

GENERALIZATIONS OF 1-ABSORBING PRIMARY IDEALS OF COMMUTATIVE RINGS

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Let R be a commutative ring with identity. In this paper, we extend the concept of 1-absorbing primary ideals to the concept of φ -1-absorbing primary ideals. Let $\varphi: S(R) \rightarrow S(R) \cup \{\emptyset\}$ be a function, where $S(R)$ is the set of all ideals of R . A proper ideal I of R is called a φ -1-absorbing primary ideal of R if whenever nonunit elements a, b, c of R and $abc \in I \setminus \varphi(I)$, then $ab \in I$ or $c \in \sqrt{I}$. A number of results concerning φ -1-absorbing primary ideals are given and some characterizations of φ -1-absorbing primary ideals in some special rings are obtained.

Keywords: 1-absorbing primary ideal, weakly 1-absorbing primary ideal, φ -1-absorbing primary ideal.

1. Introduction

Throughout this paper R denotes a commutative ring with identity and the set of all ideals of R is denoted by $S(R)$. Various generalizations of prime (primary) ideals are studied in [1]-[14]. The concept of weakly prime ideals was introduced by Anderson and Smith in [2]. According to that paper, a proper ideal I of R is called a weakly prime ideal of R if whenever $\mathbf{0} \neq ab \in I$ for some $a, b \in I$, then $a \in I$ or $b \in I$. A proper ideal I of R is called a weakly primary ideal of R as in [4] if whenever $\mathbf{0} \neq ab \in I$ for some $a, b \in I$, then $a \in I$ or $b \in \sqrt{I}$. Let $\varphi: S(R) \rightarrow S(R) \cup \{\emptyset\}$ be a function. A proper ideal I of R is called a φ -prime ideal of R as in [1] if whenever $a, b \in R$ with $ab \in I \setminus \varphi(I)$ implies $a \in I$ or $b \in I$. A proper ideal I of R is called a φ -primary ideal of R as in [12] if whenever $a, b \in R$ with $ab \in I \setminus \varphi(I)$, then $a \in I$ or $b \in \sqrt{I}$. As a different generalization of (resp. weakly prime) prime ideals, (resp. [6]) [5] that a proper ideal I of R is said to be a (resp. weakly) 2-absorbing ideal of R if whenever (resp. $\mathbf{0} \neq abc \in I$) $abc \in I$ for some $a, b, c \in R$, then $ab \in I$ or $bc \in I$ or $ac \in I$. Recall that a proper ideal I of R is called a φ -2-absorbing ideal of R as in [13] if whenever $a, b, c \in R$ with $abc \in I \setminus \varphi(I)$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Some generalizations of (resp. weakly) 2-absorbing ideals, (resp. weakly) 2-absorbing primary ideals were introduced and studied in [7] and [8], where a proper ideal of R is said to be a (resp. weakly) 2-absorbing primary ideal if whenever $a, b, c \in R$ with (resp. $\mathbf{0} \neq abc \in I$) $abc \in I$,

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then $\mathbf{ab} \in \mathbf{I}$ or $\mathbf{ac} \in \sqrt{\mathbf{I}}$ or $\mathbf{bc} \in \sqrt{\mathbf{I}}$. Later, a generalization of 2-absorbing primary ideals was introduced which covers all the above mentioned definitions. Recall from [9] that a proper ideal \mathbf{I} of \mathbf{R} is said to be a φ -2-absorbing primary ideal if whenever $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{R}$ with $\mathbf{abc} \in \mathbf{I} \setminus \varphi(\mathbf{I})$, then $\mathbf{ab} \in \mathbf{I}$ or $\mathbf{ac} \in \sqrt{\mathbf{I}}$ or $\mathbf{bc} \in \sqrt{\mathbf{I}}$. In some recent studies [10] and [11], Badawi and Yetkin Celikel introduced the concepts of 1-absorbing primary and weakly 1-absorbing primary ideals. A proper ideal \mathbf{I} of \mathbf{R} is said to be a (resp. weakly) 1-absorbing primary if whenever nonunit $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{R}$ and (resp. $\mathbf{0} \neq \mathbf{abc} \in \mathbf{I}$) $\mathbf{abc} \in \mathbf{I}$, then $\mathbf{ab} \in \mathbf{I}$ or $\mathbf{c} \in \sqrt{\mathbf{I}}$.

In this paper, we extend the concept of 1-absorbing primary ideal to the concept of φ -1-absorbing primary ideal. Let $\varphi: S(\mathbf{R}) \rightarrow S(\mathbf{R}) \cup \{\emptyset\}$ be a function. We call a proper ideal \mathbf{I} of \mathbf{R} a φ -1-absorbing primary ideal if whenever nonunit elements $\mathbf{a}, \mathbf{b}, \mathbf{c}$ of \mathbf{R} with $\mathbf{abc} \in \mathbf{I} \setminus \varphi(\mathbf{I})$, then $\mathbf{ab} \in \mathbf{I}$ or $\mathbf{c} \in \sqrt{\mathbf{I}}$. Let $\varphi(\mathbf{I}) \neq \emptyset$. Then, we have the following relations among these mentioned concepts:

$$\begin{array}{ccccc}
 \text{primary} & \Rightarrow & 1\text{-absorbing primary} & \Rightarrow & 2\text{-absorbing primary} \\
 \Downarrow & & \Downarrow & & \Downarrow \\
 \text{weakly primary} & \Rightarrow & \text{weakly 1-absorbing primary} & \Rightarrow & \text{weakly 2-absorbing primary} \\
 \Downarrow & & \Downarrow & & \Downarrow \\
 \varphi\text{-primary} & \Rightarrow & \varphi\text{-1-absorbing primary} & \Rightarrow & \varphi\text{-2-absorbing primary}
 \end{array}$$

We show that φ -1-absorbing primary ideals enjoy analogs of many of the properties of (weakly) 1-absorbing primary ideals. Among many results in this paper, we show (Theorem 2.3) that if \mathbf{I} is a φ -1-absorbing primary ideal of \mathbf{R} such that φ is an order-preserving function with $\varphi(\mathbf{I}) \neq \mathbf{I}$ and $\sqrt{\varphi(\mathbf{I})} = \varphi(\mathbf{I})$, then $\sqrt{\mathbf{I}}$ is a prime ideal of \mathbf{R} . Additionally, if \mathbf{R} is a Dedekind domain, we show (Theorem 2.5) that the converse of Theorem 2.3 is also satisfied. We show (Theorem 2.4) that in divided rings (chained rings) φ -1-absorbing primary ideals \mathbf{I} of \mathbf{R} coincide with φ -primary ideals of \mathbf{R} provided that $\varphi(\mathbf{I}) \neq \mathbf{I}$ and $\varphi(\mathbf{I})$ is a radical ideal, where φ is an order-preserving function. Furthermore, a characterization for φ -1-absorbing primary ideals in u-rings is obtained via equivalent conditions. (Theorem 2.9).

2. φ -1-absorbing primary ideals

Definition 2.1. Let I be a φ -1-absorbing primary ideal of R . We define the function $\varphi_\alpha: S(R) \rightarrow S(R) \cup \{\emptyset\}$ and the corresponding φ_α -1-absorbing primary ideals, where $\alpha \in \{\emptyset\} \cup \{0, 1, 2, \dots\} \cup \{\omega\}$ as follows:

- (1) If $\varphi(I) = \emptyset$ for every $I \in S(R)$, then $\varphi = \varphi_\emptyset$ and I is called a φ_\emptyset -1-absorbing primary ideal of R , and hence I is a 1-absorbing primary ideal of R .
- (2) If $\varphi(I) = 0$ for every $I \in S(R)$, then $\varphi = \varphi_0$ and I is called a φ_0 -1-absorbing primary ideal of R . Thus I is a weakly 1-absorbing primary ideal of R .

(3) If $\varphi(I) = I$ for every $I \in S(R)$, then $\varphi = \varphi_1$ and I is called a φ_1 -1-absorbing primary ideal of R . (any ideal.)

(4) If $n \geq 2$ and $\varphi(I) = I^n$ for every $I \in S(R)$, then $\varphi = \varphi_n$ and I is called a φ_n -1-absorbing primary ideal (n -almost 1-absorbing primary ideal) of R . In particular, if $n = 2$, then we say that I is an almost-1-absorbing primary ideal

(5) If $\varphi(I) = \bigcap_{n=1}^{\infty} I^n$ for every $I \in S(R)$, then $\varphi = \varphi_{\omega}$ and I is called a φ_{ω} -1-absorbing primary ideal of R .

Since $I \setminus \varphi(I) = I \setminus (I \cap \varphi(I))$, throughout we may assume that $\varphi(I) \subseteq I$. Given two functions $\psi_1, \psi_2 : S(R) \rightarrow S(R) \cup \{\emptyset\}$, if $\psi_1(I) \subseteq \psi_2(I)$, then it is clear that if I is a ψ_1 -1-absorbing primary, then I is a ψ_2 -1-absorbing primary ideal of R . The following example shows that the concepts of (weakly) 1-absorbing primary ideal and φ -1-absorbing primary ideal are different.

Example 2.1. Consider an ideal $J = \{0, 6, 12, 18\}$ of \mathbb{Z}_{24} and the idealization ring $R = \mathbb{Z}_{24}(+)J$. Then $I = \{(0,0), (0,6), (0,12), (0,18)\}$ is a φ_2 -1-absorbing primary ideal of R . Observe that if $abc \in I$ for some $a, b, c \in R \setminus I$, abc can be neither $(0,6)$ nor $(0,18)$. So, $abc \in I$ for some $a, b, c \in R \setminus I$ if and only if $abc \in \varphi_2(I) = I^2 = \{(0,0), (0,12)\}$. Thus I is a φ_2 -1-absorbing primary ideal of R . However, it is not weakly 1-absorbing primary since $(0,0) \neq (0,12) = (2,0)(4,6)(3,0) \in I$, but neither $(2,0)(4,6) = (8,12) \in I$ nor $(3,0) \in \sqrt{I} = I$.

Theorem 2.1. Let I be a proper ideal of R . Then

- (1) I is a primary ideal $\Rightarrow I$ is a 1-absorbing primary ideal $\Rightarrow I$ is a weakly 1-absorbing primary ideal $\Rightarrow I$ is a φ_{ω} -1-absorbing primary ideal $\Rightarrow I$ is an n -almost-1-absorbing primary ideal for $n \geq 2 \Rightarrow I$ is an almost 1-absorbing primary ideal.
- (2) If I is an idempotent ideal of R , then I is an φ_{ω} -1-absorbing primary ideal of R .
- (3) I is a φ_n -1-absorbing primary ideal of R for all $n \geq 2$ if and only if I is a φ_{ω} -1-absorbing primary ideal of R .
- (4) If $R/\varphi(I)$ is an integral domain, then I is a φ -1-absorbing primary ideal of R if and only if I is a 1-absorbing primary ideal of R .
- (5) If R is a quasilocal ring with maximal ideal $\sqrt{0}$, then I is a φ -1-absorbing primary ideal of R for all φ .

Proof. (1) It is clear that every primary ideal is 1-absorbing primary. Since $\emptyset \subseteq 0 \subseteq \bigcap_{n=1}^{\infty} I^n \subseteq \dots \subseteq I^n \subseteq \dots \subseteq I^2 \subseteq I$, the implications follow from the definitions.

(2) If I is an idempotent ideal, then $I = I^2 = I^n = \bigcap_{n=1}^{\infty} I^n$; so we are done.

(3) It is obvious from definitions.

(4) Suppose that I is a φ -1-absorbing primary ideal of R . Let $abc \in I$ for some nonunit $a, b, c \in R$ and $ab \notin I$. If $abc \notin \varphi(I)$, then $c \in \sqrt{I}$, so we are done. Assume that $abc \in \varphi(I)$. Since $ab \notin \varphi(I)$ and $R/\varphi(I)$ is an integral domain, we conclude that $c \in \varphi(I) \subseteq I \subseteq \sqrt{I}$. The converse part is clear.

(5) If R is a quasilocal ring with maximal ideal $\sqrt{0}$, then for every nonunit element c of R , $c \in \sqrt{0} \subseteq \sqrt{I}$. Therefore, I is a φ -1-absorbing primary ideal of R .

Theorem 2.2. Let $\varphi: S(R) \rightarrow S(R) \cup \{\emptyset\}$ be a function and I a proper ideal of R . Then the following statements hold:

- (1) If I is a φ -1-absorbing primary ideal of R and J is a proper ideal of R with $\varphi(I) \subseteq J \subseteq I$, then I/J is a weakly 1-absorbing primary ideal of R/J .
- (2) Suppose that the set of unit elements of $R/\varphi(I)$ is $U(R/\varphi(I)) = \{a + \varphi(I) \mid a \in U(R)\}$. Then I is a φ -1-absorbing primary ideal of R if and only if $I/\varphi(I)$ is a weakly 1-absorbing primary ideal of $R/\varphi(I)$.
- (3) Suppose that the set of unit elements of R/I^n is $U(R/I^n) = \{a + I^n \mid a \in U(R)\}$. Then I is a φ_n -1-absorbing primary ideal of R for $n \geq 2$ if and only if I/I^n is a weakly 1-absorbing primary ideal of R/I^n .

Proof. (1) Suppose that $J \neq (a + J)(b + J)(c + J) \in I/J$ for some nonunit $(a + J)$, $(b + J)$, $(c + J) \in R/J$. Hence $abc \in I \setminus J$, where a, b, c are nonunit elements of R . Since $\varphi(I) \subseteq J$, we have $abc \in I \setminus \varphi(I)$. It follows that $ab \in I$ or $c \in \sqrt{I}$. Since $\sqrt{I/J} = \sqrt{I}/J$, we have $ab + J \in I/J$ or $c + J \in \sqrt{I}/J$.

(2) Suppose that $I/\varphi(I)$ is a weakly 1-absorbing primary ideal of $R/\varphi(I)$ and $abc \in I \setminus \varphi(I)$ for some nonunit $a, b, c \in R$. Thus $\varphi(I) \neq (a + \varphi(I))(b + \varphi(I))(c + \varphi(I)) \in I/\varphi(I)$, where $a + \varphi(I)$, $b + \varphi(I)$, $c + \varphi(I)$ are nonunit elements of $R/\varphi(I)$ by hypothesis. Hence $(a + \varphi(I))(b + \varphi(I)) \in I/\varphi(I)$ or $c + \varphi(I) \in \sqrt{I}/\varphi(I)$. Thus $ab \in I$ or $c \in \sqrt{I}$. The converse part is clear by (1).

(3) Since $\varphi_n(I) = I^n$, the conclusion follows from (2).

A function $\varphi: S(R) \rightarrow S(R) \cup \{\emptyset\}$ is said to be order-preserving if whenever $I, J \in S(R)$ with $I \subseteq J$, then $\varphi(I) \subseteq \varphi(J)$.

Theorem 2.3. Let $\varphi: S(R) \rightarrow S(R) \cup \{\emptyset\}$ be an order-preserving function. If I is a φ -1-absorbing primary ideal of R such that $\varphi(I) \neq I$ and $\sqrt{\varphi(I)} = \varphi(I)$, then \sqrt{I} is a prime ideal of R .

Proof. First we show that \sqrt{I} is a φ -prime ideal of R . Suppose that $ab \in \sqrt{I} \setminus \varphi(\sqrt{I})$ for some $a, b \in R$. We may assume that a, b are nonunit. Then there exists an even positive integer $n = 2m$ ($m \geq 1$) such that $(ab)^n \in I$. If $(ab)^n \in \varphi(I)$, then $ab \in \sqrt{\varphi(I)} = \varphi(I) \subseteq \varphi(\sqrt{I})$, a contradiction. So, we have $(ab)^n = a^m a^m b^n \in I \setminus \varphi(I)$. Thus $a^m a^m = a^n \in I$ or $b^n \in \sqrt{I}$, and therefore \sqrt{I} is a φ -prime ideal of R . Thus, \sqrt{I} is a prime ideal of R by [1, Corollary 7].

Recall that a ring R is called divided if for every prime ideal P of R and for every $x \in R \setminus P$, we have $x \mid p$ for every $p \in P$. We have the following result.

Theorem 2.4. Let R be a divided ring and I a proper ideal of R with such that $\varphi(I) \neq I$ and $\sqrt{\varphi(I)} = \varphi(I)$ for some order-preserving function φ . Then I is a φ -1-absorbing primary ideal of R if and only if I is a φ -primary ideal of R .

Proof. Suppose that $ab \in I \setminus \varphi(I)$ for some $a, b \in R$ and $b \notin \sqrt{I}$. We may assume that a and b are nonunit. Since \sqrt{I} is a prime ideal of R by Theorem 2.3, we

conclude that $a \in \sqrt{I}$. Since R is divided, we conclude that $b \mid a$. Thus $a = bc$ for some $c \in R$. Observe that c is a nonunit element of R as $b \notin \sqrt{I}$ and $a \in \sqrt{I}$. Since $ab = bcb \in I \setminus \varphi(I)$, I is φ -1-absorbing primary and $b \notin \sqrt{I}$, we conclude that $bc = a \in I$. Thus I is a φ -primary ideal of R . The converse part is clear.

Remark 2.1. Recall that a ring R is called a chained ring if for every $x, y \in R$, we have $x \mid y$ or $y \mid x$. Every chained ring is divided. So, if R is a chained ring, then a proper ideal I of R with the properties that $\varphi(I) \neq I$ and $\sqrt{\varphi(I)} = \varphi(I)$, where φ is a order-preserving function, is a φ -1-absorbing primary if and only if it is φ -primary.

Theorem 2.5. Let R be a Dedekind domain and I a proper ideal of R such that $\varphi(I) \neq I$ and $\sqrt{\varphi(I)} = \varphi(I)$, where φ is an order-preserving function. Then I is a φ -1-absorbing primary ideal of R if and only if \sqrt{I} is a prime ideal of R .

Proof. Suppose that I is a φ -1-absorbing primary ideal of R . Then \sqrt{I} is a prime ideal of R by Theorem 2.3. Conversely, if \sqrt{I} is prime, then I is 1-absorbing primary by [10, Theorem 14]. Thus I is a φ -1-absorbing primary ideal of R .

The following result is similar to [11, Theorem 2].

Theorem 2.6. Let I be a φ -1-absorbing primary ideal of R . If I is not a φ -primary ideal of R , then there exist an irreducible element $x \in R$ and a nonunit element $y \in R$ such that $xy \in I \setminus \varphi(I)$, but neither $x \in I$ nor $y \in \sqrt{I}$. Furthermore, if $ab \in I \setminus \varphi(I)$ for some nonunit $a, b \in R$ such that neither $a \in I$ nor $b \in \sqrt{I}$, then a is an irreducible element of R .

Proof. Suppose that I is not a φ -primary ideal of R . Then there exist nonunit $x, y \in R$ such that $xy \in I \setminus \varphi(I)$ with $x \notin I$, $y \notin \sqrt{I}$. Assume that x is not an irreducible element of R . Then $x = cd$ for some nonunit $c, d \in R$. Since $xy = cdy \in I \setminus \varphi(I)$ and I is φ -1-absorbing primary and $y \notin \sqrt{I}$, we conclude that $cd = x \in I$, a contradiction. Hence x is an irreducible element of R .

Theorem 2.7. Let x be a nonunit element of R with $(0:x) \subseteq (x)$. Then the following statements are equivalent:

- (1) (x) is φ -1-absorbing primary ideal of R for some φ with $\varphi \leq \varphi_3$.
- (2) (x) is a 1-absorbing primary ideal of R .

Proof. (1) \Rightarrow (2) Suppose that (x) is φ_3 -1-absorbing primary and $abc \in (x)$ and $ab \notin (x)$ for some nonunit $a, b, c \in R$. If $abc \notin (x^3) = \varphi_3((x))$, we have $c \in \sqrt{(x)}$. Suppose that $abc \in (x^3)$. Hence $abc + abx = ab(c + x) \in (x)$. Since $ab \notin (x)$, $(c + x)$ is a nonunit element of R . Now, we show that $ab(c + x) \notin (x^3)$. Assume that $ab(c + x) \in (x^3)$. Since $abc \in (x^3)$, we have $abx \in (x^3)$. Hence there exists $r \in R$ such that $abx = rx^3$ which means $(ab - rx^2) \in (0:x)$. We get $ab \in (0:x) + (x) \subseteq (x)$, a contradiction. Thus $ab(c + x) \notin (x^3)$. Since (x) is φ_3 -

1-absorbing primary and $ab \notin (x)$, we have $(c+x) \in \sqrt{(x)}$. Thus $c \in \sqrt{(x)}$. Therefore, (x) is a 1-absorbing primary ideal of R . $(2) \Rightarrow (1)$ is clear.

Theorem 2.8. Let I be a φ -1-absorbing primary ideal of R with $(\varphi(I): c) \subseteq \varphi((I: c))$. If c is a nonunit element of $R \setminus I$, then $(I: c)$ is a φ -primary ideal of R .

Proof. Suppose that $ab \in (I: c) \setminus \varphi((I: c))$ and $a \notin (I: c)$. Hence b is a nonunit element of R . If a is unit, then $b \in (I: c) \subseteq \sqrt{(I: c)}$ and we are done. Assume that a is nonunit. Then $acb \in I \setminus \varphi(I)$ as $(\varphi(I): c) \subseteq \varphi((I: c))$. Since $ac \notin I$ and I is a φ -1-absorbing primary ideal of R , we conclude that $b \in \sqrt{I} \subseteq \sqrt{(I: c)}$, as needed.

Let R be a commutative ring. If an ideal of R contained in a finite union of ideals must be contained in one of those ideals, then R is called a u-ring [15]. The next theorem gives a characterization for φ -1-absorbing primary ideals in u-rings.

Theorem 2.9. Let I be a proper ideal of a u-ring R . The following are equivalent:

- (1) I is a φ -1-absorbing primary ideal of R .
- (2) For every nonunit $a, b \in R$ with $ab \notin I$, $(I: ab) = (\varphi(I): ab)$ or $(I: ab) \subseteq \sqrt{I}$.
- (3) For every nonunit $a \in R$ and every ideal I_1 of R with $I_1 \not\subseteq \sqrt{I}$, if $(I: aI_1)$ is a proper ideal of R , then $(I: aI_1) = (\varphi(I): aI_1)$ or $(I: aI_1) \subseteq (I: a)$.
- (4) For every ideals I_1, I_2 of R with $I_1 \not\subseteq \sqrt{I}$, if $(I: I_1I_2)$ is a proper ideal of R , then $(I: I_1I_2) = (\varphi(I): I_1I_2)$ or $(I: I_1I_2) \subseteq (I: I_2)$.
- (5) For every ideals I_1, I_2, I_3 of R with $I_1I_2I_3 \subseteq I \setminus \varphi(I)$, $I_1I_2 \subseteq I$ or $I_3 \subseteq \sqrt{I}$.

Proof. $(1) \Rightarrow (2)$ Suppose that $c \in (I: ab)$. Then $abc \in I$. Since $ab \notin I$, c is nonunit. If $abc \in \varphi(I)$, then $c \in (\varphi(I): ab)$. Assume that $abc \in I \setminus \varphi(I)$. Since I is φ -1-absorbing primary, we have $c \in \sqrt{I}$. Hence $(I: ab) \subseteq (\varphi(I): ab) \cup \sqrt{I}$. Since R is a u-ring, we obtain that $(I: ab) = (\varphi(I): ab)$ or $(I: ab) \subseteq \sqrt{I}$. $(2) \Rightarrow (3)$ Suppose that $c \in (I: aI_1)$. It is clear that c is nonunit. Then $acI_1 \subseteq I$. Now $I_1 \subseteq (I: ac)$. If $ac \in I$, then $c \in (I: a)$. Suppose that $ac \notin I$. Hence $(I: ac) = (\varphi(I): ac)$ or $(I: ac) \subseteq \sqrt{I}$ by (2). Thus $I_1 \subseteq (\varphi(I): ac)$ or $I_1 \subseteq \sqrt{I}$. Since $I_1 \not\subseteq \sqrt{I}$ by hypothesis, we conclude $I_1 \subseteq (\varphi(I): ac)$; i.e. $c \in (\varphi(I): aI_1)$. Thus $(I: aI_1) \subseteq (\varphi(I): aI_1) \cup (I: a)$. Since R is a u-ring, we have $(I: aI_1) = (\varphi(I): aI_1)$ or $(I: aI_1) \subseteq (I: a)$. $(3) \Rightarrow (4)$ If $I_1 \subseteq \sqrt{I}$, then we are done. Suppose that $I_1 \not\subseteq \sqrt{I}$ and $c \in (I: I_1I_2)$. Then $I_2 \subseteq (I: cI_1)$. Since $(I: I_1I_2)$ is proper, c is nonunit. Hence $I_2 \subseteq (\varphi(I): cI_1)$ or $I_2 \subseteq (I: c)$ by (3). If $I_2 \subseteq (\varphi(I): cI_1)$, then $c \in (I: I_1I_2)$. If $I_2 \subseteq (I: c)$, then $c \in (I: I_2)$. So, $(I: I_1I_2) \subseteq (\varphi(I): I_1I_2) \cup (I: I_2)$ which implies that $(I: I_1I_2) = (\varphi(I): I_1I_2)$ or $(I: I_1I_2) \subseteq (I: I_2)$. $(4) \Rightarrow (5)$ It is clear. $(5) \Rightarrow (1)$ Let $a, b, c \in R$ nonunit elements and $abc \in I \setminus \varphi(I)$. Put $I_1 = aR$, $I_2 = bR$, and $I^3 = cR$ in (5). Then the result is clear.

Definition 2.2. Let I be a φ -1-absorbing primary ideal of R . If $abc \in \varphi(I)$ for some nonunit $a, b, c \in R$ such that $ab \notin I$, $c \notin \sqrt{I}$, then we call (a, b, c) a φ -1-triple-zero of I .

Let I be a proper ideal of a ring R . Observe that I is φ -1-absorbing primary that is not 1-absorbing primary if and only if I has a φ -triple-zero. We obtain the next theorem and corollary similar to [11, Theorem 10 and Theorem 11].

Theorem 2.10. Let I be a φ -1-absorbing primary ideal of R , and (a, b, c) a φ -1-triple-zero of I . Then

- (1) $abI \subseteq \varphi(I)$.
- (2) If $a, b \notin (I: c)$, then $bcI = acI = aI^2 = bI^2 = cI^2 \subseteq \varphi(I)$.
- (3) If $a, b \notin (I: c)$, then $I^3 \subseteq \varphi(I)$.

Proof. (1) Suppose that $abI \not\subseteq \varphi(I)$. Then $abx \notin \varphi(I)$ for some nonunit $x \in I$. Hence $ab(c + x) \in I \not\subseteq \varphi(I)$. Since $ab \notin I$, $(c + x)$ is a nonunit element of R . Since I is a φ -1-absorbing primary ideal of R and $ab \notin I$, we conclude that $(c + x) \in \sqrt{I}$. Since $x \in I$, we have $c \in \sqrt{I}$, a contradiction. Thus $abI \subseteq \varphi(I)$.

(2) Suppose that $bcy \notin \varphi(I)$ for some nonunit $y \in I$. Hence $bcy = b(a + y)c \in I \setminus \varphi(I)$. Since $b \notin (I: c)$, we conclude that $a + y$ is a nonunit element of R . Since I is φ -1-absorbing primary, $ab \notin I$ and $by \in I$, we conclude that $b(a + y) \notin I$, and hence $c \in \sqrt{I}$, a contradiction. Thus $bcI \subseteq \varphi(I)$. We show that $acI \subseteq \varphi(I)$. Suppose that $acI \not\subseteq \varphi(I)$. Then $acy \notin \varphi(I)$ for some nonunit element $y \in I$. Hence $acy = a(b + y)c \in I \setminus \varphi(I)$. Since $a \notin (I: c)$, we conclude that $b + y$ is a nonunit element of R . Since I is φ -1-absorbing primary, $ab \notin I$ and $ay \in I$, we conclude that $a(b + y) \notin I$, and hence $c \in \sqrt{I}$, a contradiction. Thus $acI \subseteq \varphi(I)$. Now we prove that $al^2 \subseteq \varphi(I)$. Suppose that $axy \notin \varphi(I)$ for some $x, y \in I$. Since $abI \subseteq \varphi(I)$ by (1) and $acI \subseteq \varphi(I)$ by (2), $axy = a(b + x)(c + y) \in I \setminus \varphi(I)$. Since $ab \notin I$, we conclude that $c + y$ is a nonunit element of R . Since $a \notin (I: c)$, we conclude that $b + x$ is a nonunit element of R . Since I is φ -1-absorbing primary, we have $a(b + x) \in I$ or $c + y \in \sqrt{I}$. Since $x, y \in I$, we conclude that $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Thus $al^2 \subseteq \varphi(I)$. By a similar manner, one can easily obtain that $bl^2 \subseteq \varphi(I)$. Now, we show that $cl^2 \subseteq \varphi(I)$. Suppose that $cxy \notin \varphi(I)$ for some $x, y \in I$. Since $acI = bcI \subseteq \varphi(I)$ by (2), $cxy = (a + x)(b + y)c \in I \setminus \varphi(I)$. Since $a, b \notin (I: c)$, we conclude that $a + x$ and $b + y$ are nonunit elements of R . Since I is φ -1-absorbing primary, we have $(a + x)(b + y) \in I$ or $c \in \sqrt{I}$. Since $x, y \in I$, we conclude that $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Thus $cl^2 \subseteq \varphi(I)$.

(3) Assume that $xyz \notin \varphi(I)$ for some $x, y, z \in I$. Then $xyz = (a + x)(b + y)(c + z) \in I \setminus \varphi(I)$ by (1) and (2). Since $ab \notin I$, we conclude that $c + z$ is a nonunit element of R . Since $a, b \notin (I: c)$, we conclude that $a + x$ and $b + y$ are nonunit elements of R . Since I is φ -1-absorbing primary, $(a + x)(b + y) \in I$ or $c + z \in \sqrt{I}$. Since $x, y, z \in I$, we conclude that $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Thus $I^3 \subseteq \varphi(I)$.

Corollary 2.1. Let I be a φ -1-absorbing primary ideal of R such that $\varphi(I) \neq I$.

(1) Let $R/\varphi(I)$ be a reduced ring. If (a, b, c) is a φ -1-triple-zero of I , then $ac \in I$ or $bc \in I$.

(2) If $(a + \varphi(I), b + \varphi(I), c + \varphi(I))$ is a 1-triple-zero of $I/\varphi(I)$, then (a, b, c) is a φ -1-triple-zero of I .

Proof. (1) Assume that neither $ac \in I$ nor $bc \in I$. Hence $I^3 \subseteq \varphi(I)$ by Theorem 2.10 (3). Since $R/\varphi(I)$ is reduced, we conclude that $I \subseteq \varphi(I)$. Thus $I = \varphi(I)$, a contradiction. Therefore, $ac \in I$ or $bc \in I$.

(2) Suppose that $((a + \varphi(I), b + \varphi(I), c + \varphi(I))$ is a 1-triple-zero of $I/\varphi(I)$. It is clear that a, b, c are nonunit elements of R . Then $((a + \varphi(I))(b + \varphi(I)) \notin I/\varphi(I)$ and $c + \varphi(I) \notin \sqrt{I/\varphi(I)} = \sqrt{I}/\varphi(I)$. Thus $ab \notin I$ and $c \notin \sqrt{I}$ which means that (a, b, c) is a φ -1-triple-zero of I .

The following theorem is a generalization of [11, Theorem 15].

Theorem 2.11. Let R and R' be commutative rings with identity, $f: R \rightarrow R'$ be a ring homomorphism with $f(1_R) = 1_{R'}$, and let $\varphi: S(R) \rightarrow S(R) \cup \{\emptyset\}$ and $\varphi': S(R') \rightarrow S(R') \cup \{\emptyset\}$ be functions. Then the following statements hold:

- (1) Suppose that $f(a)$ is a nonunit element of R' for every nonunit $a \in R$ (for example if $U(R')$ is a torsion group) and J is a φ' -1-absorbing primary ideal of R' with $\varphi(f^{-1}(J)) = f^{-1}(\varphi'(J))$. Then $f^{-1}(J)$ is a φ -1-absorbing primary ideal of R .
- (2) If f is an epimorphism, I is a φ -1-absorbing primary ideal of R such that $\text{Ker}(f) \subseteq I$ and $\varphi'(f(I)) = f(\varphi(I))$, then $f(I)$ is a φ -1-absorbing primary ideal of R' .

Proof. (1) Let $abc \in f^{-1}(J) \setminus \varphi(f^{-1}(J))$ for some nonunit $a, b, c \in R$. Then $f(abc) = f(a)f(b)f(c) \in J \setminus \varphi'(J)$, where $f(a), f(b), f(c)$ are nonunit elements of R' by hypothesis. Hence $f(a)f(b) \in J$ or $f(c) \in \sqrt{J}$. Hence $ab \in f^{-1}(J)$ or $c \in \sqrt{f^{-1}(J)} = f^{-1}(\sqrt{J})$. Thus $f^{-1}(J)$ is a φ -1-absorbing primary ideal of R .

(2) Let $xyz \in f(I) \setminus \varphi'(f(I))$ for some nonunit $x, y, z \in R$. Since f is onto, there exists nonunit $a, b, c \in R$ such that $x = f(a), y = f(b), z = f(c)$. Then $f(abc) = f(a)f(b)f(c) = xyz \in f(I) \setminus f(\varphi(I))$. Since $\text{Ker}(f) \subseteq I$, we have $abc \in I \setminus \varphi(I)$. It follows $ab \in I$ or $c \in \sqrt{I}$. Thus $xy \in f(I)$ or $z \in f(\sqrt{I})$. Since f is onto and $\text{Ker}(f) \subseteq I$, we have $f(\sqrt{I}) = \sqrt{f(I)}$. Thus we are done.

Given an ideal I of R , define $\varphi_I: S(R/I) \rightarrow S(R/I) \cup \{\emptyset\}$ by $\varphi_I(J/I) = (\varphi(J) + I)/I$ for $I \subseteq J$. Note that $\varphi_I(J/I) \subseteq J/I$ and $(\varphi_\alpha)_I = \varphi_\alpha$ for $\alpha \in \{\emptyset\} \cup \{0, 1, 2, \dots\}$.

Corollary 2.2. Let I and J be proper ideals of R such that $I \subseteq J$. If J is a φ -1-absorbing primary ideal of R , then J/I is a φ_I -1-absorbing primary ideal of R/I .

Proof. Suppose that $(a + I)(b + I)(c + I) \in J/I \setminus \varphi_I(J/I) = J/I \setminus (\varphi(J) + I)/I$ for some nonunit $a + I, b + I, c + I \in R/I$. Since $I \subseteq J$, we have $abc \in J \setminus \varphi(J)$, where a, b, c are nonunit elements of R . Hence $ab \in J$ or $c \in \sqrt{J}$. Thus $(a + I)(b + I) \in J/I$ or $(c + I) \in \sqrt{J}/I = \sqrt{J/I}$.

We characterize φ -1-absorbing primary ideals of $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings with identity in the following theorem.

Theorem 2.12. Let R_1 and R_2 be commutative rings with identity, I_1, I_2 nonzero ideals of R_1, R_2 , respectively, and $R = R_1 \times R_2$. Let $\psi_i: S(R_i) \rightarrow S(R_i) \cup \emptyset$ ($i = 1, 2$) be functions such that $\psi_1(I_1) = I_1$ and $\psi_2(I_2) = I_2$. Let $\varphi = \psi_1 \times \psi_2$. If I is a proper ideal of R , then the following statements are equivalent:

- (1) $I_1 \times I_2$ is a φ -1-absorbing primary ideal of R .
- (2) $I_1 = R_1, I_2$ is a primary ideal of R_2 or $I_2 = R_2, I_1$ is a primary ideal of R_1 .
- (3) $I_1 \times I_2$ is a primary ideal of R .
- (4) $I_1 \times I_2$ is a 1-absorbing primary ideal of R .

Proof. Suppose that $\psi_1(I_1) = \emptyset$ or $\psi_2(I_2) = \emptyset$. Then $\varphi(I_1 \times I_2) = \emptyset$. Hence (1) \Leftrightarrow (2) \Leftrightarrow (3). Thus assume that $\varphi(I_1 \times I_2) = \emptyset$, and hence neither $\psi_1(I_1) = \emptyset$ nor $\psi_2(I_2) = \emptyset$. (1) \Rightarrow (2). Suppose that $I_1 \times I_2$ is a φ -1-absorbing primary ideal of R . Hence $I_1/\psi_1(I_1) \times I_2/\psi_2(I_2)$ is a weakly 1-absorbing primary ideal of $R_1/\psi_1(I_1) \times R_2/\psi_2(I_2)$ by Theorem 2.2 (1). Hence by [11, Theorem 13], we conclude that $I_1/\psi_1(I_1) = R_1/\psi_1(I_1)$ and $I_2/\psi_2(I_2)$ is a primary ideal of $R_2/\psi_2(I_2)$ or $I_2/\psi_2(I_2) = R_2/\psi_2(I_2)$ and $I_1/\psi_1(I_1)$ is a primary ideal of $R_1/\psi_1(I_1)$. Thus $I_1 = R_1$ and I_2 is a primary ideal of R_2 or $I_2 = R_2$ and I_1 is a primary ideal of R_1 . (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1). It is straightforward.

Definition 2.3. Let $\varphi: S(R) \rightarrow S(R) \cup \{\emptyset\}$ be a function, and S a multiplicatively closed subset of R . A proper ideal L_S of R_S , where L is a proper ideal of R such that $L \cap S = \emptyset$, is called a φ_S -1-absorbing primary ideal of R_S if whenever nonunit $a, b, c \in R_S$ with $abc \in L_S \setminus \varphi(L)_S$, then $ab \in L_S$ or $c \in \sqrt{L_S}$.

Let $Z(R)$ denotes the set of all zero-divisors of R and $Z_I(R) = \{a \in R: ab \in I \text{ for some } b \in R \setminus I\}$. We have the following result similar to [11, Theorem 17].

Theorem 2.13. Let I be a proper ideal of R and S a multiplicatively closed subset of R such that $S \cap Z(R) = S \cap I = S \cap Z_I(R) = \emptyset$. Then I is a φ -1-absorbing primary ideal of R if and only if I_S is a φ_S -1-absorbing primary ideal of R_S .

Proof. Suppose that $abc \in I_S \setminus \varphi(I)_S$ for some nonunit $a, b, c \in R_S$. Hence there is an $s \in S$ and nonunit $a_1, b_1, c_1, d_1 \in R$ such that $a = a_1/s, b = b_1/s, c = c_1/s$. Thus $((a_1b_1c_1)/(s^3)) = ((d_1)/(s^3)) \in I_S \setminus \varphi(I)_S$. Since $Z(R) \cap S = \emptyset$, we have $a_1b_1c_1 = d_1 \in I \setminus \varphi(I)$. Thus $ab \in I$ or $c \in \sqrt{I}$. Since $Z(R) \cap S = \emptyset$, $\sqrt{I_S} = (\sqrt{I})_S$. Thus $ab \in I_S$ or $c \in \sqrt{I_S}$. Conversely, suppose that $abc \in I \setminus \varphi(I)$ for some nonunit $a, b, c \in R$. Then $((abc)/1) = (a/1)(b/1)(c/1) \in I_S \setminus \varphi(I)_S$ as $S \cap Z(R) = \emptyset$. Hence $(a/1)(b/1) \in I_S$ or $(c/1) \in \sqrt{I_S}$. If $(a/1)(b/1) \in S^{-1}I$, then $uab \in I$ for some $u \in S$. Since $S \cap Z_I(R) = \emptyset$, we conclude that $ab \in I$. If $(c/1) \in (\sqrt{I})_S$, then $(tc)^n \in I$ for some $n \geq 1$ and $t \in S$. Since $S \cap Z_I(R) = \emptyset$, we have $t^n \notin Z_I(R)$. Hence $c^n \in I$. Thus I is a φ -1-absorbing primary ideal of R .

Definition 2.4. Let I be a φ -1-absorbing primary ideal of R and $I_1I_2I_3 \subseteq I$ for some proper ideals I_1, I_2, I_3 of R . If (a, b, c) is not φ -1-triple zero of I for every $a \in I_1, b \in I_2, c \in I_3$, then we call I a free φ -1-triple zero with respect to $I_1I_2I_3$.

Theorem 2.14. Let I be a φ -1-absorbing primary ideal of R and J a proper ideal of R with $abJ \subseteq I$ for some $a, b \in R$. If (a, b, j) is not a φ -1-triple zero of I for all $j \in J$ and $ab \notin I$, then $J \subseteq \sqrt{I}$.

Proof. The proof is similar to the proof of [11, Theorem 18].

Corollary 2.3. Let I be a φ -1-absorbing primary ideal of R and $I_1I_2I_3 \subseteq I \setminus \varphi(I)$ for some proper ideals I_1, I_2, I_3 of R . If I is free φ -1-triple zero with respect to $I_1I_2I_3$, then $I_1I_2 \subseteq I$ or $I_3 \subseteq \sqrt{I}$.

Proof. The proof is similar to the proof of [11, Theorem 19].

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R E F E R E N C E S

- [1]. D. D. Anderson and M. Bataineh, Generalizations of prime ideals, *Comm. Algebra*, **36** (2008), 686-696.
- [2]. D. D. Anderson and E. Smith, Weakly prime ideals, *Houston J. Math.*, **29** (2003), 831-840.
- [3]. D.F. Anderson and A. Badawi, On n-absorbing ideals of commutative rings, *Comm. Algebra*, **39** (2011), 1646-1672.
- [4]. S.E. Atani and F. Farzalipour, On weakly primary ideals, *Georgian Math. J.*, **12** (2005) 423-429.
- [5]. A. Badawi, On 2-absorbing ideals of commutative rings, *Bull. Austral. Math. Soc.*, **75** (2007), 417-429.
- [6]. A. Badawi and A.Y. Darani, On weakly 2-absorbing ideals of commutative rings, *Houston J. Math.*, **39** (2013), 441-452.
- [7]. A. Badawi, U. Tekir and E. Yetkin, On 2-absorbing primary ideals in commutative rings, *Bull. Korean Math. Soc.*, **51** (2014), 1163-1173.
- [8]. A. Badawi, U. Tekir and E. Yetkin, On weakly 2-absorbing primary ideals of commutative rings. *J. Korean Math. Soc.* **52** (2015), No:1, 97-111.
- [9]. A. Badawi, U. Tekir, E. A. Ugurlu, G. Ulucak, E. Yetkin Celikel, Generalizations of 2-absorbing primary ideals of commutative rings, *Turkish J. Math.* **40** (2016), 703-717.
- [10]. A. Badawi and E. Yetkin Celikel, On 1-absorbing primary ideals, *Journal of Algebra and App.* (in press) (DOI: 10.1142/S021949882050111X)
- [11]. A. Badawi and E. Yetkin Celikel, On weakly 1-absorbing primary ideals, arXiv: 2002.12608v1.
- [12]. A.Y. Darani, Generalizations of primary ideals in commutative rings, *Novi Sad. J. Math.*, **42** (2012), 27-35.
- [13]. M. Ebrahimpour and R. Nekooei, On generalizations of prime ideals, *Comm. Algebra*, **40** (2012), 1268-1279.
- [14]. A.K. Jabbar and C.A. Ahmed, On almost primary ideals, *Int. J. Algebra*, **5** (2011), 627-636.
- [15]. P. Quartararo and H. S. Butts, Finite unions of ideals and modules, *Proc. Amer. Math. Soc.*, **52** (1975), 91-96.