

## GENERAL SOLUTION OF FULL ROW RANK LINEAR SYSTEMS OF EQUATIONS VIA A NEW EXTENDED ABS MODEL

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*ABS methods have been used broadly for solving linear and nonlinear systems of equations comprising large number of constraints and variables. Also, ABS methods provide a unification of the field of finitely terminating methods for the general solution of full row rank linear systems of equations. In this paper, we theoretically describe a new ABS algorithm based on the two-step ABS methods for solving general solution of full row rank linear systems of equations. This new algorithm requires the same number of multiplications as Gaussian elimination method, but does not need pivoting. Computational complexity and numerical results indicate that our new version of ABS algorithm needs less work than the corresponding two-step ABS algorithms and Huang's method.*

**Keywords:** ABS methods, Two-step ABS algorithms, General solution of full row rank linear systems of equations

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### 1. Introduction

ABS methods were introduced by Abaffy, Broyden and Spedicato initially for solving a determined or underdetermined linear system and later extended for linear least squares, nonlinear equations, optimization problems and Diophantine equations [1, 4]. These extended ABS algorithms offer some new approaches that are better than classical ones under several respects. Also, extensive computational experience has shown that ABS methods are implementable in a stable way, being often more accurate than the corresponding initial algorithm. ABS methods can be more effective than some of the other traditional methods. See more about ABS algorithms in [5, 6, 7]. A review of ABS algorithms is observed in [8]. The basic ABS algorithm works on a system of the form

$$Ax = b \tag{1}$$

where  $A = [a_1, \dots, a_m]^T$ ,  $a_i \in \mathbb{R}^n$ ,  $1 \leq i \leq m$ ,  $x \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$ ,  $m \leq n$ .

The basic ABS methods determine the solution of (1) or signify lack of its existence in at most  $m$  iterates. Now, we present a new extended ABS algorithm based on the two-step ABS algorithms that were proposed in [2, 3] for the general solution of full row rank linear systems of equations. In order to have less numerical complexity, we modify some of the parameters of the basic two-step ABS methods. The remainder of this paper is organized as follows;

In Section 2, we construct a new two-step ABS model for solving general solution of full row rank linear systems of equations. Rank reducing process is done in two phases, for per iterate. The first phase helps us to have a solution of the  $i$ -th iteration and the next phase leads to compute general solution of that iterate. Also, we state and prove related

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theorems. In Section 3, we present a new extended two-step ABS algorithm for the general solution of full row rank linear equations. The  $i$ -th iteration solves the first  $2i$  equations in at most  $\lceil \frac{m+1}{2} \rceil$  steps. Furthermore, we propose some parameters to have less computational complexity. Computational complexity and numerical results are discussed in Section 4, in details. In Section 5, we conclude that our new algorithm requires the same number of multiplications as Gaussian elimination method, but does not need pivoting. Moreover, we need less work than those corresponding two-step ABS methods and Huang's algorithm.

## 2. Constructing a new two-step ABS model

The basic ABS algorithm starts with an initial vector  $x_0 \in \mathbb{R}^n$  and a nonsingular matrix  $H_0 \in \mathbb{R}^{n \times n}$  (Spedicato's parameter). Given that  $x_i$  is a solution of the first  $i$  equations, the ABS algorithm computes  $x_i$  as the solution of the first  $i+1$  equations as the following steps [1]:

- (1) Determine  $z_i$  (Broyden's parameter) so that  $z_i^T H_i a_i \neq 0$  and set  $P_i = H_i^T z_i$ ,
- (2) Update the solution by  $x_{i+1} = x_i + \alpha_i P_i$  where the stepsize  $\alpha_i$  is given by  $\alpha_i = \frac{b_i - a_i^T x_i}{a_i^T P_i}$ .
- (3) Update the Abaffian matrix  $H_i$  by  $H_{i+1} = H_i - \frac{H_i a_i w_i^T H_i}{w_i^T H_i a_i}$ .

Here, we are motivated to study on a method that satisfies two new equations at a time. We consider the system (1) under the assumption that  $A$  is full rank in row, i.e.,  $\text{rank}(A)=m$  and  $m \leq n$ . Suppose that  $m = 2l$  (if  $m$  is odd, we can add a trivial equation to the system). Take  $A^{2i} = [a_1, \dots, a_{2i}]^T$ ,  $b^{2i} = [b_1, \dots, b_{2i}]^T$  and  $r_j(x) = a_j^T x - b_j$  ( $j = 1, \dots, m$ ). Assume that we are at the  $i$ -th iteration and  $x_i$  satisfies  $A^{2i}x = b^{2i}$ . We determine  $H_i \in \mathbb{R}^{n \times n}$ ,  $z_i \in \mathbb{R}^n$  and  $\lambda_i \in \mathbb{R}$  so that  $x_i = x_{i-1} - \lambda_i H_i^T z_i$  is a solution of the first  $2i$  equations of the system (1), which is  $A^{2i}x_i = b^{2i}$ . As a result, we have  $r_j(x_i) = 0$ ,  $j = 1, \dots, 2i$ . Thus, for  $j = 2i-1$  and  $j = 2i$ , we have

$$\begin{cases} a_{2i-1}^T (x_{i-1} - \lambda_i H_i^T z_i) - b_{2i-1} = 0, \\ a_{2i}^T (x_{i-1} - \lambda_i H_i^T z_i) - b_{2i} = 0, \end{cases}$$

or equivalently

$$\begin{cases} \lambda_i (H_i a_{2i-1})^T z_i = r_{2i-1}(x_{i-1}), \\ \lambda_i (H_i a_{2i})^T z_i = r_{2i}(x_{i-1}). \end{cases} \quad (2)$$

Suppose that  $r_{2i-1}(x_{i-1}) \neq 0$  and  $r_{2i}(x_{i-1}) \neq 0$ . Then,  $\lambda_i$  must be nonzero and (2) is compatible if and only if we take

$$\lambda_i = \frac{\bar{r}_{2i-1}(x_{i-1})}{(H_i a_{2i-1})^T z_i} = \frac{\bar{r}_{2i}(x_{i-1})}{(H_i a_{2i})^T z_i} \quad (3)$$

where  $\bar{r}_{2i-1}(x_{i-1}) = \bar{r}_{2i}(x_{i-1}) = r_{2i-1}(x_{i-1})r_{2i}(x_{i-1})$ . There are various ways to satisfy (3). We consider the following model:

1. Choose an appropriate update for  $H_i$  so that  $H_i a_{2i-1} = H_i a_{2i} \neq 0$ , and
2. Select a vector  $z_i$  from an orthogonal space to the vector  $H_i a_{2i}$ , so that  $z_i^T H_i a_{2i} \neq 0$ .

Now, since two new equations are considered in each iterate, we use a rank one update as (4) and another rank one update as (5). Therefore, we have a new rank two update for each iterate. Here, we present  $H_i$  and  $H_{i+1}$  satisfying the following properties;

$$H_i a_{2i-1} = H_i a_{2i} \neq 0, \quad i = 1, \dots, l, \quad (4)$$

and

$$H_{i+1} a_j = 0, \quad j = 1, \dots, 2i. \quad (5)$$

Now, assume

$$c_j = \begin{cases} a_{2i} - a_j, & j \neq 2i, \\ a_{2i}, & j = 2i. \end{cases} \quad (6)$$

As relations (4), (5) and (6), we will construct (7) and (8) such that

$$H_i c_j = 0, \quad j = 1, \dots, 2i - 1, \quad (7)$$

and

$$H_{l_i} c_j = 0, \quad j = 1, \dots, 2i. \quad (8)$$

We compute  $H_{i+1}$  from  $H_{l_i}$ , such that the relations (7) and (8) hold and proceed inductively. We define  $H_{i+1} = H_{l_i} + g_{2i+1} d_{2i+1}^T$  where  $g_{2i+1}, d_{2i+1} \in \mathbb{R}^n$ . We need to have

$$\begin{cases} H_{i+1} c_j = 0, & j = 1, \dots, 2i + 1, \\ H_{l_i} c_j = 0, & j = 1, \dots, 2i. \end{cases}$$

So, we have

$$\begin{cases} (H_{l_i} + g_{2i+1} d_{2i+1}^T) c_j = 0, & j = 1, \dots, 2i + 1, \\ H_{l_i} c_j = 0, & j = 1, \dots, 2i. \end{cases}$$

Thus, we must define  $g_{2i+1}, d_{2i+1} \in \mathbb{R}^n$  in such a way that

$$H_{l_i} c_j + (d_{2i+1}^T c_j) g_{2i+1} = 0, \quad j = 1, \dots, 2i + 1, \quad (9)$$

and  $H_{l_i} c_j = 0, j = 1, \dots, 2i$ . By defining  $d_{2i+1} = H_{l_i}^T w_{2i+1}$ , the condition (9) is satisfied for  $j \leq 2i - 1$  by the induction hypothesis. Letting  $j = 2i + 1$  in (9), we get

$$(d_{2i+1}^T c_{2i+1}) g_{2i+1} = -H_{l_i} c_{2i+1}. \quad (10)$$

We consider the choice  $g_{2i+1} = -H_{l_i} c_{2i+1}$  with  $d_{2i+1}^T c_{2i+1} = 1$ , which clearly holds in (10). Now, we define  $w_{2i+1} \in \mathbb{R}^n$  such that

$$w_{2i+1}^T H_{l_i} c_{2i+1} = 1. \quad (11)$$

Later, as Theorem 2.2., we will conclude that the above system has solution and  $H_{l_i}$  is well-defined. Therefore, the updating formula for  $H_i$  is given by

$$H_{i+1} = H_{l_i} - H_{l_i} c_{2i+1} w_{2i+1}^T H_{l_i} \quad (12)$$

where  $w_{2i+1}$  can be any vector satisfying (11). Now to satisfy (12) and complete the induction,  $H_1$  should be chosen so that  $H_1 a_1 = H_1 a_2 \neq 0$  or

$$H_1 c_1 = 0. \quad (13)$$

Let  $H_0$  be an arbitrary nonsingular matrix. We obtain  $H_1$  from  $H_0$  by using a rank one update. Take  $H_1 = H_0 - u_1 v_1^T$  where  $u_1, v_1 \in \mathbb{R}^n$  are chosen so that (13) is satisfied. So, we have  $H_0 c_1 - (v_1^T c_1) u_1 = 0$ . The above equation is satisfied if we set  $u_1 = H_0 c_1, v_1 = H_0^T w_1$ , and we choose  $w_1 \in \mathbb{R}^n$  satisfying the next condition

$$w_1^T H_0 c_1 = 1. \quad (14)$$

Clearly, (14) can be held with a proper choice of  $w_1 \in \mathbb{R}^n$ , whenever  $a_1$  and  $a_2$  are linearly independent. Thus, we have a rank one update

$$H_1 = H_0 - H_0 c_1 w_1^T H_0 \quad (15)$$

where  $w_1$  is an arbitrary vector satisfying (14). In order to compute the general solution and update the second phase for the  $i$ -th iteration, we introduce a matrix  $H_{l_i}$  with properties  $H_{l_i} c_j = H_{l_i} a_j = 0, j = 1, \dots, 2i$ . So, we define the matrix  $H_{l_i}$  by a rank one update as the next formula

$$H_{l_i} = H_i - H_i a_{2i} w_{2i}^T H_i = H_i - H_i c_{2i} w_{2i}^T H_i, \quad i = 1, \dots, l. \quad (16)$$

Notice that  $w_{2i} \in \mathbb{R}^n$  is an arbitrary vector satisfying the following condition:

$$w_{2i}^T H_i a_{2i} = 1. \quad (w_{2i}^T H_i c_{2i} = 1) \quad (17)$$

Hence, the general solution of the  $i$ -th iteration is given by  $x_{l_i} = x_i - H_{l_i}^T s$  where  $s \in \mathbb{R}^n$  is arbitrary. Clearly, the general solution for the last iteration is presented by

$$x_{l_i} = x_l - H_{l_i}^T s \quad (18)$$

where  $s \in \mathbb{R}^n$  is arbitrary.

**Lemma 2.1.** *The vectors  $a_1, \dots, a_m$  are linearly independent if and only if the vectors  $c_1, \dots, c_m$  are linearly independent.*

Therefore, we proved the following theorem.

**Theorem 2.1.** *Assume that we have  $m = 2l$  arbitrary linearly independent vectors  $a_1, \dots, a_m \in \mathbb{R}^n$  and an arbitrary nonsingular matrix  $H_0 \in \mathbb{R}^{n \times n}$ . Let  $H_1$  be generated by (15) and  $w_1$  is satisfying (14) and the sequence of matrices  $H_i$ ,  $i = 2, \dots, l$ , be generated by*

$$H_i = H_{l_{i-1}} - H_{l_{i-1}} c_{2i-1} w_{2i-1}^T H_{l_{i-1}}, \quad (19)$$

with  $w_{2i-1} \in \mathbb{R}^n$  satisfying the following condition:

$$w_{2i-1}^T H_{l_{i-1}} c_{2i-1} = 1. \quad (20)$$

Also, let the sequence of matrices  $H_1, \dots, H_l$  be generated by (16) with  $w_{2i} \in \mathbb{R}^n$  satisfying (17). Then, when we are at the  $i$ -th iteration, the following properties ((i) – (iv)) hold for  $i = 1, \dots, l$ :

- (i)  $H_i a_{2i-1} = H_i a_{2i} \neq 0$ ,
- (ii)  $H_{l_i} a_j = 0$ ,  $j = 1, \dots, 2i$ ,
- (iii)  $H_i c_j = 0$ ,  $j = 1, \dots, 2i - 1$ ,
- (iv)  $H_{l_i} c_j = 0$ ,  $j = 1, \dots, 2i$ .

**Theorem 2.2.** *Assume that  $a_1, \dots, a_m$  are linearly independent vectors in  $\mathbb{R}^n$ . Let  $H_0 \in \mathbb{R}^{n \times n}$  be an arbitrary nonsingular matrix and  $H_1$  be defined by (15), with  $w_1 \in \mathbb{R}^n$  satisfying (14) and for  $i = 2, \dots, l$ , the sequence of matrices  $H_i$  be generated by (19) with  $w_{2i-1} \in \mathbb{R}^n$  satisfying (20). Then, for all  $i$ ,  $1 \leq i \leq l$ , and  $j$ ,  $2i \leq j \leq m$ , the vectors  $H_i a_j$  are nonzero and linearly independent.*

*Proof.* We proceed by induction. For  $i = 1$ , the theorem is true. Since if  $\sum_{j=2}^m \alpha_j H_1 a_j = 0$ , we have

$$\begin{aligned} \sum_{j=2}^m \alpha_j (H_0 - H_0 c_1 w_1^T H_0) a_j &= 0, \\ \sum_{j=2}^m \alpha_j H_0 a_j - \left( \sum_{j=2}^m \alpha_j w_1^T H_0 a_j \right) H_0 c_1 &= 0, \end{aligned}$$

or

$$\beta_1 H_0 a_1 - (\beta_1 - \alpha_2) H_0 a_2 + \sum_{j=3}^m \alpha_j H_0 a_j = 0$$

where  $\beta_1 = \sum_{j=2}^m \alpha_j w_1^T H_0 a_j$ . Now, since  $a_1, \dots, a_m$  are nonzero and linearly independent and  $H_0$  is nonsingular,  $H_0 a_j$  for  $1 \leq j \leq m$  are nonzero and linearly independent.

Hence,  $\beta_1 = \alpha_2 = \alpha_3 = \dots = \alpha_m = 0$ . Therefore, the vectors  $H_1 a_j$ , for  $2 \leq j \leq m$ , are nonzero and linearly independent. Now, we assume that the theorem is true up to  $1 \leq i \leq l - 1$ , and then we prove it for  $i + 1$ . From (12), for  $2i + 2 \leq j \leq m$ , we have

$$H_{i+1} a_j = H_{l_i} a_j - (w_{2i+1}^T H_{l_i} a_j) H_{l_i} c_{2i+1}. \quad (21)$$

We need to show that the relation

$$\sum_{j=2i+2}^m \alpha_j H_{i+1} a_j = 0, \quad (22)$$

implies that  $\alpha_j = 0$ , for  $2i+2 \leq j \leq m$ . Using (21) we can write (22) as follows:  $\sum_{j=2i+2}^m \alpha_j H_{l_i} a_j - \beta'_1 H_{l_i} c_{2i+1} = 0$ , where  $\beta'_1 = \sum_{j=2i+2}^m \alpha_j w_{2i+1}^T H_{l_i} a_j$ . Thus, we have  $\sum_{j=2i+2}^m \alpha_j H_{l_i} a_j - \beta'_1 H_{l_i} (a_{2i+2} - a_{2i+1}) = 0$ . As a result

$$\sum_{j=2i+3}^m \alpha_j H_{l_i} a_j + \beta'_1 H_{l_i} a_{2i+1} - (\beta'_1 - \alpha_{2i+2}) H_{l_i} a_{2i+2} = 0. \quad (23)$$

By the induction hypothesis, the vectors  $H_i a_j$ , for  $2i \leq j \leq m$ , are nonzero and linearly independent. Now as the assumption of the induction, we are going to prove that the vectors  $H_{l_i} a_j$  are nonzero and linearly independent for  $2i+1 \leq j \leq m$ . Using (16), we have

$$H_{l_i} a_j = H_i a_j - H_i a_{2i} w_{2i}^T H_i a_j. \quad (24)$$

We must prove the relation

$$\sum_{j=2i+1}^m \alpha'_j H_{l_i} a_j = 0, \quad (25)$$

implies that  $\alpha'_j = 0$ , for  $j \geq 2i+1$ . Using (24), we can write (25) as follows:

$$\sum_{j=2i+1}^m \alpha'_j H_i a_j - \left( \sum_{j=2i+1}^m \alpha'_j w_{2i}^T H_i a_j \right) H_i a_{2i} = 0.$$

As the linearly independence of  $H_i a_j$ , for  $2i \leq j \leq m$ , we conclude that  $\alpha'_j = 0$ , for  $j \geq 2i+1$ . Consequently, for relation (23), we have  $\beta'_1 = \alpha_{2i+2} = \alpha_{2i+3} = \dots = \alpha_m = 0$ . Hence, the vectors  $H_{i+1} a_j$ , for  $2i+2 \leq j \leq m$  are nonzero and linearly independent.  $\square$

**Corollary 2.1.** *Considering the assumptions of Theorem 2.2., the following statement are true;*

- (i) *When we are at the  $i$ -th iteration, we have  $H_i a_{2i-1} = H_i a_{2i} \neq 0$ . Also, there exist  $z_i \in \mathbb{R}^n$  and  $w_{2i} \in \mathbb{R}^n$  such that  $z_i^T H_i a_{2i} \neq 0$  and  $w_{2i}^T H_i a_{2i} \neq 0$ .*
- (ii) *Each of the systems (14) and (20) has solution.*
- (iii)  *$H_i$ ,  $H_{l_i}$ ,  $x_i$  and  $x_{l_i}$  are well-defined for  $i = 1, \dots, l$ .*

**Theorem 2.3.** *For the matrices  $H_i$  generated by (15) and (19) and the matrices  $H_{l_i}$  given by (16), we have*

$$\begin{aligned} \dim R(H_i) &= n - 2i + 1, & 1 \leq i \leq l, \\ \dim N(H_i) &= 2i - 1, & 1 \leq i \leq l, \\ \dim R(H_{l_i}) &= n - 2i, & 1 \leq i \leq l, \\ \dim N(H_{l_i}) &= 2i, & 1 \leq i \leq l, \\ \dim R(H_{l_l}) &= n - m, \\ \dim N(H_{l_l}) &= m. \end{aligned}$$

**Remark 2.1.** For the matrices  $H_i$  generated by (15) and (19), we have  $x_i = x_{i-1} - \lambda_i H_i^T z_i$ , ( $i = 1, \dots, l$ ), where  $x_i$  is a solution of the first  $2i$  equations of the system and  $\lambda_i$  will be discussed in Algorithm 1.

### 3. Introducing a new two-step ABS algorithm

**Algorithm 1.** Assume that  $A_{m \times n} = [a_1, a_2, \dots, a_m]^T$  is a full row rank linear matrix where  $m = 2l$  and  $Ax = b$  is a compatible full row rank linear system,  $x \in \mathbb{R}^n$  and  $b \in \mathbb{R}^m$ .

- (1) Let  $x_0 \in \mathbb{R}^n$  be an arbitrary vector and choose  $H_0 \in \mathbb{R}^{n \times n}$  (an arbitrary nonsingular matrix). Set  $i = 1$ .
- (2) (a) Compute  $r_1(x_0) = a_1^T x_0 - b_1$  and  $r_2(x_0) = a_2^T x_0 - b_2$ .  
 (b) If  $r_1(x_0)r_2(x_0) \neq 0$ , we let

$$\begin{cases} a_1 = r_2(x_0)a_1, & \begin{cases} a_2 = r_1(x_0)a_2, \\ b_2 = r_1(x_0)b_2. \end{cases} \\ b_1 = r_2(x_0)b_1, \end{cases}$$

- (c) If  $r_1(x_0)r_2(x_0) = 0$  and one of the residual values is nonzero, without loss of generality we assume that  $r_2(x_0) \neq 0$  and we let

$$\begin{cases} a_1 = a_2 + a_1, & \begin{cases} a_2 = a_2, \\ b_2 = b_2. \end{cases} \\ b_1 = b_2 + b_1, \end{cases}$$

- (d) If  $r_1(x_0)r_2(x_0) = 0$ , and both of the residual values are zero,  $x_1$  will be  $x_0$  and go to (5).
- (3) (a) Take  $c_1 = a_2 - a_1$ .  
 (b) Select  $w_1 \in \mathbb{R}^n$  such that  $w_1^T H_0 c_1 = 1$  and compute  $H_1 = H_0 - H_0 c_1 w_1^T H_0$ .  
 (c) Select  $w_2 \in \mathbb{R}^n$  such that  $w_2^T H_1 a_2 = 1$  and compute  $H_{l_1} = H_1 - H_1 a_2 w_2^T H_1$ .  
 (d) Select  $z_1 \in \mathbb{R}^n$  so that  $z_1^T H_1 a_2 \neq 0$  and compute  
 $\lambda_1 = \frac{r_1(x_0)r_2(x_0)}{z_1^T H_1 a_2}$ , if  $r_1(x_0)r_2(x_0) \neq 0$ ,  
 $\lambda_1 = \frac{r_2(x_0)}{z_1^T H_1 a_2}$ , if  $r_1(x_0) = 0$  and  $r_2(x_0) \neq 0$ .  
 (4) Take  $x_1 = x_0 - \lambda_1 H_1^T z_1$ .  
 (5) Set  $i = 2$  and go to 6.  
 (6) While  $i \leq \frac{m}{2}$ , do steps 6(a)-8(b).  
 (a) Compute  $r_{2i-1}(x_{i-1}) = a_{2i-1}^T x_{i-1} - b_{2i-1}$  and  $r_{2i}(x_{i-1}) = a_{2i}^T x_{i-1} - b_{2i}$ .  
 (b) If  $r_{2i-1}(x_{i-1})r_{2i}(x_{i-1}) \neq 0$ , then we let

$$\begin{cases} a_{2i-1} = r_{2i}(x_{i-1})a_{2i-1}, & \begin{cases} a_{2i} = r_{2i-1}(x_{i-1})a_{2i}, \\ b_{2i} = r_{2i-1}(x_{i-1})b_{2i}. \end{cases} \\ b_{2i-1} = r_{2i}(x_{i-1})b_{2i-1}, \end{cases}$$

- (c) If  $r_{2i-1}(x_{i-1})r_{2i}(x_{i-1}) = 0$  and one of the residual values is nonzero, without loss of generality we assume that  $r_{2i}(x_{i-1}) \neq 0$  and we let

$$\begin{cases} a_{2i-1} = a_{2i} + a_{2i-1}, & \begin{cases} a_{2i} = a_{2i}, \\ b_{2i} = b_{2i}. \end{cases} \\ b_{2i-1} = b_{2i} + b_{2i-1}, \end{cases}$$

- (d) If  $r_{2i-1}(x_{i-1})r_{2i}(x_{i-1}) = 0$  and both of the residual values are zero,  $x_i$  will be  $x_{i-1}$  and go to 8.(b).
- (7) (a) Take  $c_{2i-1} = a_{2i} - a_{2i-1}$ .  
 (b) Select  $w_{2i-1} \in \mathbb{R}^n$  such that  $w_{2i-1}^T H_{l_{i-1}} c_{2i-1} = 1$ , and compute  $H_i = H_{l_{i-1}} - H_{l_{i-1}} c_{2i-1} w_{2i-1}^T H_{l_{i-1}}$ .  
 (c) Select  $w_{2i} \in \mathbb{R}^n$  such that  $w_{2i}^T H_i a_{2i} = 1$  and compute  $H_{l_i} = H_i - H_i a_{2i} w_{2i}^T H_i$ .  
 (d) Select  $z_i \in \mathbb{R}^n$  so that  $z_i^T H_i a_{2i} \neq 0$ , and compute  $\lambda_i = \frac{r_{2i-1}(x_{i-1})r_{2i}(x_{i-1})}{z_i^T H_i a_{2i}}$ , if  $r_{2i-1}(x_{i-1})r_{2i}(x_{i-1}) \neq 0$ ,  
 $\lambda_i = \frac{r_{2i}(x_{i-1})}{z_i^T H_i a_{2i}}$ , if  $r_{2i-1}(x_{i-1}) = 0$  and  $r_{2i}(x_{i-1}) \neq 0$ .  
 (8) (a) Take  $x_i = x_{i-1} - \lambda_i H_i^T z_i$ .  
 (b) Set  $i = i + 1$ .  
 Endwhile.  
 (9) Stop. ( $x_l$  is a solution of the system.)

From (18), we can compute general solution of the system after the final iterate by  $x_{l_l} = x_l - H_{l_l}^T s$  where  $s \in \mathbb{R}^n$  is arbitrary.

**Remark 3.1.** To reduce computational complexity of the above algorithm, we propose the following tactics;

1. Taking  $x_0$  and  $H_0$  as the zero vector and the identity matrix  $I_n$ , respectively, is proper in step (1) as Algorithm 1.
2. To compute  $w_1 \in \mathbb{R}^n$ , we take  $t_j = H_0 c_1$  and then we define  $w_1$  as follows:

$$w_1 = \begin{cases} \frac{1}{\text{sign}(t_{j_M})t_{j_M}}, & i = j_M, \\ 0, & i \neq j_M. \end{cases}$$

Then, we take  $t_{j_M} = \max\{|t_j| : j \in \{1, \dots, n\} \text{ such that } w_1^T t_j = 1\}$ . To determine the other  $w_{2i-1} \in \mathbb{R}^n$ ,  $i = 2, \dots, l$ , we let  $t'_j = H_{l_{i-1}} c_{2i-1}$  and then, we continue the similar way.

3. We can choose  $w_{2i} = z_i \in \mathbb{R}^n$ ,  $i = 1, \dots, l$ , by defining the next parameters  $d_j = H_i a_{2i}$ , and

$$w_{2i} = z_i = \begin{cases} \frac{1}{\text{sign}(d_{j_M})d_{j_M}}, & i = j_M, \\ 0, & i \neq j_M. \end{cases}$$

Now, we take  $d_{j_M} = \max\{|d_j| : j \in \{1, \dots, n\} \text{ such that } w_{2i}^T d_j = z_i^T d_j = 1\}$ .

#### 4. Computational complexity and numerical results

At first, considering the points 2. and 3. as Remark 3.1., we compute the number of multiplications for Algorithm 1. as follows;

**Table 1**

Number of multiplications required to solve  $m$  linear equations in  $l = \frac{m}{2}$  iterates, for the main parameters.

The number of multiplications for each iterate	
residual values	$2n$
The products of residual values	$1$
$a_{2i-1}, a_{2i}$	$2n$
$b_{2i-1}, b_{2i}$	$2$
$H_i$	$(2n+1)(n-2i+2)$
$H_{l_i}$	$(2n+1)(n-2i+1)$
$x_i$	$2n$

Hence, the total number of multiplications for the  $l$  iterates is

$$N = \sum_{i=1}^l [2n+1+2n+2+(2n+1)(n-2i+2)+(2n+1)(n-2i+1)+2n],$$

$$N = 2mn^2 - m^2n + O(mn) - O(m^2) + O(m).$$

The total number for Huang's method is  $\frac{3}{2}mn^2 + O(mn)$  multiplications and  $3mn^2 - \frac{7}{4}m^2n + \frac{1}{6}m^3 + O(mn) + O(m^2) + O(n^2)$  for the two-step methods presented by Amini et al. [2, 3]. Our computations indicate that our numerical results are better than Amini et al. introduced by two-step algorithm. In addition, when  $n < 2m$ , we need the cheaper number of multiplications up to the corresponding Huang's algorithm. Also, when  $m$  tends to  $n$ , the total number for our two-step method is  $n^3$  multiplications, while it is  $\frac{3}{2}n^3$  for Huang's algorithm and  $\frac{17}{12}n^3$  for those two-step methods given by Amini et al. in 2004 and 2007.

Obviously, when  $m$  and  $n$  are not too large, the lower order terms have influence on our results.

**Remark 4.1.** We recommend using all of the points as Remark 3.1., and setting of  $H_0$  the identity matrix  $I_n$ . In this case, for the  $i$ -th iteration, the number of multiplications of the matrix  $H_i$  is  $(n - 2i + 2)(4i - 1)$  and it is  $(n - 2i + 1)(4i + 1)$  multiplications for the matrix  $H_{i+1}$ . So, our algorithm needs  $nm^2 - \frac{2}{3}m^3 + O(nm) + O(m^2)$  multiplications. Hence, for  $m = n$ ,  $\frac{1}{3}n^3$ , multiplications plus lower order terms are needed.

## 5. Conclusion

The presented work is a new extended two-step ABS algorithm for solving general solution of compatible full row rank linear systems of equations in at most  $\lceil \frac{m+1}{2} \rceil$  iterates. The number of multiplications of our new version ABS algorithm is the same as that in Gaussian elimination method, but no pivoting is necessary. Moreover, we need less computational complexity up to those corresponding two-step ABS algorithms and Huang's method. Furthermore, this work offers more flexibility for the definition of the Abaffian matrix.

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