

## THE CLASSIFICATION OF RINGS WITH GENUS TWO CLASS OF GRAPHS

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*For any commutative ring  $R$ , there is a annihilator graph, denoted  $AG(R)$ , in which the vertices are the nonzero zero-divisors of  $R$ , and two distinct vertices  $x$  and  $y$  are joined by an edge if and only if  $ann(xy) \neq ann(x) \cup ann(y)$  where  $ann(x) = \{z \in R : xz = 0\}$ . This paper investigates the genus properties of  $AG(R)$ . In particular, we determine all isomorphism classes of commutative rings with identity whose annihilator graph has genus two.*

**Keywords:** Zero divisor of a ring, Annihilator of a ring, Embedding of a graph.

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### 1. Introduction

The study of the interrelationship between algebra and graph theory by associating a graph to a ring was initiated, in 1988, by Beck in [9], who developed the notion of a zero-divisor graph of a commutative ring with identity. In Beck's definition the vertices of the graph are the elements of the ring and two distinct vertices  $x$  and  $y$  are adjacent if and only if  $xy = 0$ . Later, Anderson and Livingston (1999) slightly modified this idea, considering only the non-zero zero divisors of ring as vertices of the graph with the same adjacency condition. Redmond (2001) extended this notion of zero-divisor graph to non-commutative rings. Since then, a number of authors have studied various forms of zero-divisor graphs associated to rings and other algebraic structures. For more details on zero divisor graph, readers may refer to the survey article [2].

The present paper deals with the concept of annihilator graph of a ring and its genus. The annihilator graph is a another variant to the zero divisor graph, which was introduced by Badawi in [8]. Let  $R$  be a commutative ring with identity and  $Z(R)$  its set of zero-divisors. For  $x \in Z(R)$ ,  $ann(x) = \{z \in R : xz = 0\}$ . The *annihilator graph* of  $R$ , denoted by  $AG(R)$ , is the undirected simple graph with vertex set  $Z(R)^* = Z(R) \setminus \{0\}$  and two distinct

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vertices  $x$  and  $y$  are adjacent if and only if  $\text{ann}(xy) \neq \text{ann}(x) \cup \text{ann}(y)$ . Several properties of annihilator graphs of different general classes of rings are studied in [8], [13], [15] and [19]. It is worth to mention that the zero-divisor graph is a subgraph of the annihilator graph  $AG(R)$ . In [8], it has been shown that for any reduced ring  $R$  that is not an integral domain,  $AG(R)$  is complete if and only if zero-divisor graph is complete if and only if  $R$  is ring-isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ .

The *genus* of a graph  $G$ , denoted by  $\gamma(G)$ , is the smallest nonnegative integer  $\ell$  such that the graph  $G$  can be embedded on the surface obtained by attaching  $\ell$  handles to a sphere. The graphs of genus 0, 1 and 2 are called *planar*, *toroidal* and *double toroidal* graphs respectively. A minor of  $G$  is a graph obtained from  $G$  by contracting edges in  $G$  or deleting edges and isolated vertices in  $G$ . Also if  $G'$  is a minor of  $G$ , then  $\gamma(G') \leq \gamma(G)$ . For  $xy \in E(G)$ , we denote the contracted edge by the vertex  $[x, y]$ . For details on the notion of embedding of graphs in a surface, we refer to White [20].

Over past decades, the topological structures were widely investigated. The planar zero-divisor graphs were first explicitly characterized by Smith in [17], the characterization of commutative rings with genus one zero-divisor graph was obtained by Chiang-Hsieh et al. in [12] and Local rings with genus two zero-divisor graph were characterized by Bloomfield et al. in [10]. Also many research articles have been published on the genus of graphs constructed out of rings. For instant, the study on genus of the total graph of a commutative ring was initiated by Maimani et al. in [14] and they classified all commutative rings with genus one total graphs. Subsequently, Tamizh Chelvam et al. in [18], characterized all commutative Artinian rings whose total graph has genus two. Also Asir et al. in [4], determined all isomorphic classes of commutative Artinian rings whose ideal based total graph has genus at most two. Further Azimi et al. [7] characterized all commutative rings whose Jacobson graphs are planar and Amraei et al. [1] classified all commutative rings with genus one Jacobson graphs. The genus two Jacobson graphs are classified by Asir in [3]. Apart from this, Asir et al. in [5, 6] determined all commutative Artinian rings whose generalized unit and unitary Cayley graphs have genus either one or two.

The purpose of this paper is to explore the question of embedding a annihilator graph on higher genus, the double torus in particular. The genus properties of annihilator graph was independently studied by Tamizh Chelvam et al. in [19] and Nikmehr et al. in [15]. In particular, they have characterized all commutative rings whose annihilator graph has genus either zero or one. Motivated by these works, we set up our goal to classify all commutative rings with genus two annihilator graphs. Throughout the paper,  $R$  will be a commutative ring with identity  $1 \neq 0$ . We denote the cardinality of the set  $A$  by  $|A|$  and the set of all non zero elements of  $A$  by  $A^*$ . Also  $R^\times$  and  $\text{Nil}(R)$  denote the set of all unit elements and nilpotent elements of  $R$ , respectively.

## 2. Properties of Annihilator graphs

In this section, we state some of the basic properties of the annihilator graphs that will be used in the proof of the main result. Especially, we state the results regarding the planar and toroidal annihilator graphs.

The first result is due to Badawi which deals with the subgraph induced by the set of nilpotent elements of  $R$ .

**Lemma 2.1.** [8, Theorem 3.10] *Let  $R$  be a non-reduced commutative ring with  $|Nil(R)| \geq 2$  and let  $AG_N(R)$  be the (induced) subgraph of  $AG(R)$  with vertices  $Nil(R)^*$ . Then  $AG_N(R)$  is complete.*

The next result is used to identify the adjacency between the vertices of the annihilator graph. Note that a local ring is a ring with a unique maximal ideal.

**Lemma 2.2.** [13, Lemma 2.11] *If  $R$  is non-local ring with  $R \cong R_1 \times R_2 \times \cdots \times R_n$ , where each  $R_i$  are local ring. Then any two distinct elements which has the same number of non-zero entries but not identical are adjacent in  $AG(R)$ .*

The following theorem provides all commutative rings whose annihilator graph is planar.

**Theorem 2.1.** [19, Theorems 14,15] [15, Corollary 10] *Let  $R$  be a commutative ring with identity and let  $R \cong R_1 \times \dots \times R_n$  where each  $R_i$  is a local. Then  $AG(R)$  is planar if and only if one of the following conditions hold*

(i) *For  $n = 1$ ,  $R$  is isomorphic to one of the following rings*

$$\begin{aligned} \mathbb{Z}_4, \quad \mathbb{Z}_2[x]/(x^2), \quad \mathbb{Z}_9, \quad \mathbb{Z}_3[x]/(x^2), \quad \mathbb{Z}_8, \quad \mathbb{Z}_2[x]/(x^3), \quad \mathbb{Z}_4[x]/(x^2 - 2, x^3), \\ \mathbb{Z}_2[x, y]/(x^2, xy, y^2), \quad \mathbb{Z}_4[x]/(2x, x^2), \quad \mathbb{F}_4[x]/(x^2), \quad \mathbb{Z}_4[x]/(x^2 + x + 1), \\ \mathbb{Z}_{25}, \quad \mathbb{Z}_5[x]/(x^2); \end{aligned}$$

(ii) *For  $n = 2$ ,  $R$  is isomorphic to one of the following rings*

$$\mathbb{Z}_2 \times \mathbb{F}_{p^n}, \quad \mathbb{Z}_3 \times \mathbb{Z}_{p^n}, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2);$$

(iii) *For  $n = 3$ ,  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ .*

The next theorem gives us all commutative rings with toroidal annihilator graphs.

**Theorem 2.2.** [19, Theorems 16,17] [15, Theorems 12,15] *Let  $R$  be a commutative ring with identity and let  $R \cong R_1 \times \dots \times R_n$  where each  $R_i$  is a local. Then  $AG(R)$  is toroidal if and only if*

(i) For  $n = 1$ ,  $R$  is isomorphic to one of the following 22 rings

$$\begin{aligned} \mathbb{Z}_{49}, \mathbb{Z}_7[x]/(x^2), \mathbb{Z}_{16}, \mathbb{Z}_2[x]/(x^4), \mathbb{Z}_4[x]/(x^2 - 2, x^4), \mathbb{Z}_4[x]/(x^3 - 2, x^4), \\ \mathbb{Z}_4[x]/(x^3 + x^2 - 2, x^4), \mathbb{Z}_2[x, y]/(x^3, xy, y^2 - x^2), \mathbb{Z}_4[x]/(x^3, x^2 - 2x), \\ \mathbb{Z}_8[x]/(x^2 - 4, 2x), \mathbb{Z}_4[x, y]/(x^3, x^2 - 2, xy, y^2 - 2, y^3), \\ \mathbb{Z}_4[x, y]/(x^2, y^2, xy - 2), \mathbb{Z}_4[x]/(x^2), \mathbb{Z}_2[x, y]/(x^2, y^2), \mathbb{Z}_2[x, y]/(x^3, xy, y^2), \\ \mathbb{Z}_4[x]/(x^3, 2x), \mathbb{Z}_4[x, y]/(x^3, x^2 - 2, xy, y^2), \mathbb{Z}_8[x]/(x^2, 2x), \mathbb{F}_8[x]/(x^2), \\ \mathbb{Z}_4[x]/(x^3 + x + 1), \mathbb{Z}_4[x, y]/(2x, 2y, x^2, xy, y^2), \mathbb{Z}_2[x, y, z]/(x, y, z)^2; \end{aligned}$$

(ii) For  $n = 2$ ,  $R$  is isomorphic to one of the following 6 rings

$$\mathbb{Z}_5 \times \mathbb{Z}_5, \mathbb{F}_4 \times \mathbb{Z}_7, \mathbb{F}_4 \times \mathbb{F}_4, \mathbb{F}_4 \times \mathbb{Z}_5, \mathbb{Z}_4 \times \mathbb{Z}_3, \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_3;$$

(iii) For  $n = 3$ ,  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ .

### 3. Characterization of genus two annihilator graphs

The main result of this paper is Theorem 3.1. There we classifying all commutative rings whose annihilator graphs have genus two. For the convenience of the reader, we state without proof a few known results in the form of propositions which will be used in the proof of the main result.

**Proposition 3.1.** [20] Let  $n \geq 3$  be positive integers and for real number  $x$ ,  $\lceil x \rceil$  is the least integer that is greater than or equal to  $x$ . Then  $\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ .

**Proposition 3.2.** [20] Let  $m, n$  be positive integers and for real number  $x$ ,  $\lceil x \rceil$  is the least integer that is greater than or equal to  $x$ . Then  $\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$  if  $m, n \geq 2$ .

**Proposition 3.3.** [11] If  $G$  is a simple bipartite graph with  $n$  vertices and  $e$  edges, then

$$\gamma(G) \geq \frac{e - 2n + 4}{4}.$$

**Proposition 3.4.** [21, Proposition 2.1] If  $G$  is a graph with  $n$  vertices and genus  $\gamma$ , then

$$\delta(G) \leq 6 + \frac{12\gamma - 12}{n}.$$

We are now in the position to provides all commutative rings whose annihilator graph is double toroidal.

**Theorem 3.1.** Let  $R$  be a commutative ring with identity. Then the genus of annihilator graph is two if and only if the following conditions hold.

1. If  $R$  is a local, then  $R$  is isomorphic to one of the following 8 rings

$$\mathbb{Z}_{27}, \mathbb{Z}_9[x]/(3x, x^2 - 3), \mathbb{Z}_9[x]/(3x, x^2 - 6), \mathbb{Z}_9[x]/(3, x)^2, \\ \mathbb{Z}_9[x]/(x^2 + 1), \mathbb{F}_9[x]/(x^2), \mathbb{Z}_3[x]/(x^3), \mathbb{Z}_3[x, y]/(x, y)^2.$$

2. If  $R$  is not a local, then  $R$  is isomorphic to one of the following 10 rings

$$\mathbb{Z}_2 \times \mathbb{Z}_9, \mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2), \mathbb{Z}_4 \times \mathbb{F}_4, \mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_4, \mathbb{Z}_4 \times \mathbb{Z}_5, \\ \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_5, \mathbb{F}_4 \times \mathbb{F}_8, \mathbb{F}_4 \times \mathbb{F}_9, \mathbb{F}_4 \times \mathbb{Z}_{11}, \mathbb{Z}_5 \times \mathbb{Z}_7.$$

*Proof.* Assume that  $\gamma(AG(R)) = 2$ .

(1) Let  $(R, \mathfrak{m})$  be a local ring with maximal ideal  $\mathfrak{m}$ . Then by Lemma 2.1, the subgraph induced by the vertices  $Nil(R)^*$  in  $AG(R)$  is complete. Since  $R$  is local,  $Z(R) = Nil(R)$  and so  $AG(R)$  is complete. Thus by Proposition 3.1, we have  $\gamma(AG(R)) = 2$  if and only if  $|Z(R)^*| = 8$  i.e.,  $|\mathfrak{m}| = 9$ . There are eight local rings with size of the maximal ideal is 9 (refer [16]). They are

$$\mathbb{Z}_{27}, \mathbb{Z}_9[x]/(3x, x^2 - 3), \mathbb{Z}_9[x]/(3x, x^2 - 6), \mathbb{Z}_9[x]/(3, x)^2, \\ \mathbb{Z}_3[x]/(x^3), \mathbb{Z}_3[x, y]/(x, y)^2, \mathbb{Z}_9[x]/(x^2 + 1), \mathbb{F}_9[x]/(x^2).$$

(2) Assume that  $R$  is not a local ring and written as  $R \cong R_1 \times R_2 \times \cdots \times R_n$ , where each  $R_i$  is a local with maximal ideal  $\mathfrak{m}_i$  for  $i = 1, \dots, n$ .

For  $1 \leq i < j \leq n$ , let  $A_{ij} = \{(x_1, x_2, \dots, x_n) \in R : x_i = 0, x_j = 0 \text{ and } x_k \neq 0 \text{ for all remaining } k\text{'s}\}$ . Let  $A$  be the set consists of exactly one element form each  $A_{ij}$  for  $1 \leq i < j \leq n$ . Now by Lemma 2.2, every element in  $A$  is adjacent to each other. Therefore the subgraph induced by  $A$  in  $AG(R)$  is complete. If  $n \geq 5$ , we have  $|A| \geq 10$  so that  $AG(R)$  would contain a copy of complete subgraph  $K_{10}$ . That is, by Proposition 3.1,  $\gamma(AG(R)) \geq 3$ , a contradiction. Hence  $n \leq 4$ .

**Case (i)** Suppose  $n = 4$ . Without loss of generality, let us assume that  $|R_i| \leq |R_j|$  for all  $1 \leq i < j \leq 4$ . Let  $V_1 = \{0\} \times \{1\} \times R_3 \times R_4$  and  $V_2 = \{1\} \times \{0\} \times R_3 \times R_4$ . Then every vertex in  $V_1$  is adjacent to  $V_2$  and so  $AG(R)$  contains a complete bipartite graph  $K_{|V_1|, |V_2|}$ . If  $|R_4| \geq 3$ , then  $|V_i| \geq 6$  for  $i = 1, 2$  so that, by Proposition 3.2,  $\gamma(AG(R)) \geq 3$ , a contradiction. Hence  $R_i \cong \mathbb{Z}_2$  for  $i = 1, 2, 3, 4$ . Here  $|V(AG(R))| = 14$  and the minimum degree of  $AG(R)$  is seven, which contradicts the fact given in Proposition 3.4.

**Case (ii)** Suppose  $n = 3$ . Let us take  $|R_1| \leq |R_2| \leq |R_3|$ . Note that the vertex subsets  $V_1 = \{0\} \times \{1\} \times R_3$  and  $V_2 = \{1\} \times \{0\} \times R_3$  contains a complete bipartite graph  $K_{|V_1|, |V_2|}$ . If  $|R_3| \geq 5$ , then  $|V_i| \geq 5$  for  $i = 1, 2$ , we have  $AG(R)$  would contains a copy of  $K_{5,5}$  so that  $\gamma(AG(R)) \geq 3$ . Hence  $|R_3| \leq 4$ . Also every vertex in  $V_3 = \{1\} \times \{0\} \times R_3$  is adjacent to all the vertex of  $V_4 = \{0\} \times R_2^* \times R_3^*$ . So if  $|R_2| = 4$ , then  $K_{4,9}$  is a subgraph of  $AG(R)$  so that  $\gamma(AG(R)) \geq 3$ . Hence  $|R_2| \leq 3$ . Further the vertex subsets  $V_5 = \{0\} \times R_2^* \times R_3^*$  and  $V_6 = \{R_1^* \times \{0\} \times R_3^*\} \cup \{R_1^* \times R_2^* \times \{0\}\}$  contains  $K_{|V_5|, |V_6|}$ . Therefore if  $|R_1| \geq 3$ , then  $AG(R)$  would contain a copy of  $K_{4,8}$  so that  $\gamma(AG(R)) \geq 3$ .

Thus  $R_1 \cong \mathbb{Z}_2$  and we have the following possible candidates for  $R$ :

$$\begin{aligned} & \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, \\ & \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4, \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \\ & \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4. \end{aligned}$$

Note that by Theorem 2.1 and Theorem 2.2, we have  $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$  is planar and  $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$  is toroidal.

Let  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ . Note that the vertex  $(1, 0, 1)$  is adjacent to  $(0, 1, 0)$  and  $(0, 1, 3)$  is adjacent to  $(1, 0, 3)$  in  $AG(R)$ . We may now contract the  $(1, 0, 1)$  to  $(0, 1, 0)$  and  $(0, 1, 3)$  to  $(1, 0, 3)$  along the edge between them. Further denote the contracted edges as  $[(1, 0, 1), (0, 1, 0)], [(0, 1, 3), (1, 0, 3)]$  and call the resulting minor subgraph of  $AG(R)$  as  $H$ . Note that in  $H$ , every vertex in the set  $\{(0, 0, 1), (0, 0, 2), (0, 0, 3), [(1, 0, 1), (0, 1, 0)], [(0, 1, 3), (1, 0, 3)]\}$  is adjacent to each vertex of  $\{(1, 0, 0), (1, 1, 2), (1, 1, 0), (1, 0, 2), (0, 1, 2)\}$  and so  $K_{5,5}$  is a subgraph of  $H$ . Thus  $\gamma(AG(R)) \geq 3$ . Moreover one may note that  $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)) \cong AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4)$  and so  $\gamma(AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))) \geq 3$ .

Consider  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$ . Let  $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$  and letting  $X = \{(0, 1, 0), (0, 1, 1), (0, 1, \omega), (0, 1, \omega^2)\}$  and  $Y = \{(1, 1, 0), (1, 0, 0), (1, 0, 1), (1, 0, \omega), (1, 0, \omega^2)\}$ . Then every vertex in  $X$  is adjacent to all the vertices in  $Y$  except the edge between  $(0, 1, 0)$  and  $(1, 1, 0)$  in  $AG(R)$ . Therefore  $K_{4,5} \setminus e$  is a subgraph of  $AG(R)$ , call it as  $G'$ . In other way, the subgraph  $G'$  is equivalent to  $AG(R) \setminus \{v_1, v_2, v_3, e_1, e_2, e_3\}$  where  $v_k = (0, 0, \omega^{k-1})$  and  $e_k$  is the edge between  $(1, 1, 0)$  and  $(1, 0, \omega^{k-1})$  for  $k = 1, 2, 3$ . By Proposition 3.3,  $\gamma(G') = 2$  and by Euler characteristic formula,  $G'$  has 8 faces, say  $F'_1, \dots, F'_8$ . Assume that  $|F'_1| \leq \dots \leq |F'_8|$ . Let  $f'$  and  $f'_i$  be the number of faces and the number of  $i$ -gons in  $G'$  respectively. Since  $G'$  is a bipartite graph with 9 vertices,  $i = 4$  or 6 or 8. Then  $f' = f'_4 + f'_6 + f'_8$  and  $2|E(G')| = 4f'_4 + 6f'_6 + 8f'_8$ . By solving the two equations, we get that either  $f'_4 = 5$  and  $f'_6 = 3$  or  $f'_4 = 6$ ,  $f'_6 = 1$  and  $f'_8 = 1$ . Here we denote any embedding of  $G'$  in  $S_2$  as  $E'$ .

Suppose  $AG(R) = 2$ . Then one can recover the embedding of  $AG(R)$  on double torus  $S_2$  by inserting the vertices  $v_1, v_2, v_3$  with all its incident edges and the edges  $e_1, e_2, e_3$  into the representation corresponding to  $F'_1, \dots, F'_8$ . First let us insert the vertices  $v_k$  in the embedding of  $G'$  for  $k = 1, 2, 3$ . Here note that the neighborhood set  $N_{AG(R)}(v_k) = \{(1, 1, 0), (0, 1, 0), (1, 0, 0)\}$  for all  $k$ . Since the vertex  $(1, 1, 0)$  is not adjacent to both  $(0, 1, 0)$  and  $(1, 0, 0)$  in  $G'$ , there is no 4-gon having all the three vertices of  $N_{AG(R)}(v_k)$ . Further  $(0, 1, 0)$  is adjacent to  $(1, 0, 0)$  in  $G'$  and any edge occur exactly in two faces of an embedding implies that at most two  $i$ -gons ( $i = 6$  or 8) of  $E'$ , say  $F'_7$  and  $F'_8$ , have all the vertices of  $N_{AG(R)}(v_k)$ . Now one can able to insert  $v_1, v_2, v_3$  along with its edges inside  $F'_7$  and  $F'_8$  without any crossing in  $E'$ , refer Figure 1. Next we try to insert the edges  $e_1, e_2$  and  $e_3$ . Here one end of all the three  $e_k$ 's is  $(1, 1, 0)$ . Since the degree of  $(1, 1, 0)$  in  $G'$  is three, the number of faces in  $E'$  containing  $(1, 1, 0)$  is three. Let  $F'_j$  ( $1 \leq j \leq 6$ ) be the third face containing the

vertex  $(1, 1, 0)$ . Since we have inserted  $v_k$ 's in faces  $F'_7$  and  $F'_8$  of the bipartite graph  $G'$ , it is not possible to embed at least one of the three edges incident with the three vertices  $(1, 0, 1), (1, 0, \omega), (1, 0, \omega^2) \in Y$  in the three faces  $F'_j, F'_7$  and  $F'_8$ . Thus  $\gamma(AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4)) \geq 3$ .

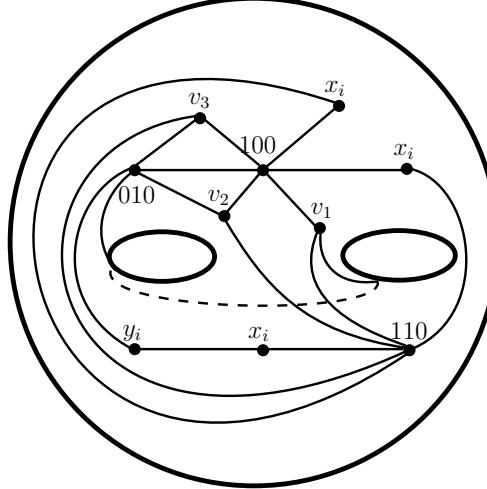


FIGURE 1. The faces  $F'_7$  and  $F'_8$  along with  $v_k$ 's of  $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4)$

If  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ , then every vertex in the set  $\{(0, 1, 1), (0, 1, 2), (0, 2, 1), (0, 2, 2), [(0, 1, 0), (0, 0, 1)]\}$  is adjacent to each vertex of the set  $\{(1, 1, 0), (1, 2, 0), (1, 0, 0), (1, 0, 1), (1, 0, 2)\}$  and so  $K_{5,5}$  is a subgraph of  $AG(R)$ . Therefore  $\gamma(AG(R)) \geq 3$ . Further one may easily verify that the graph  $AG(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$  is a subgraph of  $AG(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4)$  so that  $\gamma(AG(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4)) \geq 3$ .

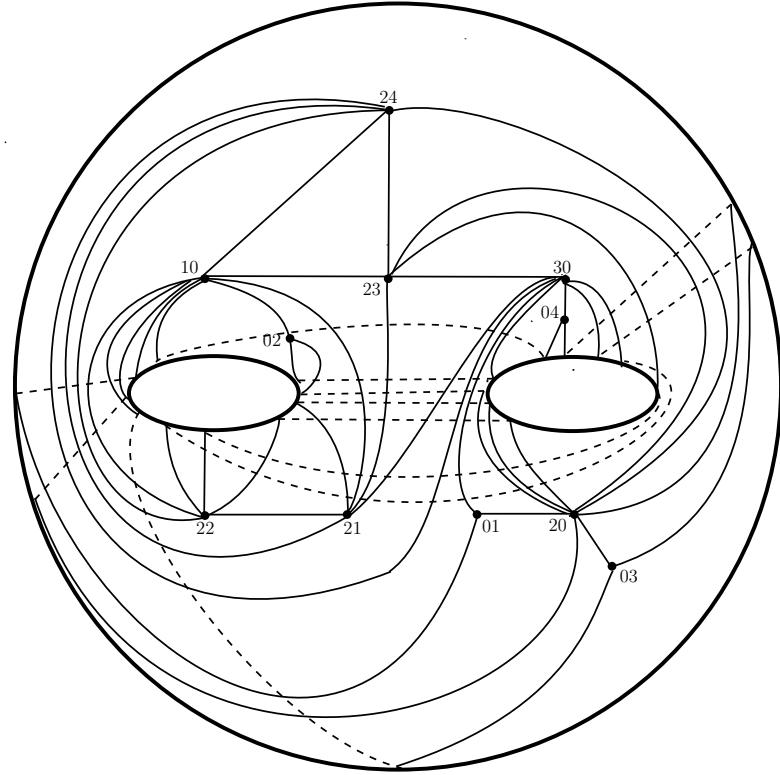
If  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$ , then the vertex subsets  $\{0\} \times \mathbb{Z}_3^* \times \mathbb{Z}_4^*$  and  $\{\mathbb{Z}_2^* \times \{0\} \times \mathbb{Z}_4^*\} \cup \{\mathbb{Z}_2^* \times \mathbb{Z}_3^* \times \{0\}\}$  induced a subgraph in  $AG(R)$  which contains  $K_{6,5}$ . So  $\gamma(AG(R)) \geq 3$ . In similar way, we get  $\gamma(AG(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2))) \geq 3$ .

**Case (iii)** Suppose  $n = 2$ .

If both  $R_1$  and  $R_2$  are fields, then  $AG(R)$  is isomorphic to  $K_{|R_1^*|, |R_2^*|}$ . Thus, by Proposition 3.2,  $\gamma(AG(R)) = 2$  if and only if  $R$  is isomorphic to one of  $\mathbb{F}_4 \times \mathbb{F}_8$ ,  $\mathbb{F}_4 \times \mathbb{F}_9$ ,  $\mathbb{F}_4 \times \mathbb{Z}_{11}$  or  $\mathbb{Z}_5 \times \mathbb{Z}_7$ .

If both  $R_1$  and  $R_2$  are not fields; i.e.,  $|\mathfrak{m}_i| \geq 2$  and  $|R_i| \geq 4$  for  $i = 1, 2$ . Then every vertex in the set  $V_1 = R_1^* \times \mathfrak{m}_2$  is adjacent to each vertex of the set  $V_2 = \{\mathfrak{m}_1 \times R_2^*\} \setminus V_1$ . Since  $|V_1| \geq 6$  and  $|V_2| \geq 5$ ,  $K_{6,5}$  is a subgraph of  $AG(R)$  so that  $\gamma(AG(R)) \geq 3$ .

Assume that either  $R_1$  or  $R_2$  is a field; say  $R_2$  is a field. Suppose  $|\mathfrak{m}_1^*| \geq 3$ , then  $|R_1^*| \geq 4$ . Let  $a, b, c \in \mathfrak{m}_1^*$  such that  $ab = ac = 0$  and  $\{u_1, u_2, u_3, u_4\} \subseteq R_1^*$ . Consider  $S_1 = \{z_1, z_2, \dots, z_7, w_1, \dots, w_4\} \subseteq Z(R)^*$ , where  $z_1 = (a, 0)$ ,  $z_2 = (b, 0)$ ,  $z_3 = (c, 0)$ ,  $z_4 = (u_1, 0)$ ,  $z_5 = (u_2, 0)$ ,  $z_6 = (u_3, 0)$ ,  $z_7 = (u_4, 0)$ ,  $w_1 = (a, 1)$ ,  $w_2 = (b, 1)$ ,  $w_3 = (c, 1)$ ,  $w_4 = (0, 1)$ . Then the subgraph induced by  $S_1$  of  $AG(R)$  contains  $K_{4,7}$  and so  $\gamma(AG(R)) \geq 3$ . Thus we conclude that  $|\mathfrak{m}_1^*| \leq 2$ .

FIGURE 2. Embedding of  $AG(\mathbb{Z}_4 \times \mathbb{Z}_5)$  in  $S_2$ 

Suppose  $|\mathfrak{m}_1^*| = 1$ . i.e.,  $R_1 \cong \mathbb{Z}_4$  or  $\mathbb{Z}_2[x]/(x^2)$ . Note that, by Theorems 2.1 and 2.2,  $R_2 \not\cong \mathbb{Z}_2$  and  $\mathbb{Z}_3$ . Therefore  $|R_2| \geq 4$ . Let us take  $|R_2| \geq 7$ . For  $a \in \mathfrak{m}_1^*$  with  $a^2 = 0$  and  $u_1, u_2 \in R_1^\times$ ,  $b_i \in R_2^*$ , where  $i = 1, \dots, 6$ . Let  $S_2 = \{x_1, \dots, x_{15}\} \subseteq Z(R)^\ast$ , where  $x_1 = (a, 0)$ ,  $x_2 = (u_1, 0)$ ,  $x_3 = (u_2, 0)$ ,  $x_4 = (a, b_1)$ ,  $x_5 = (a, b_2)$ ,  $x_6 = (a, b_3)$ ,  $x_7 = (a, b_4)$ ,  $x_8 = (a, b_5)$ ,  $x_9 = (a, b_6)$ ,  $x_{10} = (0, b_1)$ ,  $x_{11} = (0, b_2)$ ,  $x_{12} = (0, b_3)$ ,  $x_{13} = (0, b_4)$ ,  $x_{14} = (0, b_5)$ ,  $x_{15} = (0, b_6)$ . Then the subgraph induced by  $S_2$  of  $AG(R)$  contains  $K_{3,12}$  so that  $\gamma(AG(R)) \geq 3$ . Hence  $R_2$  is isomorphic to  $\mathbb{F}_4$  or  $\mathbb{Z}_5$ . If  $R \cong \mathbb{Z}_4 \times \mathbb{Z}_5$ , then by Figure 2,  $\gamma(AG(R)) = 2$ . Note that in any figure of this paper, an element  $(x, y)$  of  $R_1 \times R_2$  is denoted by  $xy$ . By the fact  $AG(\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_5) \cong AG(\mathbb{Z}_4 \times \mathbb{Z}_5)$ , we have  $\gamma(AG(\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_5)) = 2$ . Similarly, by Figure 3,  $\gamma(AG(\mathbb{Z}_4 \times \mathbb{F}_4)) = \gamma(AG(\mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_4)) = 2$ .

Suppose  $|\mathfrak{m}_1^*| = 2$ ; i.e.,  $R_1 \cong \mathbb{Z}_9$  or  $\mathbb{Z}_3[x]/(x^2)$ . Assume  $|R_2| \geq 3$ . For  $a, c \in \mathfrak{m}_1^*$  with  $ac = 0$ , and  $u_i \in R_1^\times$  for  $i = 1, \dots, 6$ ;  $b_1, b_2 \in R_2^*$ . Let  $S_3 = \{x_1, \dots, x_{12}\} \subseteq Z(R)^\ast$ , where  $x_1 = (a, 0)$ ,  $x_2 = (c, 0)$ ,  $x_3 = (u_1, 0)$ ,  $x_4 = (u_2, 0)$ ,  $x_5 = (u_3, 0)$ ,  $x_6 = (u_4, 0)$ ,  $x_7 = (u_5, 0)$ ,  $x_8 = (u_6, 0)$ ,  $x_9 = (a, b_1)$ ,  $x_{10} = (a, b_2)$ ,  $x_{11} = (c, b_1)$ ,  $x_{12} = (c, b_2)$ . Then the subgraph induced by  $S_3$  of  $AG(R)$  contains  $K_{4,8}$  and so  $\gamma(AG(R)) \geq 3$ . Hence  $R_2$  is isomorphic to  $\mathbb{Z}_2$ . Therefore  $R \cong \mathbb{Z}_9 \times \mathbb{Z}_2$  or  $\mathbb{Z}_3[x]/(x^2) \times \mathbb{Z}_2$ . Now by Figure 4,  $\gamma(AG(R)) = 2$ .

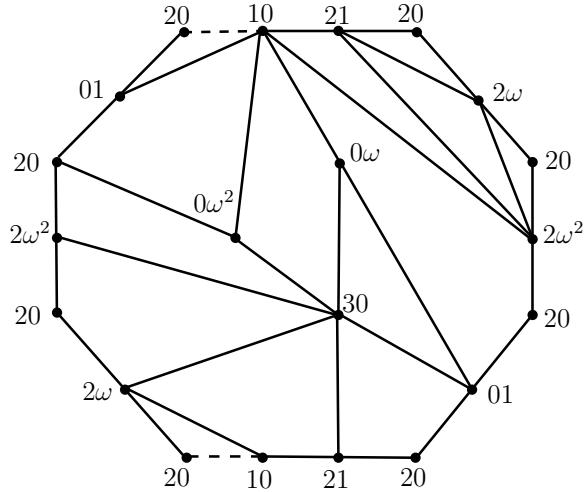


FIGURE 3. Embedding of  $AG(\mathbb{Z}_4 \times \mathbb{F}_4)$  in  $S_2$

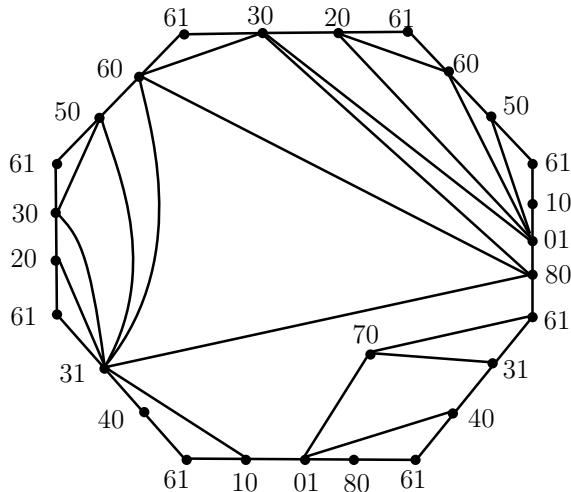


FIGURE 4. Embedding of  $AG(\mathbb{Z}_9 \times \mathbb{Z}_2)$  in  $S_2$

Hence, in this case,  $\gamma(AG(R)) = 2$  if and only if  $R$  is isomorphic to one of  $\mathbb{Z}_4 \times \mathbb{F}_4$ ,  $\mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_4$ ,  $\mathbb{Z}_4 \times \mathbb{Z}_5$ ,  $\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_5$ ,  $\mathbb{Z}_9 \times \mathbb{Z}_2$  or  $\mathbb{Z}_3[x]/(x^2) \times \mathbb{Z}_2$ .  $\square$

## 4. Conclusions

We obtained all isomorphism classes of commutative rings with identity  $R$  for which the annihilator graph of  $R$  has genus two.

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