

## A DUALITY-TYPE METHOD FOR THE FOURTH ORDER OBSTACLE PROBLEM

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*In this paper we study by duality the fourth order obstacle problem. The main idea is to use Fenchel duality theorem. We apply the duality principle to the approximate problem as well and the dual is a finite dimensional minimization problem, which can be solved efficiently. The method developed here is easy to implement. The obtained results are superior to other known methods, in the considered examples.*

**Keywords:** biharmonic obstacle problem, dual problem, Fenchel theorem, approximate problem.

**MSC2010:** 49N25, 49N15, 65K15

### 1. Introduction

The biharmonic obstacle problem is subject to intensive research activity due to many authors, starting with the pioneering works of Landau and Lifshitz [16], Brezis and Stampacchia [7], Duvaut and Lions [11, 12], Glowinski et al. [13] and Comodi [10]. There are many applications in elasticity theory and fluid mechanics: bending of plates and beams, Stokes problem, free boundary problems.

The well known paper by Caffarelli and Friedman [8] from 1979 proves regularity of the solution up to the boundary in the case  $n \leq 4$ . They proved that the solution is  $C^2(\Omega)$  if  $n = 2$ . We also mention a paper on the stability of the solution of the obstacle problem for plates by Pozzolini and Lèger [21], proving the existence of a strong derivative of the solution. The Neumann boundary value problems for the biharmonic obstacle problems was studied in the paper of Karachik, Turmetov and Bekaeva [15], in which they give the necessary and sufficient conditions for the solution to exist. We also find the problem of bending a plate over an obstacle as an example of in the monograph by Rodrigues [22]. In Neittaanmaki, Sprekels and Tiba [19], Sprekels and Tiba [23] a duality approach was used in the study of Kirchhoff-Love arches and explicit solutions formulas were obtained. Other recent papers that study the biharmonic obstacle problem are Chuquipoma, Raposo and Bastos [9], Yau and Gao [26].

From the numerical point of view, there also are many articles that treat the fourth order obstacle problem. One of the recent works in this area develops a Morley

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finite element method for the displacement obstacle problem for the clamped Kirchhoff plates on polygonal domains, Brenner et al. [5]. There are many other articles treating numerically the biharmonic obstacle problems in very different manners, such as [1, 20, 2, 4].

In this paper we treat the biharmonic obstacle problem by the duality point of view. We apply a similar algorithm as for the second order obstacle problem in Merlușcă [17, 18]. Constructing the approximate problem and passing by the Fenchel duality theorem, we obtain a finite dimensional dual problem. In Section 2, we consider the simply supported plate problem and we construct an approximate problem. In Section 3, by using Fenchel's duality theorem we obtain the dual approximate problem as a finite dimensional minimization problem. Section 4 is dedicated to some numerical modelling and numerical results which point out that in the cases considered here the duality method could generate better approximate solutions in comparison with other methods. The argument is that the value of the cost functional to be minimized is strictly less than the one obtained by other methods.

## 2. The simply supported plate problem and its approximation

We consider that  $\Omega \subset \mathbb{R}^n$ , with  $n \leq 3$ , a bounded open set with the strong local Lipschitz property. We denote by  $V$  the space  $H^2(\Omega) \cap H_0^1(\Omega)$  endowed with the scalar product

$$(u, v)_V = \int_{\Omega} \Delta u \Delta v.$$

$V$  is a Hilbert space and the norm

$$|y|_V = \left( \int_{\Omega} (\Delta y)^2 \right)^{\frac{1}{2}}$$

is equivalent to the usual Sobolev norm.

We extend the duality method presented in Merlușcă [17, 18] to the following obstacle problem

$$\min_{y \in K} \left\{ \frac{1}{2} \int_{\Omega} (\Delta y)^2 - \int_{\Omega} f y \right\} \quad (1)$$

where  $f \in L^2(\Omega)$ , and  $K = \{y \in V : y \geq 0 \text{ in } \Omega\}$ .

The problem considered here is a simplified model of the simply supported plate problem.

By the Sobolev theorem, and using the fact that  $\dim \Omega \leq 3$ , we have  $H^2(\Omega) \cap H_0^1(\Omega) \rightarrow C(\bar{\Omega})$  and thus we may consider the following approximate problem

$$\min \left\{ \frac{1}{2} \int_{\Omega} (\Delta y)^2 - \int_{\Omega} f y : y \in V; y(x_i) \geq 0, i = 1, 2, \dots, k \right\} \quad (2)$$

where  $\{x_i\}_{i \in \mathbb{N}} \subseteq \Omega$  is a dense set in  $\Omega$ . For each  $k \in \mathbb{N}$ , we denote the closed convex cone

$$C_k = \{y \in V : y(x_i) \geq 0, i = 1, 2, \dots, k\}.$$

**Proposition 2.1.** The following assertions are true

- (i): Problem (1) has a unique solution  $\bar{y} \in K$ .
- (ii): Problem (2) has a unique solution  $\bar{y}_k \in C_k$ , for any  $k \in \mathbb{N}$ .

The above Proposition can be easily proved using the compact imbedding  $H^2(\Omega) \cap H_0^1(\Omega) \rightarrow L^2(\Omega)$ .

Furthermore, we have the following approximation result

**Theorem 2.1.** The sequence  $\{\bar{y}_k\}_k$  of the solutions of problems (2), for  $k \in \mathbb{N}$ , is a strongly convergent sequence in  $V$  to the unique solution  $\bar{y}$  of the problem (1).

*Proof.* Let  $\{\bar{y}_k\}_{k \in \mathbb{N}} \subseteq V$  be the sequence of the solutions of the problems (2). Consider  $y \in K$  arbitrary. Then  $y \in C_k$ , for every  $k \in \mathbb{N}$ . Thus, since  $\bar{y}_k = \operatorname{argmin}(P_k)$ , we have

$$\frac{1}{2} \int_{\Omega} (\Delta y)^2 - \int_{\Omega} f y \geq \frac{1}{2} \int_{\Omega} (\Delta \bar{y}_k)^2 - \int_{\Omega} f \bar{y}_k, \quad \forall y \in K, \forall k \in \mathbb{N}. \quad (3)$$

Then, there is a constant  $M > 0$  such that

$$M \geq \frac{1}{2} \int_{\Omega} (\Delta y_k)^2 - \int_{\Omega} f y_k \geq \frac{1}{2} |\bar{y}_k|_V^2 - c \|f\|_{L^2(\Omega)} |\bar{y}_k|_V^2.$$

Then the sequence  $\{|\bar{y}_k|_V\}_k$  is bounded, which means that the sequence  $\{\bar{y}_k\}_k \subseteq V$  is weakly convergent, on a subsequence, to an element  $\hat{y} \in V$ .

Since  $\bar{y}_k(x_i) \geq 0$  and  $\bar{y}_k \rightarrow \hat{y}$  uniformly on  $\bar{\Omega}$ , then for every  $x \in \bar{\Omega}$  we have  $\bar{y}_k(x) \rightarrow \hat{y}(x)$ . Then  $\hat{y}(x_i) \geq 0$ ,  $\forall i \in \mathbb{N}$ . As we assumed the set  $\{x_i : i \in \mathbb{N}\}$  to be dense in  $\Omega$ , it yields that  $\hat{y} \in K$ . Thus,  $\hat{y}$  is admissible for (1).

Since  $\bar{y} \in K$ , we can write (3) for  $\bar{y}$ ,

$$\frac{1}{2} \int_{\Omega} (\Delta \bar{y})^2 - \int_{\Omega} f \bar{y} \geq \frac{1}{2} \int_{\Omega} (\Delta \bar{y}_k)^2 - \int_{\Omega} f \bar{y}_k. \quad (4)$$

Considering the weak inferior semicontinuity of the norm, we pass to the limit

$$\frac{1}{2} \int_{\Omega} (\Delta \bar{y})^2 - \int_{\Omega} f \bar{y} \geq \frac{1}{2} \int_{\Omega} (\Delta \hat{y})^2 - \int_{\Omega} f \hat{y}.$$

As we already stated in Proposition 2.1, the solution of problem (1) is unique, it follows that  $\bar{y} = \hat{y}$ . Then  $\bar{y}_k \rightarrow \bar{y}$  weakly in  $V$ .

To prove the strong convergence, we use (4) to get that

$$\frac{1}{2} |\bar{y}|_V^2 \geq \limsup_{k \rightarrow \infty} \frac{1}{2} |\bar{y}_k|_V^2. \quad (5)$$

Again, by the weak convergence we have the other inequality too

$$\frac{1}{2} |\bar{y}|_V^2 \leq \liminf_{k \rightarrow \infty} \frac{1}{2} |\bar{y}_k|_V^2. \quad (6)$$

Then, using Proposition 3.32, page 78, Brezis, [6] and the equality given by the relations (5) and (6)

$$\frac{1}{2} |\bar{y}|_V^2 = \lim_{k \rightarrow \infty} \frac{1}{2} |\bar{y}_k|_V^2,$$

it follows that  $\bar{y}_k \rightarrow \bar{y}$  strongly in  $V$ . Note that, since the limit is unique, the convergence is valid without taking subsequences.  $\square$

### 3. The dual problem

In this section we construct the dual continuous and approximate problems which will help us to solve problem (1) easier.

We denote  $V^*$  the dual space of  $V$ . By the Riesz representation theorem, for every  $y^* \in V^*$  we find a unique element  $v \in V$  such that

$$(y^*, y)_{V^* \times V} = (v, y)_V, \quad \forall y \in V.$$

And, moreover,  $|y^*|_{V^*} = |v|_V$ .

Notice that  $H^{-2}(\Omega)$  is not dense in  $V^*$ , since  $H_0^2(\Omega)$  is not dense in  $V$ . But the inclusion  $H_0^2(\Omega) \subset V$  is continuous, then for every  $y^* \in V^*$  the restriction  $y^*|_{H_0^2(\Omega)} \in H^{-2}(\Omega)$ .

Consider a sequence  $\{v_n\}_n \subset H^4(\Omega) \cap H_0^1(\Omega)$ , such that  $v_n \rightarrow v \in H^2(\Omega)$  and  $\Delta v_n = 0$  on  $\partial\Omega$ . This is possible by taking  $v_n$  as the solution of an appropriate system of type (8), obtained from the similar system satisfied by  $v$  in a weak sense and by regularizing its right-hand side. Let  $y_n^* = I(v_n)$  where  $I : V \rightarrow V^*$  is the canonical isomorphism. Obviously, if we denote  $I(v) = y^*$  we get that  $y_n^* \rightarrow y^*$  strongly in  $V^*$ . Then

$$(y_n^*, y)_{V^* \times V} = \int_{\Omega} \Delta v_n \Delta y = \int_{\Omega} (\Delta \Delta v_n) y, \quad \forall y \in V$$

Thus  $\Delta \Delta v_n = y_n^*$  converges strongly to an element  $x \in V^*$ . We denote this element  $x$  with  $\Delta \Delta v$ .

With the above arguments, the duality mapping  $J : V \rightarrow V^*$  may be written as  $J(v) = \Delta \Delta v$ .

We consider the functional

$$F(y) = \frac{1}{2} \int_{\Omega} (\Delta y)^2 - \int_{\Omega} f y, \quad y \in V. \quad (7)$$

The convex conjugate of  $F$  is

$$F^*(y^*) = \sup \left\{ (y^*, y)_{V^* \times V} - \frac{1}{2} \int_{\Omega} (\Delta y)^2 + \int_{\Omega} f y : y \in V \right\}.$$

Since  $f \in L^2(\Omega)$ , with the arguments above we find a unique  $y_f \in V$ , which is the weak solution of the problem

$$\begin{cases} \Delta \Delta y_f = f, & \text{on } \Omega \\ y_f = 0, \Delta y_f = 0, & \text{on } \partial\Omega \end{cases} \quad (8)$$

such as

$$(f, y)_{V^* \times V} = \int_{\Omega} \Delta \Delta y_f y = \int_{\Omega} f y, \quad \forall y \in V.$$

Then

$$F^*(y^*) = \sup \left\{ (y^* + f, y)_{V^* \times V} - \frac{1}{2} \int_{\Omega} (\Delta y)^2 : y \in V \right\}.$$

Using the inequality  $(y^* + f, y)_{V^* \times V} \leq \frac{1}{2} |y^* + f|_{V^*}^2 + \frac{1}{2} |y|_V^2$  we obtain

$$F^*(y^*) \leq \frac{1}{2} |y^* + f|_{V^*}^2$$

Since the duality mapping is bijective, for each  $y^* + f \in V^*$  there exists an element  $v \in V$  such that  $|v|_V^2 = |y^* + f|_{V^*}^2 = (J(v), v)_{V^* \times V}$ . Then  $(y^* + f, v)_{V^* \times V} - \frac{1}{2}|v|_V^2 = \frac{1}{2}|y^* + f|_{V^*}^2$ . It yields that the convex conjugate of  $F$  is

$$F^*(y^*) = \frac{1}{2}|y^* + f|_{V^*}^2.$$

We now consider the functional  $g = -I_K$ . From the concave conjugate definition we have

$$g^\bullet(y^*) = \begin{cases} 0, & y^* \in K^* \\ -\infty, & y^* \notin K^* \end{cases}$$

with  $K^* = \{y^* \in V^* : (y, y^*)_{V \times V^*} \geq 0, \forall y \in K\} = V_+^*$ .

We can apply Fenchel duality Theorem (see Barbu and Precupanu, [3], pp 189), since  $F$  and  $-g$  are convex and proper functionals on  $H^2(\Omega) \cap H_0^1(\Omega)$ , the domain of  $g$  is  $D(g) = K$ , and  $F$  is continuous everywhere on  $K$ . Then

$$\min_{y \in K} \left\{ \frac{1}{2} \int_{\Omega} (\Delta \bar{y})^2 - \int_{\Omega} f y \right\} = \max \left\{ -\frac{1}{2} |f + y^*|_{V^*}^2 : y^* \in K^* \right\}.$$

The dual problem associated to problem (1) is stated as

$$\min \left\{ \frac{1}{2} |f + y^*|_{V^*}^2 : y^* \in K^* \right\}.$$

We compute the concave conjugated of  $g_k = -I_{C_k}$  which is needed for the dual approximate problem. Thus, by definition, the concave conjugate is

$$g_k^\bullet(y^*) = \inf \{(y, y^*)_{V \times V^*} - g_k(y) : y \in C_k\} = \begin{cases} 0, & y^* \in C_k^* \\ -\infty, & y^* \notin C_k^* \end{cases}$$

where  $C_k^* = \{y^* \in V^* : (y^*, y)_{V^* \times V} \geq 0, \forall y \in C_k\}$ .

**Lemma 3.1.** Consider the Dirac distributions concentrated in  $x_i \in \Omega$ , i.e.  $\delta_{x_i}(y) = y(x_i)$ ,  $\forall y \in H^2(\Omega) \cap H_0^1(\Omega)$ .

The polar cone of  $C_k$  is

$$C_k^* = \left\{ u = \sum_{i=1}^k \alpha_i \delta_{x_i} : \alpha_i \geq 0 \right\}$$

*Proof.* We denote

$$T = \left\{ u = \sum_{i=1}^k \alpha_i \delta_{x_i} : \alpha_i \geq 0 \right\}$$

The Dirac distributions  $\delta_{x_i}$  are linear and continuous functionals on  $V$  due to the fact that  $H^2(\Omega) \cap H_0^1(\Omega) \rightarrow C(\bar{\Omega})$ . This yields that  $T \subset V^*$ .

We compute the polar of the cone  $T$ , which is, by definition,

$$T^* = \{y \in V : (y, u)_{V \times V^*} \geq 0, \forall u \in T\}.$$

Note that

$$(y, u)_{V \times V^*} = (y, \sum_{i=1}^k \alpha_i \delta_{x_i})_{V \times V^*} = \sum_{i=1}^k \alpha_i (y, \delta_{x_i})_{V \times V^*} = \sum_{i=1}^k \alpha_i y(x_i)$$

and  $\alpha_i \geq 0, \forall i = \overline{1, k}$ . Here we also use that  $T \subset H^{-2}(\Omega)$ . Then we have the equivalence  $(y, u)_{V \times V^*} \geq 0, \forall u \in T \Leftrightarrow y(x_i) \geq 0, \forall i = \overline{1, k}$ . Thus

$$T^* = \{y \in V : y(x_i) \geq 0, \forall i = \overline{1, k}\} = C_k.$$

Applying the polar to the above relation, we have  $(T^*)^* = C_k^*$ .

The Bipolar theorem (Barbu and Precupanu, [3], pp 88) states that

$$T^{**} = \overline{\text{conv}(T \cup \{0\})}. \quad (9)$$

We have to prove just that  $T$  is a closed cone, because it is obvious that  $0 \in T$  and the cone  $T$  is convex.

Take  $u \in \overline{T}$ . Then there exists a sequence  $(u_n)_n \in T$  convergent to  $u$  in  $V$ . Since  $u_n \in T$ , we have

$$u_n = \sum_{i=1}^k \alpha_i^n \delta_{x_i} \rightarrow u \text{ in } V^*.$$

We consider  $S(x_i, r) \subset \Omega$  such that  $x_j \notin S(x_i, r)$ , for  $i \neq j$ . For every  $i \in \{1, 2, \dots, k\}$ , let  $\rho_i \in \mathcal{D}(S(x_i, r)) \subset \mathcal{D}(\Omega)$  such that  $\rho_i(x_i) = 1$ . Then, the convergence above gives us

$$\left( \sum_{i=1}^k \alpha_i^n \delta_{x_i}, \rho_j \right)_{V^* \times V} \rightarrow (u, \rho_j)_{V^* \times V}, \quad \forall j = \overline{1, k}.$$

We obtain

$$\alpha_j^n \rightarrow (u, \rho_j)_{V^* \times V}, \quad \forall j = \overline{1, k}.$$

We denote  $\alpha_j = \lim_{n \rightarrow +\infty} \alpha_j^n$ , and clearly it is independent of  $\rho_j$ .

Then

$$u = \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \sum_{i=1}^k \alpha_i^n \delta_{x_i} = \sum_{i=1}^k \left( \lim_{n \rightarrow \infty} \alpha_i^n \right) \delta_{x_i} = \sum_{i=1}^k \alpha_i \delta_{x_i}$$

which implies that  $u \in T$ .

Then  $T$  is closed and, from relation (9), it follows that  $T^{**} = T$ . Then  $T = C_k^*$  as claimed.  $\square$

Since the domain of  $g_k$  is  $D(g_k) = C_k$  and the functional  $F$  is still continuous on the closed convex cone  $C_k$ , the hypothesis of Fenchel duality Theorem are satisfied and

$$\min \left\{ \frac{1}{2} \int_{\Omega} (\Delta y)^2 - \int_{\Omega} f y : y \in C_k \right\} = \max \left\{ -\frac{1}{2} \|y^* + f\|_{V^*}^2 : y^* \in C_k^* \right\}$$

We write the dual approximate problem associated to problem (2)

$$\min \left\{ \frac{1}{2} \|y^* + f\|_{V^*}^2 : y^* \in C_k^* \right\}. \quad (10)$$

**Theorem 3.1.** Consider  $\bar{y}_k$  to be the solution of the approximate problem (2) and  $\bar{y}_k^*$  the solution of the dual approximate problem (10). Then

$$\bar{y}_k = J^{-1}(\bar{y}_k^* + f) \quad (11)$$

where  $J$  is the duality mapping  $J : V \rightarrow V^*$ .

Moreover,  $(\bar{y}_k^*, \bar{y}_k)_{V^* \times V} = 0$ .

*Proof.* We get the following system of equations from Theorem 2.4 (Barbu and Precupanu, [3], pp 188)

$$\bar{y}_k^* \in \partial F(\bar{y}_k), \quad (12)$$

$$-\bar{y}_k^* \in \partial I_{C_k}(\bar{y}_k) \quad (13)$$

where the functional  $F$  is the functional defined as in (7).

Using the definition of the subdifferential of a convex function, from (12), we obtain  $\bar{y}_k^* + f \in J(\bar{y}_k)$ . Since the duality mapping is single-valued and bijective, we get that  $\bar{y}_k = J^{-1}(\bar{y}_k^* + f)$ .

From (13), we get for the indicator functions

$$I_{C_k}(\bar{y}_k) - I_{C_k}(z) \leq (-\bar{y}_k^*, \bar{y}_k - z)_{V^* \times V}, \quad \forall z \in C_k$$

Take  $z = \frac{1}{2}\bar{y}_k$  and we get

$$I_{C_k}(\bar{y}_k) \leq -(\bar{y}_k^*, \bar{y}_k)_{V^* \times V}$$

Then, take  $z = 2\bar{y}_k \in C_k$  and we have the opposite inequality

$$I_{C_k}(\bar{y}_k) \geq -(\bar{y}_k^*, \bar{y}_k)_{V^* \times V}$$

But, since  $\bar{y}_k \in C_k$ , we can conclude that  $(\bar{y}_k^*, \bar{y}_k)_{V^* \times V} = 0$  □

**Remark 3.1.** Since  $\bar{y}_k^* \in C_k^*$ , by Lemma 3.1, we know

$$\bar{y}_k^* = \sum_{i=1}^k \alpha_i^* \delta_{x_i} \in H^{-2}(\Omega)$$

where  $\alpha_i^* \geq 0$  for all  $i = 1, 2, \dots, k$ . Then, from the Theorem 3.1,

$$0 = (\bar{y}_k^*, \bar{y}_k)_{V^* \times V} = \left( \sum_{i=1}^k \alpha_i^* \delta_{x_i}, \bar{y}_k \right)_{V^* \times V} = \sum_{i=1}^k \alpha_i^* (\delta_{x_i}, \bar{y}_k)_{V^* \times V} = \sum_{i=1}^k \alpha_i^* \bar{y}_k(x_i)$$

Thus,

$$\sum_{i=1}^k \alpha_i^* \bar{y}_k(x_i) = 0$$

Again  $\bar{y}_k \in C_k$ , using the definition of the cone  $C_k$ , we have  $\bar{y}_k(x_i) \geq 0$  for all  $i = 1, 2, \dots, k$ . Then

$$\alpha_i^* \bar{y}_k(x_i) = 0, \quad \forall i = 1, 2, \dots, k.$$

In conclusion, the Lagrange multipliers  $\alpha_i^*$  are zero if the constraint is inactive, i.e.  $\bar{y}_k(x_i) > 0$  and they can be positive only when the constraint is active, i.e.  $\bar{y}_k(x_i) = 0$ .

#### 4. Numerical applications and comparison of the dual method with other methods

In this section we discuss the numerical implementation of the algorithm and we explain how to solve the dual approximate problem.

In the space  $V = H^2(\Omega) \cap H_0^1(\Omega)$  with the scalar product  $(\cdot, \cdot)_V$  given in Section 3, we have already seen that the duality mapping  $J : V \rightarrow V^*$  is defined by  $J(y) = \Delta \Delta y$  and it is a linear, single-valued, bijective operator.

For every  $y^* \in C_k^*$

$$|y^* + f|_{V^*}^2 = |J^{-1}(y^* + f)|_V^2 = \left| \sum_{i=1}^k \alpha_i J^{-1}(\delta_{x_i}) + J^{-1}(f) \right|_V^2$$

Due to the definition of the duality mapping, we can denote  $\Phi_i = J^{-1}(\delta_{x_i})$ , for all  $i \in \{1, 2, \dots, k\}$ . Since  $\delta_{x_i} \in V^*$ ,  $\Phi_i$  are the weak solution of the problem

$$\begin{cases} \Delta \Delta \Phi_i = \delta_{x_i}, & \text{in } \Omega \\ \Phi_i = 0, \Delta \Phi_i = 0, & \text{on } \Omega \end{cases} \quad (14)$$

We have already introduced  $J^{-1}(f) = y_f$  in (8), Section 3. Then

$$|y^* + f|_{V^*}^2 = \left| \sum_{i=1}^k \alpha_i \Phi_i + y_f \right|_V^2$$

Computing this norm, with respect to the scalar product on  $V$ , we obtain

$$|y^* + f|_{V^*}^2 = \sum_{i,j=1}^k \alpha_i \alpha_j \int_{\Omega} \Delta \Phi_i \Delta \Phi_j + \sum_{i=1}^k \alpha_i \int_{\Omega} \Delta \Phi_i \Delta y_f + \int_{\Omega} (\Delta y_f)^2$$

We denote

$$a_{ij} = \int_{\Omega} \Delta \Phi_i \Delta \Phi_j, \quad \forall i, j \in \{1, \dots, n\}, \quad b_i = \int_{\Omega} \Delta \Phi_i \Delta y_f, \quad i \in \{1, \dots, n\} \quad (15)$$

Then the dual approximate problem is equivalent to the quadratic optimization problem

$$\min \left\{ \frac{1}{2} \alpha^T A \alpha + b^T \alpha : \alpha \in \mathbb{R}^n, \alpha_i \geq 0, \forall i = 1, 2, \dots, n \right\} \quad (16)$$

where  $A = [a_{ij}]$  and  $b = [b_i]$ .

We compute now  $a_{ij}$  and  $b_i$ . To this end, we remark that (14) can be rewritten as

$$\begin{cases} \Delta y = z, & \text{in } \Omega \\ y = 0, & \text{on } \Omega \end{cases} \quad \begin{cases} \Delta z = \delta_{x_i}, & \text{in } \Omega \\ z = 0, & \text{on } \Omega \end{cases} \quad (17)$$

Then we denote by  $\varphi_i$  the weak solutions of the second equation in (17) for all  $i \in \{1, 2, \dots, k\}$ . Then we obtain that

$$a_{ij} = \int_{\Omega} \varphi_i \varphi_j, \quad \forall i, j \in \{1, 2, \dots, n\}$$

As for the components of the vector  $b$ , we have

$$b_i = \int_{\Omega} \Delta \Phi_i \Delta y_f = \int_{\Omega} \Delta \Delta \Phi_i y_f = (\delta_{x_i}, y_f)_{V^* \times V} = y_f(x_i).$$

For  $i \neq j$ , we get

$$a_{ij} = \int_{\Omega} \Delta \Phi_i \Delta \Phi_j = \Phi_i(x_j)$$

and, since  $\delta_{x_i} \in V^*$ , it yields that  $\Phi_i \in V$ , hence, its norm is finite in  $V$ ,

$$a_{ii} = \int_{\Omega} (\Delta \Phi_i)^2 = |\Phi_i|_V^2$$



After finding the solution  $\alpha^*$  for the quadratic problem (16) we apply the formula stated in Theorem 3.1, i.e.

$$\bar{y}_k(x_i) = \sum_{j=1}^k \alpha_j^* \phi_j(x_i) + y_f(x_i).$$

**Example 4.1.** We consider  $\Omega = (-1, 1)$  and we take the function

$$f(x) = 1680x^4 - 1170x^2 + 90.$$

We consider the simply supported beam conditions at the limit. We solve the following problem

$$\min_{y \in K} \left\{ \frac{1}{2} \int_{\Omega} (y'')^2 - \int_{\Omega} f y \right\}$$

where  $K = \{y \in H_0^1(\Omega) \cap H^2(\Omega) : y \geq 0 \text{ in } \Omega\}$ .

In figure 1 we represent the two solutions, one computed using the dual method presented above, the other computed using the direct method (IPOPT optimizer for large scale, non-linear, constrained optimization, implemented in *Freefem++*; for details see Wächter and Biegler [25] and Hatch [14]).

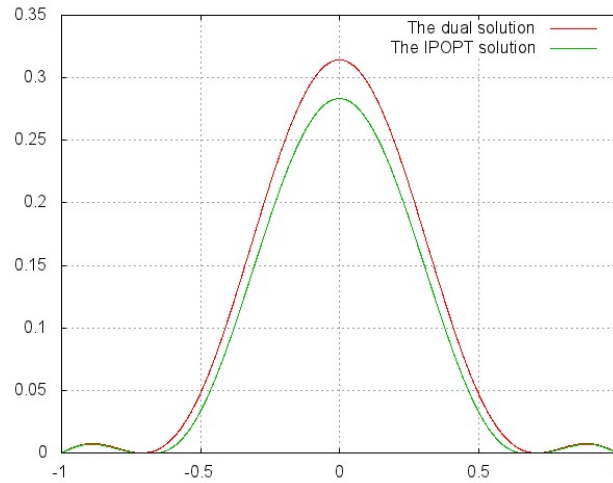


FIGURE 1. Comparison of the two solutions.

We computed the values of the cost functional at the two approximate solutions. Table 1 shows that the cost functional has lower values for the solutions computed by the dual method.

TABLE 1. The values of the cost functional for different partitions  $k$  of  $(-1, 1)$ .

k	801	1401	1601	1801	2001
IPOPT	-0.373085	-0.373096	-0.373098	-0.373099	-0.3731
Dual	-0.391613	-0.391625	-0.391626	-0.391627	-0.391628

**Example 4.2.** We take  $\Omega$  the unit disc in  $\mathbb{R}^2$  and we solve the obstacle problem

$$\min_{y \in K} \left\{ \frac{1}{2} \int_{\Omega} (\Delta y)^2 - \int_{\Omega} f y \right\} \quad (18)$$

where  $K = \{y \in H_0^1(\Omega) \cap H^2(\Omega) : y \geq 0 \text{ in } \Omega\}$  and

$$f(x_1, x_2) = 100(-x_1^2 + 3x_1).$$

We computed again the two solutions. The one by the dual method is represented in Figure 2 and the one by the IPOPT method is represented in Figure 3.

FIGURE 2. The solution obtain using the dual method.

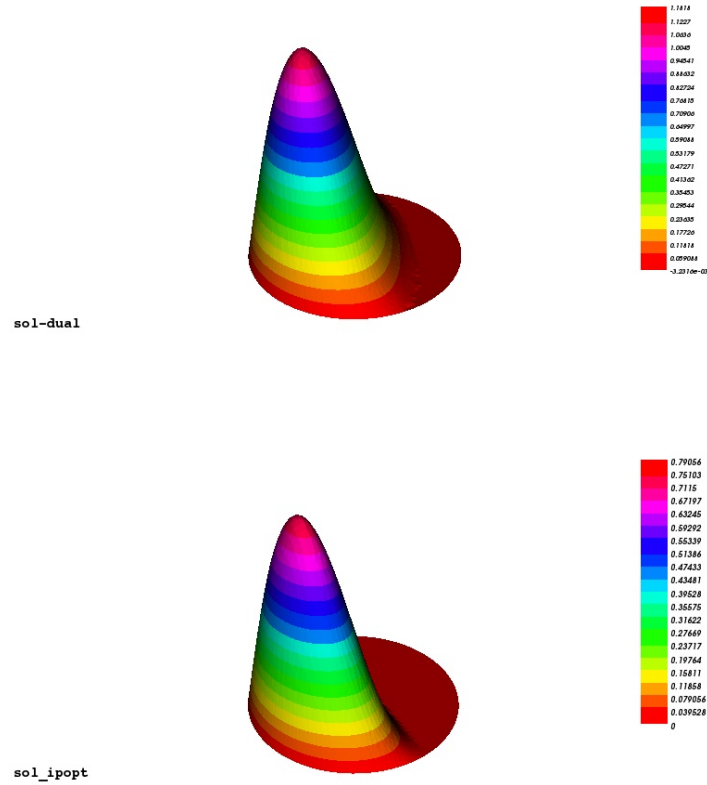


FIGURE 3. The solution given by IPOPT method.

We considered the points  $\{x_i\}_i$  as the vertices of the mesh. The two solutions have been computed on the same mesh and with the same default tolerance parameters for the IPOPT minimization method. We also mention that we used a double P1 finite element space to compute both solutions.

Comparing the values in Table 2 we conclude that, also in this example, the minimum values of the cost functional are lower when applying the dual method.

TABLE 2. The values of the cost functional for different meshes with the number of vertices denoted by  $k$ .

k	205	682	1031	1431	1912	2797
IPOPT	-55.8069	-57.9099	-58.168	-58.3493	-58.4457	-58.5392
Dual	-78.0675	-80.5279	-80.8705	-81.113	-81.2397	-81.3977

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