is no other character on  $L^1(G)$ ; that is,  $\Delta(L^1(G)) = \{\varphi_{\rho} : \rho \in \hat{G}\}$ ; for more detail see Theorem 23.7 of [6].

By using Proposition 3.2 and Theorem 3.1, we can proof following corollary.

**Corollary 3.2.** *Let  $G$  be a locally compact group and  $\varphi_\rho \in \Delta(L^1(G))$ . Then the following assertions are equivalent:*

- (i)  $L^1(G)$  is essentially left  $\varphi_\rho$ -contractible;
- (ii)  $L^2(G)$  is essentially left  $\varphi_\rho|_{L^2(G)}$ -contractible;
- (iii)  $S(G)$  is essentially left  $\varphi_\rho|_{S(G)}$ -contractible;
- (iv)  $G$  is finite.

Let  $L^\infty(G)$  be the usual Lebesgue space with the essentially supremum norm  $\|\cdot\|_\infty$  and  $L_0^\infty(G)$  be the space of all  $f \in L^\infty(G)$  which vanish at infinity (see [11]).

**Corollary 3.3.** *Let  $G$  be a locally compact group. Then  $L_0^\infty(G)^*$  is essentially left character contractible if and only if  $G$  is finite.*

*Proof.* Let  $L_0^\infty(G)^*$  be essentially left character contractible. By Theorem 2.11 of [11], for each right identity  $E$  of  $L_0^\infty(G)^*$  with norm one, the map  $F \mapsto E \boxtimes F$  ( $F \in L_0^\infty(G)$ ), where  $\boxtimes$  is the first Arens product on  $L_0^\infty(G)$ , is a continuous epimorphism from  $L_0^\infty(G)^*$  onto  $M(G)$ , and so by Corollary 3.1,  $M(G)$  is essentially left character contractible. Consequently  $G$  is finite, by Theorem 3.1. Now suppose that  $G$  is finite. Then  $L_0^\infty(G)^*$  is left character contractible, by Corollary 6.2 of [15]. Therefore  $L_0^\infty(G)^*$  is essentially left character contractible.  $\square$

**Proposition 3.3.** *Let  $A$  be a Banach algebra,  $\varphi \in \Delta(A)$  and let  $B$  be a closed subalgebra of  $A$  that contains  $AA$ . If  $B$  is essentially left  $\varphi|_B$ -contractible, then  $A$  is essentially left  $\varphi$ -contractible.*

*Proof.* Let  $X$  be a neo-unital Banach  $A$ -bimodule with  $x \cdot a = \varphi(a)x$  ( $a \in A$ ,  $x \in X$ ), and let  $D: A \rightarrow X$  be a continuous derivation. Since  $X$  is a neo-unital Banach  $A$ -bimodule,

$$X = A \cdot X = A(A \cdot X) = AA \cdot X \subseteq B \cdot X \subseteq X.$$

Hence  $X$  is also a neo-unital Banach  $B$ -bimodule. Since the map  $D|_B: B \rightarrow X$  is a continuous derivation and  $B$  is essentially left  $\varphi|_B$ -contractible, there exists  $x_0 \in X$  such that  $D|_B = ad_{x_0}$ . Define  $\bar{D} = D - ad_{x_0}$ . Clearly  $\bar{D}$  is a derivation from  $A$  into  $X$ , and  $\bar{D}|_B = 0$ . Fix  $a_0 \in A$  such that  $\varphi(a_0) = 1$ . Let  $b_0 = a_0^2$ . Then  $b_0 \in B$  and  $\varphi(b_0) = 1$ . Therefore for every  $a \in A$ , we have

$$\bar{D}(a) = \varphi(b_0)\bar{D}(a) = \bar{D}(a) \cdot b_0 = \bar{D}(ab_0) - a \cdot \bar{D}(b_0) = 0.$$

That is  $\bar{D} = 0$ . So  $D = ad_{x_0}$ . This means that  $A$  is essentially left  $\varphi$ -contractible.  $\square$

Let  $S$  be a semigroup. A semi-character on  $S$  is a non-zero homomorphism  $\varphi: S \rightarrow \bar{D}$ . The space of semi-characters on  $S$  is denoted by  $\Phi_S$ . The semi-character  $(t \in S)$ ,  $\varphi_S(t) = 1$  defined by  $\varphi_S: S \rightarrow \bar{D}$  is called the augmentation character on  $S$ . For every  $\varphi \in \Phi_S$ , define  $\hat{\varphi}: l^1(S) \rightarrow \mathbb{C}$  by

$$\hat{\varphi}(f) = \sum_{s \in S} \varphi(s) f(s) \quad (f = \sum_{s \in S} f(s) \delta_s \in l^1(S)).$$

It is easily verified that  $\hat{\varphi} \in \Delta(l^1(S))$  and every character on  $l^1(S)$  arises in this way. Indeed,  $\Delta(l^1(S)) = \{\hat{\varphi} : \varphi \in \Phi_S\}$ .

**Example 3.1.** Let  $S = \{0\}$  (null semigroup) with the semigroup operation  $s_1 s_2 = 0$  ( $s_1, s_2 \in S$ ). Let  $B = l^1(S) * l^1(S) = \mathbb{C} \delta_0$ . Choose  $m = \delta_0$  and note that  $\varphi_S(m) = 1$  and  $bm = \varphi_S(b)m$  ( $b \in B$ ). This means that  $m$  is a topological left invariant  $\varphi_S$ -mean in  $B$ . Hence  $B$  is left  $\varphi_S$ -contractible by Theorem 2.1 of [15]. Therefore  $B$  is essentially left  $\varphi_S$ -contractible and Proposition 3.3, shows that  $A = l^1(S)$  is essentially left  $\varphi_S$ -contractible.

#### 4. Essential left character contractibility of certain Banach algebras

Let  $A$  and  $B$  be Banach algebras with  $\Delta(B) \neq \emptyset$  and let  $\theta \in \Delta(B)$ . Then the direct product  $A \times B$  equipped with the algebra multiplication

$$(a, b) \cdot (a', b') = (aa' + \theta(b')a + \theta(b)a', bb'), \quad (a, a' \in A, b, b' \in B),$$

and with the norm  $\|(a, b)\| = \|a\| + \|b\|$  is a Banach algebra which is called the  $\theta$ -Lau product of  $A$  and  $B$  and is denoted by  $A \times_{\theta} B$ . This type of product was introduced by Lau [10] for certain class of Banach algebras and was extended by Sangani Monfared [12] for the general case. For example, the unitization  $A^{\#} = A \times_{\theta} \mathbb{C}$  of  $A$  can be regarded as the  $\theta$ -Lau product of  $A$  and  $\mathbb{C}$  where  $\theta: \mathbb{C} \rightarrow \mathbb{C}$  is the identity map. It is easy to verify that the dual space  $(A \times_{\theta} B)^*$  can be identified with  $A^* \times B^*$  via

$$\langle (f, g), (a, b) \rangle = f(a) + g(b), \quad g \in B^*, f \in A^*, b \in B, (a \in A,$$

To prove our next result we need to quote the following proposition from [12].

**Proposition 4.1.** Let  $A$  and  $B$  be two Banach algebras with  $\Delta(B) \neq \emptyset$ . Let  $\theta \in \Delta(B)$ ,  $E = \{(\varphi, \theta) : \varphi \in \Delta(A)\}$  and  $F = \{(0, \psi) : \psi \in \Delta(B)\}$ . Set  $E = \emptyset$  if  $\Delta(A) = \emptyset$ . Then  $\Delta(A \times_{\theta} B) = E \cup F$ .

**Theorem 4.1** Let  $A$  and  $B$  be Banach algebras and let  $\theta \in \Delta(B)$ . If  $A \times_{\theta} B$  is essentially left character contractible, then so are  $A$  and  $B$ .



*Proof.* Let  $\psi \in \Delta(B) \cup \{0\}$ . Define  $h: A \times_{\theta} B \rightarrow B$  by  $h(a, b) = b$  ( $a \in A, b \in B$ ). It is clear that  $h$  determines a continuous epimorphism. Since  $\psi \circ h = (0, \psi)$  and  $A \times_{\theta} B$  is essentially left  $(0, \psi)$ -contractible, from Proposition 3.1, it follows that  $B$  is essentially left  $\psi$ -contractible. Therefore  $B$  is essentially left character contractible.

Let  $\varphi \in \Delta(A) \cup \{0\}$  and let  $D: A \rightarrow X$  be a continuous derivation for some neo-unital Banach  $A$ -bimodule  $X$  with the right module action  $x \cdot a = \varphi(a)x$  ( $a \in A, x \in X$ ). Clearly  $X$  is a neo-unital Banach  $A \times_{\theta} B$ -bimodule with the actions  $(a, b) \cdot x = a \cdot x + \theta(b)x$ ,  $x \in X$ ,  $b \in B$ , ( $a \in A$ ,  $x \cdot (a, b) = (\varphi, \theta)(a, b)x$ ). Define  $\bar{D}: A \times_{\theta} B \rightarrow X$  by  $\bar{D}(a, b) = D(a)$  ( $a \in A, b \in B$ ). For every  $a, a' \in A$  and  $b, b' \in B$ , we have

$$\begin{aligned} \bar{D}((a, b)(a', b')) &= D(aa' + \theta(b')a + \theta(b)a') = D(aa') + \theta(b')D(a) + \theta(b)D(a') \\ &= D(a) \cdot a' + a \cdot D(a') + \theta(b')D(a) + \theta(b)D(a') \\ &= (\varphi(a') + \theta(b'))D(a) + a \cdot D(a') + \theta(b)D(a') \\ &= (\varphi, \theta)(a', b')\bar{D}(a, b) + (a, b) \cdot \bar{D}(a', b') \\ &= \bar{D}(a, b) \cdot (a', b') + (a, b) \cdot \bar{D}(a', b'). \end{aligned}$$

Thus  $\bar{D}$  is a continuous derivation. From the essentially left  $(\varphi, \theta)$ -contractibility of  $A \times_{\theta} B$ , it follows that  $\bar{D}$  is inner. So  $D$  is inner. Therefore  $A$  is essentially left character contractible.  $\square$

Let  $A$  be a Banach algebra,  $A^{\#}$  the unitization of  $A$  and  $\varphi \in \Delta(A)$ . We denote by  $\varphi^{\#} \in \Delta(A^{\#})$  the unique extension of  $\varphi$ . It is well-known that  $\Delta(A^{\#}) = \{\varphi^{\infty}\} \cup \{\varphi^{\#} : \varphi \in \Delta(A)\}$ , where  $\varphi^{\infty}(a, \lambda) = \lambda$  for all  $a \in A$  and  $\lambda \in \mathbb{C}$ .

The next example shown that the converse of the Theorem 4.1 is not true.

**Example 4.1.** Let  $A$  be an essentially left character contractible Banach algebra which is not left character contractible. Then by Corollary 3.3 of [15],  $A^{\#} = A \times_{\theta} \mathbb{C}$  is not left character contractible. So Theorem 2.2 implies that  $A^{\#}$  is not essentially left character contractible.

**Theorem 4.2.** Let  $A$  and  $B$  be Banach algebras,  $\varphi \in \Delta(A)$  and  $\psi \in \Delta(B)$ . If  $A \bar{\otimes} B$  is essentially left  $\varphi \otimes \psi$ -contractible, then  $A$  is essentially left  $\varphi$ -contractible and  $B$  is essentially left  $\psi$ -contractible.

*Proof.* Let  $A \bar{\otimes} B$  be essentially left  $\varphi \otimes \psi$ -contractible. Define  $h: A \bar{\otimes} B \rightarrow A$  by  $h(a \otimes b) = \psi(b)a$  ( $a \in A, b \in B$ ). Since  $h$  is an epimorphism and  $\varphi \circ h = \varphi \otimes \psi$  from Proposition 3.1, it follows that  $A$  is essentially left  $\varphi$ -contractible. Similarly, one can show that  $B$  is essentially left  $\psi$ -contractible.  $\square$

We do not know if the converse of Theorem 4.2 is true.

Let  $A$  be a Banach algebra and  $X$  be a Banach  $A$ -bimodule. The  $l^1$ -direct sum of  $A$  and  $X$ , denoted by  $A \oplus_1 X$ , with the product defined by

$$(a, x)(a', x') = (aa', a \cdot x' + x \cdot a') \quad x, x' \in X, \quad (a, a' \in A,$$

is a Banach algebra that is called the module extension Banach algebra of  $A$  and  $X$ . Using the fact that the element  $(0, x)$  is nilpotent in  $A \oplus_1 X$  for all  $x \in X$ , it is easy to verify that  $\Delta(A \oplus_1 X) = \{\bar{\varphi} : \varphi \in \Delta(A)\}$ , where  $\bar{\varphi}(a, x) = \varphi(a)$  for all  $a \in A$  and  $x \in X$ .

**Proposition 4.2.** *Let  $A$  be a Banach algebra,  $\varphi \in \Delta(A)$  and let  $X$  be a Banach  $A$ -bimodule. If  $A \oplus_1 X$  is essentially left  $\bar{\varphi}$ -contractible, then  $A$  is essentially left  $\varphi$ -contractible.*

*Proof.* It is clear that the map  $h: A \oplus_1 X \rightarrow A$ , defined by  $h(a, x) = a$  for all  $a \in A$  and  $x \in X$ , determines an epimorphism with  $\varphi \circ h = \bar{\varphi}$ . The proof is complete, by Proposition 3.1.  $\square$

The following theorem shows that under certain conditions, the converse of the above proposition is also true. Let  $\text{ann}(X) = \{a \in A : X \cdot \{a\} = \{0\}\}$ .

**Theorem 4.3.** *Let  $A$  be a Banach algebra,  $\varphi \in \Delta(A)$  and let  $X$  be a Banach  $A$ -bimodule. Then the following statements hold.*

(i) If  $X = \{0\}$  and  $A$  is essentially left  $\varphi$ -contractible, then  $A \oplus_1 X$  is essentially left  $\bar{\varphi}$ -contractible.

(ii) If  $A$  is left  $\varphi$ -contractible and  $\text{ann}(X) \cap (\ker \varphi)^c \neq \emptyset$ , then  $A \oplus_1 X$  is essentially left  $\bar{\varphi}$ -contractible.

*Proof.* (i): Let  $D: A \oplus_1 X \rightarrow Y$  be a continuous derivation for some neo-unital Banach  $A \oplus_1 X$ -bimodule  $Y$  with the right module action

$$y \cdot (a, 0) = \bar{\varphi}(a, 0)y = \varphi(a)y \quad (y \in Y, a \in A, x \in X).$$

Then  $Y$  can be considered as a neo-unital Banach  $A$ -bimodule with the following actions

$$a \bullet y = (a, 0) \cdot y, \quad y \bullet a = y \cdot (a, 0) = \varphi(a)y \quad (y \in Y, a \in A). \quad (2)$$

Define  $\bar{D}: A \rightarrow Y$  by  $\bar{D}(a) = D(a, 0)$  ( $a \in A$ ). Clearly,  $\bar{D}$  is a continuous derivation. Now from the essentially left  $\varphi$ -contractibility of  $A$ , it follows that there exists  $y_0 \in Y$  such that  $\bar{D} = ad_{y_0}$ . This yields that  $D$  is inner. Thus,  $A \oplus_1 X$  is essentially left  $\bar{\varphi}$ -contractible.

(ii): Let  $A$  be left  $\varphi$ -contractible and let  $D: A \oplus_1 X \rightarrow Y$  be a continuous derivation for some neo-unital Banach  $A \oplus_1 X$ -bimodule  $Y$  with the right module action  $y \cdot (a, x) = \bar{\varphi}(a, x)y = \varphi(a)y$  ( $y \in Y, a \in A, x \in X$ ). By actions defined as (i),  $Y$  can be considered as a Banach  $A$ -bimodule.

Similar as (i) we define continuous derivation  $\bar{D}: A \rightarrow Y$  by  $\bar{D}(a) = D(a, 0)$  ( $a \in A$ ). From the left  $\varphi$ -contractibility of  $A$ , it follows that there exists  $y_0 \in Y$  such that  $\bar{D} = ad_{y_0}$ . Since  $ann(X) \cap (\ker \varphi)^c \neq \emptyset$ , we can suppose that there is  $a_0 \in ann(X)$  such that  $\varphi(a_0) = 1$ . So, for every  $x \in X$ ,

$$0 = D((0, x)(a_0, 0)) = D(0, x) + (0, x) \cdot D(a_0, 0),$$

which implies that  $D(0, x) = -(0, x) \cdot D(a_0, 0)$  ( $x \in X$ ). Hence,

$$\begin{aligned} D(0, x) &= -(0, x) \cdot D(a_0, 0) = -(0, x) \cdot ((a_0, 0) \cdot y_0 - y_0 \cdot (a_0, 0)) \\ &= -(0, x \cdot a_0) \cdot y_0 + (0, x) \cdot y_0 \cdot (a_0, 0) = (0, x) \cdot y_0 = (0, x) \cdot y_0 - y_0 \cdot (0, x \cdot a_0) \\ &= (0, x) \cdot y_0 - (y_0 \cdot (0, x)) \cdot (a_0, 0) = (0, x) \cdot y_0 - y_0 \cdot (0, x), \end{aligned}$$

for all  $x \in X$ . Now for every  $a \in A$  and  $x \in X$ , we have

$$\begin{aligned} D(a, x) &= D(a, 0) + D(0, x) = ((a, 0) \cdot y_0 - y_0 \cdot (a, 0)) + ((0, x) \cdot y_0 - y_0 \cdot (0, x)) \\ &= (a, x) \cdot y_0 - y_0 \cdot (a, x). \end{aligned}$$

Thus  $D$  is inner. Therefore  $A \oplus_1 X$  is essentially left  $\varphi$ -contractible.

**Example 4.2.** Let  $S = \mathbb{N} \cup \{0\}$  be the set of non-negative integer. Define the

semigroup operation on  $S$  by  $m \bullet n = \begin{cases} m & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$

It is easy to see that  $\Phi_S = \{\varphi_n : n \in \mathbb{N}\} \cup \{\varphi_S\}$ , where  $\varphi_S$  is the augmentation character and for each  $n, m \in \mathbb{N}$ ,  $\varphi_n(m) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$

Let  $\varphi \in \Delta(l^1(S)) = \{\hat{\varphi} : \varphi \in \Phi_S\}$ , and  $X$  be a Banach  $l^1(S)$ -bimodule with the right module action  $x \cdot f = \hat{\varphi}(f)x$  ( $x \in X$ ,  $f \in l^1(S)$ ). If  $\varphi$  is of the form  $\varphi_n$  for some  $n \in \mathbb{N}$  and  $D: l^1(S) \rightarrow X$  is a continuous derivation, then by a similar argument as in Example 2.1 of [3], we may show that  $D = ad_{D(\delta_0) - D(\delta_n)}$ . Thus  $D$  is an inner derivation. Now, let  $\varphi = \varphi_S$  and  $D: l^1(S) \rightarrow X$  is a continuous derivation. Then for every  $m, n \in S$ , we have  $D(\delta_m) \cdot \delta_n = \varphi_S(\delta_n)D(\delta_m) = D(\delta_m)$ . Thus, for every  $m \neq 0$ ,

$$\begin{aligned} D(\delta_m) &= D(\delta_m) \cdot \delta_0 = D(\delta_m * \delta_0) - \delta_m \cdot D(\delta_0) = D(\delta_0) - \delta_m \cdot D(\delta_0) \\ &= D(\delta_0) \cdot \delta_m - \delta_m \cdot D(\delta_0) = ad_{-D(\delta_0)}(\delta_m). \end{aligned}$$

Moreover,  $\delta_0 \cdot D(\delta_0) = D(\delta_0 * \delta_0) - D(\delta_0) \cdot \delta_0 = D(\delta_0) - D(\delta_0) = 0$ . Therefore

$$ad_{-D(\delta_0)}(\delta_0) = D(\delta_0) \cdot \delta_0 - \delta_0 \cdot D(\delta_0) = D(\delta_0).$$

Hence  $D$  is an inner derivation. Indeed, we have proved that  $l^1(S)$  is left  $\hat{\varphi}$ -contractible for every  $\varphi \in \Phi_S$ .

**Example 4.3.** Let  $S = (\mathbb{N} \cup \{0\}, \bullet)$  be as in Example 4.2, and  $\varphi_S$  be the augmentation character on  $S$ . We define an action of  $l^1(S)$  on  $X = l^1(S)$  by

$$\pi_l(f, g) = f * g, \quad \pi_r(f, g) = \varphi_S(g)f \quad (f, g \in l^1(S)),$$

where  $\pi_l$  and  $\pi_r$  denote the left and right module actions, respectively. Take  $n \in \mathbb{N}$  and suppose that  $\varphi_n$  is the semi-character on  $S$  defined in Example 4.2. Thus  $\pi_r(f, \delta_n - \delta_m) = \varphi_S(\delta_n - \delta_m)f = 0$ , for all  $m \neq n$  and  $f \in l^1(S)$ . It follows that  $\text{ann}(X) \cap (\ker \varphi)^c \neq \emptyset$ . Since by Example 4.2,  $l^1(S)$  is left  $\varphi_n$ -contractible, Theorem 4.3 implies that the module extension Banach algebra  $l^1(S) \oplus_1 l^1(S)$  is essentially left  $\varphi_n$ -contractible.

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### REFERENCES

- [1] *H. G. Dales*, Banach Algebras and Automatic Continuity, London Math. Soc. Monogr. Ser., 24, The Clarendon Press, Oxford University Press, New York, New York 2000.
- [2] *P. Cohen*, Factorization in group algebras, Duke Math. J., **26** (1959), 199-205.
- [3] *M. Essmaili and M. Filali*,  $\phi$ -Amenability and character amenability of some classes of Banach algebras, Houston Journal of Mathematics, **39** No. 2 (2013), 515-529.
- [4] *F. Ghahramani, R.J. Loy*, Generalized notions of amenability, J. Funct. Anal., **208** (2004), 229-260.
- [5] *E. Hewitt*, The ranges of certain convolution operators, Math. Scand., **15** (1964), 147-155.
- [6] *E. Hewitt and K. A. Ross*, Abstract Harmonic Analysis I, Springer, Berlin, 1970.
- [7] *Z. Hu, M. S. Monfared and T. Traynor*, On character amenable Banach algebras, Studia Math., **193** (2009), 53-78.
- [8] *B. E. Johnson*, Cohomology in Banach algebras, harmonic problems, Mem. Amer. Math. Soc., **127**, 1972.
- [9] *E. Kaniuth, A. T. Lau and J. Pym*, On  $\varphi$ -amenability of Banach algebras, Math. Proc. Camb. Phil. Soc., **144** (2008), 85-96.
- [10] *A. T. Lau*, Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups, Fund. Math., **118** No. 3 (1983), 161-175.
- [11] *A. T. Lau and J. Pym*, Concerning the second dual of the group algebra of a locally compact group, J. London Math. Soc., **41** (1990), 445-460.
- [12] *M. S. Monfared*, On certain products of Banach algebras with applications to harmonic analysis, Studia Math., **178** No. 3 (2007), 277-294.
- [13] *M. S. Monfared*, Character amenability of Banach algebras, Math. Proc. Camb. Phil. Soc., **144** (2008), 697-706.
- [14] *R. Nasr-Isfahani and M. Nemati*, Essential character amenability of Banach algebras, Bull. Aust. Math. Soc., **84** (2011), 372-386.
- [15] *R. Nasr-Isfahani and S. Soltani Renani*, Character contractibility of Banach algebras and homological properties of Banach modules, Studia Math., **202** No. 3 (2011), 205-225.