

SOME PROPERTIES OF $(\in_\gamma, \in_\gamma \vee q_\delta)$ -FUZZY SUB-COMMUTATIVE IDEALS IN BCI-ALGEBRAS

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In this paper, we define the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal and $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$ -fuzzy sub-commutative ideal in BCI-algebra and investigate some of their related properties.

Keywords: BCI-algebra; $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal; $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$ -fuzzy sub-commutative ideal.

1. Introduction

The concept of fuzzy set, which was published by Zadeh in his classic paper [24] of 1965, was applied by many researchers to generalize some of the basic concepts of algebra. The fuzzy algebraic structures play a central role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic, set theory, real analysis, measure theory etc. In 1991, Xi applied fuzzy subsets in BCK-algebras [23] and got some interesting results. In 1993, Ahmad [1] and Jun [12] applied the concept of fuzzy sets to BCI-algebras.

The concept of sub-commutative ideals in a BCI-algebra was initiated by Liu and Meng [14]. Jun studied fuzzy sub-implicative ideals of BCI-algebras in [9]. Liu et al. [15] discussed FSI-ideals and FSC-ideals of BCI-algebras. In [8], Hedayati studied connections between generalized fuzzy ideals and sub-implicative ideals of BCI-algebras. Zhan et al. discussed characterizations of generalized fuzzy ideals of BCI-algebras in [26].

In 1971, Rosenfeld [22] laid the foundations of fuzzy groups. In [20], Murali defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set given in [21], plays a vital role to generate some different types of fuzzy subgroups, called (α, β) -fuzzy subgroups, introduced by Bhakat and Das [4]. In particular, $(\in, \in \vee q)$ -fuzzy subgroup is an important and useful generalization of the Rosenfeld's fuzzy subgroup. Bhakat [2, 3] studied

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$(\in \vee q)$ -level subsets, $(\in, \in \vee q)$ -fuzzy normal, quasi-normal and maximal subgroups. Jun introduced the concept of $(\in, \in \vee q)$ -fuzzy subalgebras in BCK/BCI-algebras and investigated some related results in [13]. In [10], Jun discussed (α, β) -fuzzy ideals of BCK/BCI-algebras. Zhan et al. [25], studied $(\in, \in \vee q)$ -fuzzy ideals of BCI-algebras. In [11], Jun defined the concept of (α, β) -fuzzy subalgebras of BCK/BCI-algebras. Davvaz [6] studied $(\in, \in \vee q)$ -fuzzy subnear-rings and ideals of nearrings. Davvaz and Corsini [7] redefined fuzzy H_v -submodule and many-valued implications. In [16, 17], Ma et al. discussed some kinds of $(\in, \in \vee q)$ -interval-valued fuzzy ideals of BCI-algebras. In [18], Ma et al. studied new types of fuzzy ideals of BCI-algebras.

In this paper, we introduce the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal and $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$ -fuzzy sub-commutative ideal in BCI-algebra and investigate some of their related properties.

2. Preliminaries

Throughout this paper X always mean a BCI-algebra without any specification. We also include some basic results that are necessary for this paper.

Definition 2.1. [1] A BCI-algebra X is a general algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions:

- (BCI-I) $((x * y) * (x * z)) * (z * y) = 0$
- (BCI-II) $(x * (x * y)) * y = 0$
- (BCI-III) $x * x = 0$
- (BCI-IV) $x * y = 0$ and $y * x = 0$ imply $x = y$
for all $x, y, z \in X$.

We can define a partial order “ \leq ” on X by $x \leq y$ if and only if $x * y = 0$.

Proposition 2.2. [1, 19] In any BCI-algebra X , the following are true:

- (i) $(x * y) * z = (x * z) * y$
- (ii) $(x * z) * (y * z) \leq x * y$
- (iii) $(x * y) * (x * z) \leq z * y$
- (iv) $x * 0 = x$
- (v) $x * (x * (x * y)) = x * y$
- (vi) $x * y \leq x$
for all $x, y, z \in X$.

Definition 2.3. [9] A non-empty subset I of a BCI-algebra X is called an ideal of X if it satisfies (I1) and (I2), where

- (I1) $0 \in I$, (I2) $x * y \in I$ and $y \in I$ imply $x \in I$,
for all $x, y \in X$.

For any elements x and y of a BCI-algebra, $x^n * y$ denotes

$x * (\dots * (x * (x * y)) \dots)$, in which x occurs n times.

Definition 2.4. [14] A non-empty subset I of a BCI-algebra X is called a sub-commutative ideal of X if it satisfies (I1) and (I3), where

$$(I1) \quad 0 \in I,$$

$$(I3) \quad (y * (y * (x * (x * y)))) * z \in I \text{ and } z \in I \text{ imply } x * (x * y) \in I, \\ \text{for all } x, y, z \in X.$$

3. Fuzzy sub-commutative ideals in BCI-algebras

We now review some fuzzy logic concepts.

Definition 3.1. [24] A fuzzy set f of a universe X is a function from X to the unit closed interval $[0, 1]$, that is $f : X \rightarrow [0, 1]$.

Definition 3.2. [5] For a fuzzy set f of a BCI-algebra X and $t \in (0, 1]$, the crisp set $f_t = \{x \in X \mid f(x) \geq t\}$ is called the level subset of f .

Definition 3.3. [10] A fuzzy set f of a BCI-algebra X is called a fuzzy ideal of X if it satisfies (F1) and (F2), where

$$(F1) \quad f(0) \geq f(x), \quad (F2) \quad f(x) \geq f(x * y) \wedge f(y), \quad \text{for all } x, y \in X.$$

Definition 3.4. [14] A fuzzy set f of a BCI-algebra X is called a fuzzy sub-commutative ideal of X if it satisfies (F1) and (F3), where

$$(F1) \quad f(0) \geq f(x),$$

$$(F3) \quad f(x * (x * y)) \geq f((y * (y * (x * (x * y)))) * z) \wedge f(z), \\ \text{for all } x, y, z \in X.$$

Theorem 3.5. [26] A fuzzy set f of a BCI-algebras X is a fuzzy sub-commutative ideal of X if and only if for every $t \in (0, 1]$, $f_t = \{x \in X \mid f(x) \geq t\}$ is a sub-commutative ideal of X .

Definition 3.6. [10] A fuzzy set f of a BCI-algebra X of the form

$$f(y) = \begin{cases} t & (\neq 0) & \text{if } y = x, \\ 0 & & \text{if } y \neq x, \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t .

A fuzzy point x_t is said to belong to (resp., quasi-coincident with) a fuzzy set f , written as $x_t \in f$ (resp., $x_t qf$) if $f(x) \geq t$ (resp., $f(x) + t > 1$).

If $x_t \in f$ or $x_t qf$, then we write $x_t \in \vee qf$. If $f(x) < t$ (resp. $f(x) + t \leq 1$), then we say that $x_t \overline{\in} f$ (resp. $x_t \overline{q}f$). The symbol $\overline{\in \vee q}$ means that $\in \vee q$ does not hold.

Let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For a fuzzy point x_r and a fuzzy set f of a BCI-algebra X , Ma et al. [18] defined as follows:

- (1) $x_r \in_\gamma f$ if $f(x) \geq r > \gamma$.
- (2) $x_r q_\delta f$ if $f(x) + r > 2\delta$.
- (3) $x_r \in_\gamma \vee q_\delta f$ if $x_r \in_\gamma f$ or $x_r q_\delta f$.
- (4) $x_r \overline{\in}_\gamma \vee \overline{q}_\delta f$ if $x_r \overline{\in} f$ or $x_r \overline{q}_\delta f$.

4. $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideals in BCI-algebras

In this section, we define the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal in BCI-algebra and investigate some of their related properties.

Definition 4.1. [18] A fuzzy set f of a BCI-algebra X is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy ideal of X if it satisfies (A) and (B), where

- (A) $x_t \in_\gamma f \Rightarrow 0_t \in_\gamma \vee q_\delta f$,
- (B) $(x * y)_t \in_\gamma f, y_r \in_\gamma f \Rightarrow x_{t \wedge r} \in_\gamma \vee q_\delta f$,
for all $t, r \in (\gamma, 1]$ and for all $x, y \in X$.

Theorem 4.2. [18] Every fuzzy ideal of a BCI-algebra X is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy ideal of X .

Definition 4.3. A fuzzy set f of a BCI-algebra X is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal of X if it satisfies (A) and (C), where

- (A) $x_t \in_\gamma f \Rightarrow 0_t \in_\gamma \vee q_\delta f$,
- (C) $((y * (y * (x * (x * y)))) * z)_t \in_\gamma f, z_r \in_\gamma f$
 $\Rightarrow (x * (x * y))_{t \wedge r} \in_\gamma \vee q_\delta f$,

for all $t, r \in (\gamma, 1]$ and for all $x, y, z \in X$.

Theorem 4.4. Every fuzzy sub-commutative ideal of a BCI-algebra X is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal of X .

Proof. Straightforward.

Theorem 4.5. A fuzzy set f of a BCI-algebra X is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal of X if and only if it satisfies (D) and (E), where

- (D) $f(0) \vee \gamma \geq f(x) \wedge \delta$,
- (E) $f(x * (x * y)) \vee \gamma \geq f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta$,
for all $x, y, z \in X$.

Proof. (A) \Rightarrow (D)

Let $x \in X$ be such that

$$f(0) \vee \gamma < f(x) \wedge \delta.$$

Then

$$f(0) \vee \gamma < t < f(x) \wedge \delta$$

for some $\gamma < t < \delta$. This implies $x_t \in f$ but $0_t \overline{\in} f$. Since

$$f(x) + t < 2\delta$$

we have $0_t \overline{q}_\delta f$. It follows that $0_t \overline{\in_\gamma \vee q_\delta} f$, which is a contradiction. Hence

$$f(0) \vee \gamma \geq f(x) \wedge \delta.$$

(D) \Rightarrow (A)

Let $x_t \in_\gamma f$. Then $f(x) \geq t > \gamma$. If $0_t \in_\gamma f$, then (A) holds. If $0_t \overline{\in_\gamma} f$, then $f(0) < t \leq f(x)$. Since

$$f(0) \vee \gamma \geq f(x) \wedge \delta \geq t \wedge \delta$$

it follows that $f(0) \geq \delta$. Hence

$$f(0) + t > f(0) + f(0) = 2f(0) > 2\delta.$$

Thus $0_t q_\delta f$. Hence $0_t \in_\gamma \vee q_\delta f$.

(C) \Rightarrow (E)

Suppose (E) does not hold. Then there exist $x, y, z \in X$ such that

$$f(x * (x * y)) \vee \gamma < f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta.$$

Then

$$f(x * (x * y)) \vee \gamma < t < f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta$$

for some $t \in (\gamma, \delta]$ and so $((y * (y * (x * (x * y)))) * z)_t \in_\gamma f$ and $z_t \in_\gamma f$,

but $(x * (x * y))_t \overline{\in_\gamma \vee q_\delta} f$. This is a contradiction. Hence

$$f(x * (x * y)) \vee \gamma \geq f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta.$$

(E) \Rightarrow (C)

Let $((y * (y * (x * (x * y)))) * z)_t \in_\gamma f$ and $z_r \in_\gamma f$.

Then

$$f((y * (y * (x * (x * y)))) * z) \geq t \text{ and } f(z) \geq r.$$

If $(x * (x * y))_{t \wedge r} \in_\gamma f$, then (C) holds. If $(x * (x * y))_{t \wedge r} \overline{\in_\gamma} f$, then $f(x * (x * y)) < t \wedge r$. Since

$$\begin{aligned} f(x * (x * y)) \vee \gamma &\geq f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta \\ &\geq t \wedge r \wedge \delta \end{aligned}$$

it follows that $f(x * (x * y)) \geq \delta$ and $t \wedge r > \delta$. Thus

$$\begin{aligned} f(x * (x * y)) + t \wedge r &> \delta + \delta = 2\delta \\ \Rightarrow (x * (x * y))_{t \wedge r} &q_\delta f. \end{aligned}$$

Hence

$$(x * (x * y))_{t \wedge r} \in_\gamma \vee q_\delta f.$$

Remark 4.6. For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal f of a BCI-algebra X , we have

- (i) If $\gamma = 0$ and $\delta = 1$, then f is a fuzzy sub-commutative ideal of X .
- (ii) If $\gamma = 0$ and $\delta = 0.5$, then f is an $(\in, \in \vee q)$ -fuzzy sub-commutative ideal of X .

For any fuzzy set f of a BCI-algebra X , we define

$$\begin{aligned} f_r^\gamma &= \{x \in X \mid x_r \in_\gamma f\} \\ f_r^\delta &= \{x \in X \mid x_r q_\delta f\} \end{aligned}$$

and

$$[f]_r^\delta = \{x \in X \mid x_r \in_\gamma \vee q_\delta f\} \text{ for all } r \in [0, 1].$$

It is clear that

$$[f]_r^\delta = f_r^\gamma \cup f_r^\delta.$$

The relationship between $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideals and the crisp sub-commutative ideals of a BCI-algebra X can be expressed in the form of the following theorem.

Theorem 4.7. Let f be a fuzzy set of a BCI-algebra X . Then

- (1) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal of X if and only if $f_r^\gamma (\neq \phi)$ is a sub-commutative ideal of X for all $r \in (\gamma, \delta]$.
- (2) If $2\delta = 1 + \gamma$, then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal of X if and only if $f_r^\delta (\neq \phi)$ is a sub-commutative ideal of X for all $r \in (\delta, 1]$.
- (3) If $2\delta = 1 + \gamma$, then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal of X if and only if $[f]_r^\delta (\neq \phi)$ is a sub-commutative ideal of X for all $r \in (\gamma, 1]$.

Proof. (1) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal of X and $x \in f_r^\gamma$ for $r \in (\gamma, \delta]$. Since

$$f(0) \vee \gamma \geq f(x) \wedge \delta \geq r \wedge \delta = r > \gamma$$

we have $f(0) \geq r$. Hence $0 \in f_r^\gamma$. Let $(y * (y * (x * (x * y)))) * z, z \in f_r^\gamma$.

Then

$$f((y * (y * (x * (x * y)))) * z) \geq r > \gamma, \quad f(z) \geq r > \gamma.$$

Since

$$\begin{aligned} f(x * (x * y)) \vee \gamma &\geq f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta \\ &\geq r \wedge r \wedge \delta = r \wedge \delta = r > \gamma \end{aligned}$$

we have $f(x * (x * y)) \geq r$. Thus $x * (x * y) \in f_r^\gamma$. Therefore f_r^γ is a sub-commutative ideal of X .

Conversely, assume that f_r^γ is a sub-commutative ideal of X for all $r \in (\gamma, \delta]$. If there exists $x \in X$ such that

$$f(0) \vee \gamma < r = f(x) \wedge \delta$$

then $x_r \in_\gamma f$, but $0_r \overline{\in_\gamma \vee q_\delta} f$, which is a contradiction. Suppose $x, y, z \in X$ be such that

$$f(x * (x * y)) \vee \gamma < f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta.$$

Select $t \in (\gamma, \delta]$ such that

$$f(x * (x * y)) \vee \gamma < t = f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta.$$

Then

$$((y * (y * (x * (x * y)))) * z)_t \in_\gamma f, z_t \in_\gamma f, \text{ but } (x * (x * y))_t \overline{\in_\gamma \vee q_\delta} f.$$

which is a contradiction. Hence

$$f(x * (x * y)) \vee \gamma \geq f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta.$$

Therefore f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal of X .

(2) Suppose f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal of X .

Let $(y * (y * (x * (x * y)))) * z, z \in f_r^\delta$. Then

$$((y * (y * (x * (x * y)))) * z)_r q_\delta f, z_r q_\delta f.$$

This implies that

$$f((y * (y * (x * (x * y)))) * z) + r > 2\delta, f(z) + r > 2\delta$$

$$f((y * (y * (x * (x * y)))) * z) > 2\delta - r, f(z) > 2\delta - r$$

$$f((y * (y * (x * (x * y)))) * z) > 2\delta - r \geq 2\delta - 1 = \gamma, f(z) > 2\delta - r \geq 2\delta - 1 = \gamma.$$

By hypothesis

$$\begin{aligned} f(x * (x * y)) \vee \gamma &\geq f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta \\ &> (2\delta - r) \wedge (2\delta - r) \wedge \delta \\ &= (2\delta - r) \wedge \delta > 2\delta - r. \end{aligned}$$

So that $f(x * (x * y)) > 2\delta - r$. Thus $f(x * (x * y)) + r > 2\delta$. This implies that

$(x * (x * y))_r q_\delta f$, that is, $x * (x * y) \in f_r^\delta$. Hence f_r^δ is a sub-commutative ideal of X .

Conversely, assume that f_r^δ is a sub-commutative ideal of X for all $r \in (\delta, 1]$. Let $x \in X$ be such that

$$f(0) \vee \gamma < f(x) \wedge \delta.$$

Then

$$2\delta - (f(x) \wedge \delta) < 2\delta - (f(0) \vee \gamma).$$

This implies that

$$(2\delta - f(x)) \vee \delta < (2\delta - f(0)) \wedge (2\delta - \gamma).$$

Select $r \in (\delta, 1]$ such that

$$(2\delta - f(x)) \vee \delta < r \leq (2\delta - f(0)) \wedge (2\delta - \gamma).$$

Then

$$\begin{aligned} 2\delta - f(x) &< r \quad \text{and} \quad r \leq 2\delta - f(0) \\ \Rightarrow 2\delta &< f(x) + r \quad \text{and} \quad f(0) + r \leq 2\delta. \end{aligned}$$

Thus $x_r q_\delta f$ but $0_r \bar{q}_\delta f$, that is, $x \in f_r^\delta$ but $0 \notin f_r^\delta$, a contradiction. Hence $f(0) \vee \gamma \geq f(x) \wedge \delta$.

Now, suppose $x, y, z \in X$ be such that

$$f(x * (x * y)) \vee \gamma < f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta.$$

Then

$$2\delta - (f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta) < 2\delta - (f(x * (x * y)) \vee \gamma).$$

This implies that

$$\begin{aligned} (2\delta - (f((y * (y * (x * (x * y)))) * z)) \vee (2\delta - f(z)) \vee \delta) \\ < (2\delta - f(x * (x * y))) \wedge (2\delta - \gamma). \end{aligned}$$

Select some $r \in (\delta, 1]$ such that

$$\begin{aligned} (2\delta - (f((y * (y * (x * (x * y)))) * z)) \vee (2\delta - f(z)) \vee \delta) \\ < r \leq (2\delta - f(x * (x * y))) \wedge (2\delta - \gamma). \end{aligned}$$

Then

$$\begin{aligned} 2\delta - f((y * (y * (x * (x * y)))) * z) &< r, \quad 2\delta - f(z) < r \text{ and} \\ r &\leq 2\delta - f(x * (x * y)) \end{aligned}$$

$$\Rightarrow f((y * (y * (x * (x * y)))) * z) + r > 2\delta, \quad f(z) + r > 2\delta$$

and

$$f(x * (x * y)) + r \leq 2\delta$$

Thus

$((y * (y * (x * (x * y)))) * z)_r q_\delta f$, $z_r q_\delta f$, but $(x * (x * y))_r \bar{q}_\delta f$, that is, $(y * (y * (x * (x * y)))) * z$ and z are in f_r^δ but $x * (x * y) \notin f_r^\delta$, a contradiction. Hence

$$f(x * (x * y)) \vee \gamma \geq f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta.$$

This shows that f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal of X .

(3) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal of X and $r \in (\gamma, 1]$. Then for all $x \in [f]_r^\delta$, we have $x_r \in_r \vee q_\delta f$, that is

$$f(x) \geq r > \gamma \text{ or } f(x) > 2\delta - r > 2\delta - 1 = \gamma.$$

Since f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy ideal of X , therefore

$$f(0) \vee \gamma \geq f(x) \wedge \delta \geq \gamma \wedge \delta = \gamma$$

and so $f(0) \geq \gamma$, that is, $f(0) \geq f(x) \wedge \delta$.

Case 1: If $r \in (\gamma, \delta]$, then $2\delta - r \geq \delta \geq r$ and so

$$f(0) \geq f(x) \wedge \delta \geq r \wedge \delta = r.$$

Thus, $0_r \in_\gamma f$.

Case 2: If $r \in (\delta, 1]$, then $2\delta - r < \delta < r$ and so

$$f(0) \geq f(x) \wedge \delta \geq r \wedge \delta = \delta > 2\delta - r.$$

Hence, $0_r q_\delta f$. Thus in any case, we have $0_r \in_\gamma \vee q_\delta f$.

Let $(y * (y * (x * (x * y)))) * z, z \in [f]_r^\delta$, so we have

$$((y * (y * (x * (x * y)))) * z)_r, z_r \in_\gamma \vee q_\delta f$$

that is, $f((y * (y * (x * (x * y)))) * z) \geq r > \gamma$ and $f(z) \geq r > \gamma$. Since f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal of X , therefore

$$\begin{aligned} f(x * (x * y)) \vee \gamma &\geq f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta \\ &> \gamma \wedge \gamma \wedge \delta = \gamma \wedge \delta = \gamma \end{aligned}$$

and so

$$f(x * (x * y)) \geq f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta.$$

Case 1: If $r \in (\gamma, \delta]$, then $2\delta - r \geq \delta \geq r$ and so

$$\begin{aligned} f(x * (x * y)) &\geq f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta \\ &\geq r \wedge r \wedge \delta = r \wedge \delta = r. \end{aligned}$$

Hence, $(x * (x * y))_r \in_\gamma f$.

Case 2: If $r \in (\delta, 1]$, then $2\delta - r < \delta < r$ and so

$$\begin{aligned} f(x * (x * y)) &\geq f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta \\ &\geq r \wedge r \wedge \delta = r \wedge \delta = \delta = 2\delta - r. \end{aligned}$$

Thus, $(x * (x * y))_r q_\delta f$. Hence in all cases we have $(x * (x * y))_r \in_\gamma \vee q_\delta f$, that is, $x * (x * y) \in [f]_r^\delta$. Therefore $[f]_r^\delta$ is a sub-commutative ideal of X .

Conversely, suppose that $[f]_r^\delta$ is a sub-commutative ideal of a BCI-algebra X , for all $r \in (\gamma, \delta]$. If there exists $x \in X$ be such that

$$f(0) \vee \gamma < r = f(x) \wedge \delta,$$

then $x_r \in_\gamma f$ but $0_r \in_\gamma \overline{\vee q_\delta f}$. Since $[f]_r^\delta$ is a sub-commutative ideal of X , we have $0 \in [f]_r^\delta$, which is a contradiction. Suppose $x, y, z \in X$ be such that

$$f(x * (x * y)) \vee \gamma < f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta.$$

Select some $r \in (\gamma, 1]$ such that

$$f(x * (x * y)) \vee \gamma < r = f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta.$$

Then

$((y * (y * (x * (x * y)))) * z)_r \in_{\gamma} f$, $z_r \in_{\gamma} f$ but $(x * (x * y))_r \overline{\in_{\gamma} \vee q_{\delta} f}$. Since $[f]_r^{\delta}$ is a sub-commutative ideal of X , we have $x * (x * y) \in [f]_r^{\delta}$, which is a contradiction. Hence

$$f(x * (x * y)) \vee \gamma \geq f((y * (y * (x * (x * y)))) * z) \wedge f(z) \wedge \delta.$$

Therefore f is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy sub-commutative ideal of X .

By setting $\gamma = 0$ and $\delta = 0.5$ in Theorem 4.7, the following corollary is obtained.

Corollary 4.8. Let f be a fuzzy set of a BCI-algebra X . Then

- (1) f is an $(\in, \in \vee q)$ -fuzzy sub-commutative ideal of X if and only if $f_r (\neq \phi)$ is a sub-commutative ideal of X for all $r \in (0, 0.5]$.
- (2) f is an $(\in, \in \vee q)$ -fuzzy sub-commutative ideal of X if and only if $Q(f; r) (\neq \phi)$ is a sub-commutative ideal of X for all $r \in (0.5, 1]$, where

$$Q(f; r) = \{x \in X \mid x_r q f\}.$$

- (3) f is an $(\in, \in \vee q)$ -fuzzy sub-commutative ideal of X if and only if $[f]_r (\neq \phi)$ is a sub-commutative ideal of X for all $r \in (0, 1]$.

5. $(\overline{\in}_{\gamma}, \overline{\in}_{\gamma} \vee \overline{q}_{\delta})$ -fuzzy sub-commutative ideals in BCI-algebras

In this section, we define the concept of $(\overline{\in}_{\gamma}, \overline{\in}_{\gamma} \vee \overline{q}_{\delta})$ -fuzzy sub-commutative ideals in BCI-algebras and investigate some of their related properties.

Definition 5.1. A fuzzy set f of a BCI-algebra X is called an $(\overline{\in}_{\gamma}, \overline{\in}_{\gamma} \vee \overline{q}_{\delta})$ -fuzzy sub-commutative ideal of X if it satisfies (F) and (G), where

- (F) $0_t \overline{\in}_{\gamma} f \Rightarrow x_t \overline{\in}_{\gamma} \vee \overline{q}_{\delta} f$,
- (G) $(x * (x * y))_{t \wedge r} \overline{\in}_{\gamma} f \Rightarrow ((y * (y * (x * (x * y)))) * z)_t \overline{\in}_{\gamma} \vee \overline{q}_{\delta} f$ or $z_r \overline{\in}_{\gamma} \vee \overline{q}_{\delta} f$,

for all $t, r \in (\gamma, 1]$ and for all $x, y, z \in X$.

Theorem 5.2. A fuzzy set f of a BCI-algebra X is an $(\overline{\in}_{\gamma}, \overline{\in}_{\gamma} \vee \overline{q}_{\delta})$ -fuzzy sub-commutative ideal of X if and only if it satisfies (H) and (I), where

- (H) $f(0) \vee \delta \geq f(x)$,
- (I) $f(x * (x * y)) \vee \delta \geq f((y * (y * (x * (x * y)))) * z) \wedge f(z)$,
for all $x, y, z \in X$.

Proof. (F) \Rightarrow (H)

Suppose there exists $x \in X$ such that

$$f(0) \vee \delta < f(x).$$

Take $f(x) = t$, then $t > \delta$ and $f(0) < t$. Now $0_t \in_\gamma f$. Then by hypothesis $x_t \in_\gamma \vee \bar{q}_\delta f$. But $f(x) = t$, so $x_t \in_\gamma f$. Hence $x_t \bar{q}_\delta f$, that is, $f(x) + t \leq 2\delta$. This implies $t \leq \delta$, which is a contradiction to $t > \delta$. Hence

$$f(0) \vee \delta \geq f(x).$$

(H) \Rightarrow (F)

Let $0_t \in_\gamma f$. Then $f(0) < t$.

- (a) If $f(0) \geq f(x)$, then $f(x) < t$, $f(x) < t$, and so $x_t \in_\gamma f$.
- (b) If $f(0) < f(x)$, then by (H), $\delta \geq f(x)$.
- (i) If $f(x) < t$, then $x_t \in_\gamma f$.
- (ii) If $f(x) \geq t$, then $t \leq f(x) \leq \delta$. It follows that $f(x) + t \leq 2\delta$ and so $x_t \bar{q}_\delta f$. Hence in any case $x_t \in_\gamma \vee \bar{q}_\delta f$.

(G) \Rightarrow (I)

Suppose there exist $x, y, z \in X$ such that

$$f(x * (x * y)) \vee \delta < f((y * (y * (x * (x * y)))) * z) \wedge f(z) = t.$$

$$t > \delta, f(x * (x * y)) < t, f((y * (y * (x * (x * y)))) * z) \geq t \text{ and } f(z) \geq t.$$

This implies

$$(x * (x * y))_t \in_\gamma f, ((y * (y * (x * (x * y)))) * z)_t \in_\gamma f \text{ and } z_t \in_\gamma f.$$

Thus by hypothesis either $((y * (y * (x * (x * y)))) * z)_t \bar{q}_\delta f$ or $z_t \bar{q}_\delta f$.

If $((y * (y * (x * (x * y)))) * z)_t \bar{q}_\delta f$, then

$$f((y * (y * (x * (x * y)))) * z) + t \leq 2\delta.$$

This implies $t \leq \delta$, which is a contradiction to $t > \delta$. If $z_t \bar{q}_\delta f$, then

$$f(z) + t \leq 2\delta. \text{ This also implies } t \leq \delta, \text{ which is again a contradiction to } t > \delta.$$

Hence

$$f(x * (x * y)) \vee \delta \geq f((y * (y * (x * (x * y)))) * z) \wedge f(z).$$

(I) \Rightarrow (G)

Let $(x * (x * y))_{t \wedge r} \in_\gamma f$. Then $f(x * (x * y)) < t \wedge r$.

- (a) If $f(x * (x * y)) \geq f((y * (y * (x * (x * y)))) * z) \wedge f(z)$, then
 $f((y * (y * (x * (x * y)))) * z) \wedge f(z) < t \wedge r$,

and consequently, $f((y * (y * (x * (x * y)))) * z) < t$ or $f(z) < r$. It follows that

$$((y * (y * (x * (x * y)))) * z)_t \in_\gamma f \text{ or } z_r \in_\gamma f.$$

- (b) If $f(x * (x * y)) < f((y * (y * (x * (x * y)))) * z) \wedge f(z)$, then by (I),
 $\delta \geq f((y * (y * (x * (x * y)))) * z) \wedge f(z)$.
- (i) If $f((y * (y * (x * (x * y)))) * z) \wedge f(z) < t \wedge r$, then either

$$f((y * (y * (x * (x * y)))) * z) < t \text{ or } f(z) < r,$$

that is, either

$$((y * (y * (x * (x * y)))) * z)_t \overline{\in}_\gamma f \text{ or } z_r \overline{\in}_\gamma f.$$

(ii) If $f((y * (y * (x * (x * y)))) * z) \wedge f(z) \geq t \wedge r$, then either
 $\delta \geq f((y * (y * (x * (x * y)))) * z) \geq t$ or $\delta \geq f(z) \geq r$.

This implies either $((y * (y * (x * (x * y)))) * z)_t \overline{q}_\delta f$ or $z_r \overline{q}_\delta f$. Hence in any case

$$((y * (y * (x * (x * y)))) * z)_t \overline{\in}_\gamma \vee \overline{q}_\delta f \text{ or } z_r \overline{\in}_\gamma \vee \overline{q}_\delta f.$$

Remark 5.3. For any $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$ -fuzzy sub-commutative ideal f of a BCI-algebra X , we can conclude that if $\delta = 0.5$, then f is the $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy sub-commutative ideal of X .

The relationship between $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$ -fuzzy sub-commutative ideals and the crisp sub-commutative ideals of a BCI-algebra X can be expressed in the form of the following theorem.

Theorem 5.4. Let f be a fuzzy set of a BCI-algebra X . Then f is an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$ -fuzzy sub-commutative ideal of X if and only if $f_r^\gamma (\neq \phi)$ is a sub-commutative ideal of X for all $r \in (\delta, 1]$.

Proof. Let f be an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$ -fuzzy sub-commutative ideal of X and $x \in f_r^\gamma$ for $r \in (\delta, 1]$. Since

$$f(0) \vee \delta \geq f(x) \geq r > \delta$$

we have $f(0) \geq r$. Hence $0 \in f_r^\gamma$. Let $(y * (y * (x * (x * y)))) * z, z \in f_r^\gamma$.

Then $f((y * (y * (x * (x * y)))) * z) \geq r > \delta$ and $f(z) \geq r > \delta$. Since

$$\begin{aligned} f(x * (x * y)) \vee \delta &\geq f((y * (y * (x * (x * y)))) * z) \wedge f(z) \\ &\geq r \wedge r = r > \delta \end{aligned}$$

we have $f(x * (x * y)) \geq r$. Thus $x * (x * y) \in f_r^\gamma$. Hence f_r^γ is a sub-commutative ideal of X .

Conversely, assume that f_r^γ is a sub-commutative ideal of X for all $r \in (\delta, 1]$. Suppose there exists $x \in X$ such that

$$f(0) \vee \delta < f(x) = r.$$

Then $r > \delta$ and $x \in f_r^\gamma$ but $0 \notin f_r^\gamma$, a contradiction. Hence

$$f(0) \vee \delta \geq f(x).$$

Now, suppose $x, y, z \in X$ be such that

$$f(x * (x * y)) \vee \delta < f((y * (y * (x * (x * y)))) * z) \wedge f(z).$$

Take $t = f((y * (y * (x * (x * y)))) * z) \wedge f(z)$. Then

$t > \delta$ and $(y * (y * (x * (x * y)))) * z \in f_r^\gamma$ and $z \in f_r^\gamma$ but $x * (x * y) \notin f_r^\gamma$, which is a contradiction. Hence

$$f(x * (x * y)) \vee \delta \geq f((y * (y * (x * (x * y)))) * z) \wedge f(z).$$

This shows that f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal of X .

Corollary 5.5. Let f be a fuzzy set of a BCI-algebra X . Then

- (1) f is an $(\in, \in \vee \bar{q})$ -fuzzy sub-commutative ideal of X if and only if $f_r(\neq \phi)$ is a sub-commutative ideal of X for all $r \in (0.5, 1]$.
- (2) f is an $(\in, \in \vee \bar{q})$ -fuzzy sub-commutative ideal of X if and only if $Q(f; r)(\neq \phi)$ is a sub-commutative ideal of X for all $r \in (0, 0.5]$, where

$$Q(f; r) = \{x \in X \mid x_r q f\}.$$

6. Conclusions

In the study of fuzzy algebraic system, we see that the fuzzy sub-commutative ideals with special properties always play a central role.

In this paper, we define the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy sub-commutative ideal of BCI-algebras and give several properties of fuzzy sub-commutative ideals in BCI-algebras in terms of these notions.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of fuzzy BCI-algebras and their applications in other branches of algebra. In the future study of fuzzy BCI-algebras, perhaps the following topics are worth to be considered:

- (1) To characterize other classes of BCI-algebras by using this notion;
- (2) To apply this notion to some other algebraic structures;
- (3) To consider these results to some possible applications in computer sciences and information systems in the future.

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