

**ON UNCERTAINTY PRINCIPLE OF ORTHONORMAL SEQUENCES
FOR THE q -DUNKL TRANSFORM**

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In this work, using some elements of the q -harmonic analysis and the q -Dunkl transform, for fixed $q \in]0, 1[$, we establish a q -analogue of uncertainty inequalities for orthonormal sequences and prove a quantitative version of Shapiro's uncertainty principle for the q -Dunkl transform. As a side results, we prove a variation of Donoho-Stark's uncertainty inequality, in particular, if f is ϵ_S -concentrated on S and $F_D^{\alpha, q}(f)$ is ϵ_{Σ} -concentrated on Σ with $\|f\|_{2, q} = 1$ and $\epsilon_S + \epsilon_{\Sigma} < 1$, then $|S||\Sigma| \geq (1 - (\epsilon_S + \epsilon_{\Sigma}))^2$.

1. Notations and preliminaries

A Fourier uncertainty principle is an inequality or uniqueness theorem concerning the joint localization of a function f and its Fourier transform $\mathcal{F}(f)$. Every discussion of the uncertainty principle must necessarily begin with the classical uncertainty principle, called the Heisenberg-Pauli-Weil inequality in which concentration is measured in terms of dispersions. It states that for $f \in L^2(\mathbb{R}^d)$

$$\|x|f\|_{L^2(\mathbb{R}^d)}^2 \|\xi|\mathcal{F}(f)\|_{L^2(\mathbb{R}^d)}^2 \geq \frac{d}{2} \|f\|_{L^2(\mathbb{R}^d)}^2, \quad (1.1)$$

with equality if and only if f is a multiple of a suitable Gaussian function, where \mathcal{F} is the classical Fourier transform defined by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle x, \xi \rangle} f(x) dx,$$

and $\langle \cdot, \cdot \rangle$, $|\cdot|$ are the usual inner product and norm on \mathbb{R}^d .

Generalizations of this result in both classical and quantum analysis have been treated and many versions of Heisenberg-Pauli-Weil type uncertainty inequalities have been obtained for several generalized Fourier transforms.

Recently, considerable attention has been devoted to proving new mathematical formulations and new contexts for the uncertainty principle.

In [18], H. Shapiro proved a number of uncertainty inequalities for orthonormal sequences that are stronger than corresponding inequalities for a single function. More precisely, if $(\varphi_n)_{n=1}^{\infty}$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$, then

$$\sup_n \left(\|x|\varphi_n\|_{L^2(\mathbb{R}^d)}^2 + \|\xi|\mathcal{F}(\varphi_n)\|_{L^2(\mathbb{R}^d)}^2 \right) = \infty. \quad (1.2)$$

A quantitative version of Shapiro's result (1.2) has been proved by Jaming and Powell [11] and then by Malinnikova [14],

$$\forall s > 0, \quad \forall N \geq 1, \quad \sum_{n=1}^N \left(\|x|^s \varphi_n\|_{L^2(\mathbb{R}^d)}^2 + \|\xi|^s \mathcal{F}(\varphi_n)\|_{L^2(\mathbb{R}^d)}^2 \right) \geq C N^{1+s/d}. \quad (1.3)$$

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In [5, 8], the authors established a Heisenberg uncertainty inequality for the q -Dunkl transform. In particular we have

$$\| |x|^s f \|_{2,q}^2 + \| |\lambda|^s F_D^{\alpha,q}(f) \|_{2,q}^2 \geq c_1(s, q) \| f \|_{2,q}^2, \quad (1.4)$$

or equivalently

$$\| |x|^s f \|_{2,q} \| |\lambda|^s F_D^{\alpha,q}(f) \|_{2,q} \geq c_2(s, q) \| f \|_{2,q}^2. \quad (1.5)$$

Motivated by these results, our purpose in this paper is to prove a quantitative version of Shapiro's mean dispersion theorem for the q -Dunkl Rubin transform. Also, the Donoho-Stark's uncertainty principle for the q -Dunkl transform is proved.

This paper is organized as follows: in this section, we present some notations and results used in q -theory and useful in the sequel. Also, we recall some basic properties of the q -Dunkl operator and the q -Dunkl transform introduced in [3]. In Section 2, we prove some Donoho-Stark's inequalities and finally in Section 3, we prove a q -analogue of uncertainty inequalities for orthonormal sequences and show a quantitative version of Shapiro's uncertainty principle for the q -Dunkl transform.

For the convenience of the reader, we collect here some usual notions and notations used in the q -theory and useful in the sequel. For more information on the q -theory we refer the reader to [10, 12, 17]. Throughout this paper, we will fix $q \in]0, 1[$ such that $\frac{Ln(1-q)}{Ln(q)} \in 2\mathbb{Z}$. We note

$$\mathbb{R}_q = \{\pm q^n; n \in \mathbb{Z}\}, \mathbb{R}_{q,+} = \{q^n; n \in \mathbb{Z}\} \text{ and } \tilde{\mathbb{R}}_q = \mathbb{R}_q \cup \{0\}.$$

For $a \in \mathbb{C}$; the q -shifted factorials are defined by

$$(a, q)_0 = 1; \quad (a, q)_n = \prod_{k=0}^{n-1} (1 - aq^k); \quad (a, q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n).$$

We also denote for all $x \in \mathbb{C}$ and $n \in \mathbb{N}$

$$[x]_q = \frac{1 - q^x}{1 - q}; \quad [n]_q! = \frac{(q, q)_n}{(1 - q)^n}.$$

In [16], Rubin defined a q -analogue differential operator by

$$\partial_q(f)(z) = \frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(z)}{2(1 - q)z}. \quad (1.6)$$

The q -Jackson integrals are defined by [10]

$$\int_0^a f(x) d_q x = (1 - q)a \sum_{k \in \mathbb{Z}} f(aq^n) q^n, \quad \int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x$$

and

$$\int_{-\infty}^{\infty} f(x) d_q x = (1 - q) \sum_{n \in \mathbb{Z}} q^n (f(q^n) + f(-q^n))$$

provided the sums converge absolutely.

Using the q -Jackson integrals, we denote by:

- $L_{\alpha,q}^p(\mathbb{R}_q) = \{f : \|f\|_{p,q} = (\int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} d_q x)^{\frac{1}{p}} < \infty\}.$
- $L_{\alpha,q}^{\infty}(\mathbb{R}_q) = \{f : \|f\|_{\infty,q} = \sup\{|f(x)| : x \in \mathbb{R}_q\} < \infty\}.$

For the particular case $p = 2$, we denote by $\langle \cdot, \cdot \rangle$ the inner product of the Hilbert space $L_{\alpha,q}^2(\mathbb{R}_q)$ as

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(x) g(x) |x|^{2\alpha+1} d_q x.$$

In the following, we recall some basic properties of the q -Dunkl operator and the q -Dunkl transform introduced in [1].

For $\alpha \geq -\frac{1}{2}$, the q -Dunkl operator is defined by

$$\Lambda_{\alpha,q}(f)(x) = \partial_q[H_{\alpha,q}(f)](x) + [2\alpha + 1]_q \frac{f(x) - f(-x)}{2x},$$

where

$$H_{\alpha,q} : f = f_e + f_o \rightarrow f_e + q^{2\alpha+1}f_o,$$

f_e and f_o are the even and odd parts respectively of the function f .

It was shown in [1] that for each $\lambda \in \mathbb{C}$, the function

$$\psi_{\lambda}^{\alpha,q} : x \rightarrow j_{\alpha}(\lambda x, q^2) + \frac{i\lambda x}{[2\alpha + 2]_q} j_{\alpha+1}(\lambda x, q^2)$$

is the unique solution of the q -differential-difference equation

$$\Lambda_{\alpha,q}(f) = i\lambda f, \quad f(0) = 1$$

where $j_{\alpha}(x, q^2)$ is the normalized third Jackson's q -Bessel function given by

$$j_{\alpha}(x, q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^2, q^2)_n (q^{2(\alpha+1)}, q^2)_n} ((1-q)x)^{2n}. \quad (1.7)$$

The q -Dunkl transform $F_D^{\alpha,q}$ is defined on $L_{\alpha,q}^1(\mathbb{R}_q)$ by (see [1])

$$F_D^{\alpha,q}(f)(\lambda) = \frac{c_{\alpha,q}}{2} \int_{-\infty}^{\infty} f(x) \psi_{-\lambda}^{\alpha,q}(x) |x|^{2\alpha+1} d_q x \quad (1.8)$$

for all $\lambda \in \mathbb{R}_q$ where $c_{\alpha,q} = \frac{(1+q)^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}$.

This transform satisfies the following properties (see [17]):

- For $f \in L_{\alpha,q}^1(\mathbb{R}_q)$:

$$\|F_D^{\alpha,q}(f)\|_{\infty,q} \leq \frac{2c_{\alpha,q}}{(q, q)_{\infty}} \|f\|_{1,q}. \quad (1.9)$$

- For all $f \in L_{\alpha,q}^1(\mathbb{R}_q)$ such that $xf \in L_{\alpha,q}^1(\mathbb{R}_q)$

$$F_D^{\alpha,q}(\Lambda_{\alpha,q}(f)) = i\lambda F_D^{\alpha,q}(f)(\lambda), \quad \Lambda_{\alpha,q}(F_D^{\alpha,q}(f)) = -iF_D^{\alpha,q}(xf). \quad (1.10)$$

- The q -Dunkl transform $F_D^{\alpha,q}$ is an isomorphism from $L_{\alpha,q}^2(\mathbb{R}_q)$ onto itself and satisfies the following Plancherel formula:

$$\|F_D^{\alpha,q}(f)\|_{2,q} = \|f\|_{2,q}. \quad (1.11)$$

- $\forall x \in \mathbb{R}_q, \quad (F_D^{\alpha,q})^{-1}(F_D^{\alpha,q}(f))(x) = f(x) = \frac{c_{\alpha,q}}{2} \int_{-\infty}^{\infty} F_D^{\alpha,q}(f)(\lambda) \psi_x^{\alpha,q}(\lambda) |\lambda|^{2\alpha+1} d_q \lambda$

The q -analogue of the Heisenberg uncertainty principle is given by (see [5])

Theorem 1.1. *Let $s > 0$, $\alpha \geq -\frac{1}{2}$ and $f \in L_{\alpha,q}^2(\mathbb{R}_q)$ then*

$$\| |x|^s f \|_{2,q} \| |\lambda|^s F_D^{\alpha,q}(f) \|_{2,q} \geq c(s, q) \|f\|_{2,q} \quad (1.12)$$

where $c(s, q)$ is a constant which depends on s and q .

2. An $L_{\alpha,q}^2(\mathbb{R}_q)$ Donoho-Stark's uncertainty principle

In this section we will prove a q -analogue of the variation on Donoho-Stark's uncertainty principle for $L_{\alpha,q}^2(\mathbb{R}_q)$.

A subset $T \subset \mathbb{R}_q$ is said to be measurable subset of \mathbb{R}_q if

$$|T| = \int_{-\infty}^{\infty} \chi_T(x) |x|^{2\alpha+1} d_q x < \infty,$$

where χ_T is the characteristic function of T .

The time-limiting and the frequency-limiting operators on $L_{\alpha,q}^2(\mathbb{R}_q)$ are defined by

$$E_S f = \chi_S f; \quad F_{\Sigma} f = (F_D^{\alpha,q})^{-1} [\chi_{\Sigma} F_D^{\alpha,q}(f)]$$

where S and Σ are measurable subsets of \mathbb{R}_q .

Definition 2.1. Let $0 < \epsilon < 1$ and $f \in L_{\alpha,q}^2(\mathbb{R}_q)$ then

- (1) A function f is said to be ϵ -concentrated on S if: $\|E_{S^c} f\|_{2,q} \leq \epsilon \|f\|_{2,q}$.
- (2) $F_D^{\alpha,q}(f)$ is said to be ϵ -concentrated on Σ if: $\|F_{\Sigma^c} f\|_{2,q} \leq \epsilon \|f\|_{2,q}$.

Lemma 2.1. $F_{\Sigma} E_S$ is a Hilbert-Schmidt operator with the kernel:

$$k(t, \lambda) = \frac{c_{\alpha,q}}{2} \chi_S(t) \int_{\Sigma} \psi_{\lambda}^{\alpha,q}(x) \psi_{-x}^{\alpha,q}(t) |x|^{2\alpha+1} d_q x \quad (2.13)$$

$$\|F_{\Sigma} E_S\|_{HS}^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |k(t, x)|^2 |t|^{2\alpha+1} |x|^{2\alpha+1} d_q t d_q x \leq \frac{4c_{\alpha,q}^2}{(q, q)_{\infty}^2} |S| |\Sigma|. \quad (2.14)$$

Proof. We have

$$\begin{aligned} (F_{\Sigma} E_S f)(\lambda) &= \frac{c_{\alpha,q}}{2} \int_{\Sigma} \psi_{\lambda}^{\alpha,q}(x) (F_D^{\alpha,q}(E_S f))(x) |x|^{2\alpha+1} d_q x \\ &= \frac{c_{\alpha,q}^2}{4} \int_{\Sigma} \psi_{\lambda}^{\alpha,q}(x) \left(\int_S \psi_{-x}^{\alpha,q}(t) f(t) |t|^{2\alpha+1} d_q t \right) |x|^{2\alpha+1} d_q x. \end{aligned}$$

By Fubini's theorem we obtain

$$(F_{\Sigma} E_S f)(\lambda) = \frac{c_{\alpha,q}^2}{4} \int_S f(t) \left(\int_{\Sigma} \psi_{\lambda}^{\alpha,q}(x) \psi_{-x}^{\alpha,q}(t) |x|^{2\alpha+1} d_q x \right) |t|^{2\alpha+1} d_q t \quad (2.15)$$

so that

$$(F_{\Sigma} E_S f)(\lambda) = \int_{-\infty}^{\infty} f(t) k(t, \lambda) |t|^{2\alpha+1} d_q t \quad (2.16)$$

where

$$k(t, \lambda) = \frac{c_{\alpha,q}^2}{4} \chi_S(t) \int_{\Sigma} \psi_{\lambda}^{\alpha,q}(x) \psi_{-x}^{\alpha,q}(t) |x|^{2\alpha+1} d_q x. \quad (2.17)$$

Let $g_t(\lambda) = k(t, \lambda)$ then the inversion formula shows that

$$F_D^{\alpha,q}(g_t)(\lambda) = \frac{c_{\alpha,q}}{2} \chi_S(t) \chi_{\Sigma}(\lambda) \psi_{-\lambda}^{\alpha,q}(t). \quad (2.18)$$

By the Plancherel formula it follows

$$\int_{-\infty}^{\infty} |g_t(s)|^2 |s|^{2\alpha+1} d_q s = \int_{-\infty}^{\infty} |F_D^{\alpha,q}(g_t)(\lambda)|^2 |\lambda|^{2\alpha+1} d_q \lambda \leq \frac{4c_{\alpha,q}^2}{(q, q)_{\infty}^2} |\Sigma|. \quad (2.19)$$

Hence, integrating over $t \in S$, we obtain the desired result. \square

Theorem 2.1. Let S and Σ be a q -mesurable sets such that f is ϵ_S -concentrated on S and $F_D^{\alpha,q}(f)$ is ϵ_{Σ} -concentrated on Σ and suppose that $\|f\|_{2,q} = 1$ and $\epsilon_S + \epsilon_{\Sigma} < 1$. Then

$$|S| |\Sigma| \geq (1 - (\epsilon_S + \epsilon_{\Sigma}))^2.$$

Proof. The norm of an operator Q is defined by:

$$\|Q\| = \sup_{g \in L^2_{\alpha,q}(\mathbb{R}_q)} \frac{\|Q(g)\|_{2,q}}{\|g\|_{2,q}}.$$

By the triangle inequality we have

$$\begin{aligned} \|f - F_\Sigma E_S f\|_{2,q} &\leq \|f - F_\Sigma f\|_{2,q} + \|F_\Sigma f - F_\Sigma E_S f\|_{2,q} \\ &\leq \epsilon_\Sigma + \|F_\Sigma\| \|f - E_S f\|_{2,q} \leq \epsilon_\Sigma + \epsilon_S \end{aligned}$$

then

$$\|F_\Sigma E_S f\|_{2,q} \geq 1 - (\epsilon_\Sigma + \epsilon_S).$$

Therefore

$$\|F_\Sigma E_S\|_q \geq 1 - (\epsilon_\Sigma + \epsilon_S).$$

Using Holder's inequality, we obtain

$$\begin{aligned} \|F_\Sigma E_S f\|_{2,q}^2 &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(t) k(t, \xi) |t|^{2\alpha+1} d_q t \right|^2 |\xi|^{2\alpha+1} d_q \xi \\ &\leq \|f\|_{2,q}^2 \int_{-\infty}^{+\infty} \int_T |K(t, \xi)|^2 |t|^{2\alpha+1} d_q t |\xi|^{2\alpha+1} d_q \xi. \end{aligned}$$

By Fubini-Tonnelli's theorem and Lemma 2.1

$$\|F_\Sigma E_S f\|_{2,q} \leq \|f\|_{2,q} \|F_\Sigma E_S\|_{HS} \leq \frac{2c_{\alpha,q} \|f\|_{2,q}}{(q, q)_\infty} \sqrt{|S||\Sigma|}.$$

Thus, the proof is complete. \square

Definition 2.2. For $f \in L^1_{\alpha,q}(\mathbb{R}_q) \cap L^2_{\alpha,q}(\mathbb{R}_q)$. We say that

(1) f is ϵ -timelimited on S if:

$$\|f \chi_{S^c}\|_{1,q} \leq \epsilon \|f\|_{1,q}.$$

(2) f is ϵ -bandlimited on Σ if

$$\|F_D^{\alpha,q}(f) \chi_{\Sigma^c}\|_{2,q} \leq \epsilon \|f\|_{2,q}. \quad (2.20)$$

Theorem 2.2. Let $0 < \epsilon_1, \epsilon_2 < 1$, S, Σ be a pair of q -measurable subsets of \mathbb{R}_q and $f \in L^1_{\alpha,q}(\mathbb{R}_q) \cap L^2_{\alpha,q}(\mathbb{R}_q)$. If f is ϵ_1 -time limited on S and ϵ_2 -bandlimited on Σ then :

$$|S||\Sigma| \geq (1 - \epsilon_1)^2 (1 - \epsilon_2^2) (q, q)_\infty^2.$$

Proof. We have

$$\|f \chi_S\|_{1,q} = \|f\|_{1,q} - \|f \chi_{S^c}\|_{1,q}$$

hence

$$\|f \chi_S\|_{1,q} \geq (1 - \epsilon_1) \|f\|_{1,q}.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\|f \chi_S\|_{1,q}^2 \leq |S| \|f\|_{2,q}^2.$$

So

$$|S| \|f\|_{2,q}^2 \geq (1 - \epsilon_1)^2 \|f\|_{1,q}^2. \quad (2.21)$$

On the other hand, we have

$$\|F_D^{\alpha,q}(f) \chi_\Sigma\|_{2,q}^2 = \|F_D^{\alpha,q}(f)\|_{2,q}^2 - \|F_D^{\alpha,q}(f) \chi_{\Sigma^c}\|_{2,q}^2$$

From (1.11) and (2.20) we get

$$\|F_D^{\alpha,q}(f) \chi_\Sigma\|_{2,q}^2 \geq (1 - \epsilon_2^2) \|f\|_{2,q}^2$$

Then

$$\|F_D^{\alpha,q}(f)\chi_{\Sigma}\|_{2,q}^2 \leq |\Sigma| \|F_D^{\alpha,q}(f)\|_{\infty,q}^2 \leq \frac{4c_{\alpha,q}^2 |\Sigma|}{(q,q)_{\infty}^2} \|f\|_{1,q}^2.$$

Consequently

$$\frac{4c_{\alpha,q}^2 |\Sigma|}{(q,q)_{\infty}^2} \|f\|_{1,q}^2 \geq (1 - \epsilon_2^2) \|f\|_{2,q}^2.$$

By (2.21) we obtain:

$$|\Sigma| |S| \geq \frac{(q,q)_{\infty}^2}{4c_{\alpha,q}^2} (1 - \epsilon_1)^2 (1 - \epsilon_2^2).$$

□

3. Uncertainty principle for orthonormal bases

In this section we will prove an uncertainty principle for orthonormal bases for $L_{\alpha,q}^2(\mathbb{R}_q)$.

Theorem 3.1. *Let $(\varphi_n)_{n=1}^N$ be an orthonormal system in $L_{\alpha,q}^2(\mathbb{R}_q)$ and let S and Σ be two measurable subsets of \mathbb{R}_q . Assume that*

$$\|E_{S^c}\varphi_n\|_{2,q} \leq a_n \quad ; \quad \|F_{\Sigma^c}\varphi_n\|_{2,q} \leq b_n$$

then

$$\sum_{n=1}^N \left(1 - \frac{3}{2}a_n - \frac{3}{2}b_n\right) \leq \frac{4c_{\alpha,q}^2}{(q,q)_{\infty}^2} |S||\Sigma|.$$

Proof. We consider the corresponding self-adjoint operator

$$Q = (F_{\Sigma}E_S)^*(F_{\Sigma}E_S) = E_S F_{\Sigma} E_S.$$

Since

$$\text{tr}(Q) = \|F_{\Sigma}E_S\|_{HS}^2 \leq \frac{4c_{\alpha,q}^2}{(q,q)_{\infty}^2} |S||\Sigma|$$

we have

$$\sum_{n=1}^N \langle Q\varphi_n, \varphi_n \rangle \leq \text{tr}(Q) = \|F_{\Sigma}E_S\|_{HS}^2 \leq \frac{4c_{\alpha,q}^2}{(q,q)_{\infty}^2} |S||\Sigma|.$$

On the other hand,

$$\begin{aligned} \langle Q\varphi_n, \varphi_n \rangle &= \langle F_{\Sigma}E_S\varphi_n, E_S\varphi_n \rangle \\ &= \langle \varphi_n, \varphi_n \rangle - \langle \varphi_n - E_S\varphi_n, \varphi_n \rangle \\ &\quad - \langle E_S\varphi_n, \varphi_n - F_{\Sigma}\varphi_n \rangle - \langle F_{\Sigma}E_S\varphi_n, \varphi_n - E_S\varphi_n \rangle. \end{aligned}$$

It follows that

$$\langle Q\varphi_n, \varphi_n \rangle \geq 1 - 2a_n - b_n$$

and

$$\sum_{n=1}^N (1 - 2a_n - b_n) \leq \frac{4c_{\alpha,q}^2}{(q,q)_{\infty}^2} |S||\Sigma|. \quad (3.22)$$

Now, if we consider the operator

$$\tilde{Q} = (E_S F_{\Sigma})^*(E_S F_{\Sigma})$$

we get similarly

$$\sum_{n=1}^N (1 - a_n - 2b_n) \leq \frac{4c_q^2}{(q,q)_{\infty}^2} |S||\Sigma|. \quad (3.23)$$

The desired result follows by combining (3.22) and (3.23). \square

The following corollary is an immediate consequence of Theorem 3.1

Corollary 3.1. *Let $r_0, r_1 > 0$ and let $0 < \epsilon_1, \epsilon_2 < 1$ such that $\epsilon_1 + \epsilon_2 < \frac{2}{3}$, let $(\varphi_n)_{n=1}^N$ be an orthonormal system in $L_{\alpha,q}^2(\mathbb{R}_q)$ such that φ_n is ϵ_1 -concentrated on a set $B_{r_0} = \mathbb{R}_q \cap]-r_0, r_0[$ and $F_D^{\alpha,q}(\varphi_n)$ is ϵ_2 -concentrated on a set $B_{r_1} = \mathbb{R}_q \cap]-r_1, r_1[$ for each $n = 1, \dots, N$, i.e.*

$$\int_{B_{r_0}} |\varphi_n(t)|^2 |t|^{2\alpha+1} d_q t \geq 1 - \epsilon_1^2 \quad ; \quad \int_{B_{r_1}} |F_D^{\alpha,q}(\varphi_n)(w)|^2 |w|^{2\alpha+1} d_q w \geq 1 - \epsilon_2^2. \quad (3.24)$$

Then

$$N \leq \frac{16c_{\alpha,q}^2(r_0 r_1)^{2\alpha+2}}{(1 - 3\frac{\epsilon_1 + \epsilon_2}{2})(q, q)_\infty^2 [2\alpha + 2]_q^2}. \quad (3.25)$$

Another immediate application of the localization inequality is the q -analogue quantitative version of Shapiro Umbrella Theorem.

Corollary 3.2. *let $A, B > 0$ and $\{\varphi_n\}_{n=1}^N$ be an orthonormal system in $L_{\alpha,q}^2(\mathbb{R}_q)$ such that*

$$\| |x|^s \varphi_n \|_{2,q} \leq A^s \quad \text{and} \quad \| |\xi|^s F_D^{\alpha,q}(\varphi_n) \|_{2,q} \leq B^s.$$

Then

$$N \leq c(s, q)(AB)^{2\alpha+2},$$

$$\text{where } c(s, q) = 64 \frac{c_{\alpha,q}^2}{(q, q)_\infty^2 [2\alpha + 2]_q^2} 4^{\frac{4(\alpha+1)}{s}}.$$

Proof. Let $r_0 = 4^{\frac{1}{s}} A$ and $r_1 = 4^{\frac{1}{s}} B$, we have

$$\begin{aligned} & \int_{B_{r_0}^c} |\varphi_n(x)|^2 |x|^{2\alpha+1} d_q x \\ &= \int_{B_{r_0}^c} |x|^{-2s} |x|^{2s} |\varphi_n(x)|^2 |x|^{2\alpha+1} d_q x \leq \frac{1}{16} \frac{1}{A^{2s}} \| |x|^s \varphi_n \|_{2,q}^2 \leq \frac{1}{16} \end{aligned}$$

so φ_n is $\frac{1}{4}$ -concentrated on B_{r_0} and in the same way we prove that $F_D^{\alpha,q}$ is $\frac{1}{4}$ -concentrated on B_{r_1} . The desired result follows from Corollary 3.1. \square

Theorem 3.2. *Let s be a positive real and $(\varphi_n)_n$ be an orthonormal sequence in $L_{\alpha,q}^2(\mathbb{R}_q)$. Then there exists a constant $C(\alpha, s, q)$ such that*

$$\sum_{n=1}^N (\| |x|^s \varphi_n \|_{2,q}^2 + \| |\lambda|^s F_D^{\alpha,q}(\varphi_n) \|_{2,q}^2) \geq C(\alpha, s, q) N^{1 + \frac{s}{2(\alpha+1)}}. \quad (3.26)$$

This theorem implies in particular that, if the elements of an orthonormal sequence and their q -Dunkl-Fourier transforms have uniformly bounded dispersions then the sequence is finite.

Proof. Let $(\varphi_n)_n$ be an orthonormal sequence in $L_{\alpha,q}^2(\mathbb{R}_q)$. For each $k \in \mathbb{Z}$ we define

$$P_k = \left\{ n : \max \left\{ \| |x|^s \varphi_n \|_{2,q}, \| |\lambda|^s F_D^{\alpha,q}(\varphi_n) \|_{2,q} \right\} \in (2^{s(k-1)}, 2^{sk}] \right\}.$$

then

$$\| |x|^s \varphi_n \|_{2,q} \leq 2^{sk} \quad \text{and} \quad \| |\lambda|^s F_D^{\alpha,q}(\varphi_n) \|_{2,q} \leq 2^{sk}$$

whenever $n \in P_k$. From Corollary 3.2, deduce that the number of elements in $\bigcup_{j=-\infty}^k P_j$ is less than $c_1(\alpha, s, q) 4^{2k(\alpha+1)}$ where $c_1(\alpha, s, q)$ is a constant that does not depend on k . This shows that when $c_1(\alpha, s, q) 4^{2k(\alpha+1)} < 1$, the number of elements in $\bigcup_{j=-\infty}^k P_j$ is null.

Consequently, there exists k_0 such that P_k is empty for all $k < k_0$.

For given $N > 2c_1(\alpha, s, q)$ choose k such that

$2c_1(\alpha, s, q)4^{2k(\alpha+1)} \geq N > 2c_1(\alpha, s, q)4^{2(k-1)(\alpha+1)}$. Then at least half of $\{1, \dots, N\}$ does not belong to $\bigcup_{j=1}^{k-1} P_j$ and we obtain

$$\sum_{n=0}^N (\| |x|^s \varphi_n \|_{2,q}^2 + \| |\lambda|^{2\alpha+1} F_D^{\alpha,q}(\varphi_n) \|_{2,q}^2) \geq \frac{N}{2} 4^{s(k-1)} \geq a(\alpha, s, q) N^{1+\frac{s}{2(\alpha+1)}}$$

For $N \leq 2c_1(\alpha, s, q)$ we have

$$\sum_{n=1}^N (\| |x|^s \varphi_n \|_{2,q}^2 + \| |\lambda|^{2\alpha+1} F_D^{\alpha,q}(\varphi_n) \|_{2,q}^2) \geq N 4^{s(k_0-1)} \geq \frac{4^{s(k_0-1)}}{(2c_1(\alpha, s, q))^{\frac{s}{2(\alpha+1)}}} N^{1+\frac{s}{2(\alpha+1)}}$$

this achieve the proof. \square

We deduce the following result

Corollary 3.3. *Let $(\varphi_n)_{n \geq 1}$ be an orthonormal sequence in $L_{\alpha,q}^2(\mathbb{R}_q)$, then*

$$\sup_n (\| |x|^s \varphi_n \|_{2,q} + \| |\xi|^s F_D^{\alpha,q}(\varphi_n) \|_{2,q}) = \infty. \quad (3.27)$$

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