

## A NEW ITERATED TIKHONOV REGULARIZATION METHOD FOR FREDHOLM INTEGRAL EQUATION OF FIRST KIND

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*We consider Fredholm integral equation of the first kind, and present an efficient new iterated Tikhonov method to solve it. The new Tikhonov iteration method has been proved which can achieve the optimal order under a-priori assumption. In numerical experiments, the new iterated Tikhonov regularization method is compared with the classical iterated Tikhonov method, Landweber iteration method to solve the corresponding discrete problem, which indicates the validity and efficiency of the proposed method.*

**Keywords:** Fredholm integral equation, Tikhonov regularization, optimal order, discrete ill-posed problem, Landweber iteration.

**MSC2020:** 65K05, 90C46.

### 1. Introduction

In recent years, the research about inverse problems or ill-posed problems draw scientists' attention. It can be emerged in earth physics, engineering technology and many other fields, such as geophysical problems [19], resistivity inversion problem [17], and computed tomography [18]. Therefore, the investigation of ill-posed problem not only has great scientific innovation significance, but also has certain practical importance.

In general, the inverse problem is much more difficult to solve than the forward problem, owing to its ill-posed feature. In the mid-1960s, the regularization method for dealing with ill-posed problems proposed by Tikhonov, brought the study of ill-posed problems into a new stage. Later, Landweber rewrote the equation (2) into an iteration form. Afterwards, many other technologies applied to regularization method came along successively, including precondition technique [13], adding contraction or penalty [5], multi-parameter regularization methods [6], filter based methods [9], methods coupling of them [10] or other methods [14, 15]. Klann et al. [11] and Hochstenbach et al. [9] discussed measuring the residual error in Tikhonov regularization with a seminorm that uses a fractional power of the Moore-Penrose pseudo inverse of  $A^T A$  as weighting matrix, which lead to fractional filter methods. The former gave the fractional Landweber method and the latter presented fractional Tikhonov method. In [7], Huckle and Sedlacek also derived seminorms for the Tikhonov-Phillips regularization based on the underlying blur operator, that is using discrete smoothing-norms of the form  $\|Lx\|_2$  to substitute the classical 2-norm  $\|x\|_2$  for obtaining regularity, with  $L$  being a discrete approximation to a derivative operator. Using a differential operator in the Tikhonov functional, it will be smoother and get a more accurate reconstruction. In [6], Gazzola and Reichel proposed two multi-parameter regularization methods for linear discrete ill-posed problems, which are based on the projection of a Tikhonov-regularized problem onto Krylov subspaces of increasing dimension. By selecting a proper set of regularization parameters and maximizing a suitable quantity, they can get the approximate solution. Stefano et al. proposed a nested primal-dual method for the efficient solution of regularized convex optimization problem in [1], under a relaxed monotonicity assumption on the scaling matrices and a shrinking condition on the extrapolation parameters, they gave the convergence result for the iteration sequence. Up to now, regularization methods are still powerful tools to settle inverse problems.

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In this paper, our goal is to give a new iterated regularization method based on (17), for solving Fredholm integral equation of the first kind. This paper is organized as follows. In Section 2, we recall some basic definitions and preliminaries about the classical Tikhonov and Landweber method, the filter based regularization methods and the optimal order of a regularization method. In Section 3, we present a new iterated Tikhonov method and give the convergence result. Some numerical examples are reported in Section 4. Finally, Section 5 gives the conclusion.

## 2. Preliminaries

Fredholm integral equation of the first kind will be reviewed in this section firstly. Then, we recall some classical results about Tikhonov method and Landweber iteration.

### 2.1. Fredholm integral equation of the first kind

Many mathematical physics inverse problems, such as the backwards heat equation problem [2] and the image restoration problem [16], can be reduced to the following Fredholm integral equation of the first kind

$$\int_a^b K(s, t)x(s)ds = y(t), \quad s \in [a, b], \quad (1)$$

where  $a, b$  is finite or infinite, and  $x(s)$  is unknown,  $K(s, t) \in C(a, b)$  is known as a kernel function. If  $K(s, t)$  is a continuous kernel function, (1) will be written as the linear operator form

$$Kx = y. \quad (2)$$

As we all know, (1) is ill-posed, that is to say, at least one of the existence, uniqueness and stability of the solution is not satisfied. Here it mainly refers to instability, that is, small perturbations in the data on the right hand side will lead to infinite variations in the solution. We consider the following example to illustrate.

#### Example 2.1.

$$\int_0^1 (1+ts)e^{ts}x(s)ds = y(t) = e^t, \quad 0 \leq t \leq 1.$$

This equation has the unique solution  $x(t) = 1$ . If we use the Simpson's rule to approximate the integral, and the step size  $h = \frac{1}{n}$ , then we can get the linear system of equations

$$\sum_{j=0}^n w_j (1+t_i t_j) e^{t_i t_j} x(t_j) = y(ih), \quad i = 0, 1, \dots, n,$$

where  $w$  denotes the corresponding weight vector. Table 1 presents the error about  $x(ih) - x_i$  in different nodal point.

TABLE 1. The error between numerical solution and true solution at different points

$t$	$n = 4$	$n = 8$	$n = 164$	$n = 32$
0	-0.0774	-0.1667	-4.9063	12
$\frac{1}{4}$	1.0765	-0.4535	-13.0625	-32
$\frac{1}{2}$	0.7730	-2.0363	16.5000	13
$\frac{3}{4}$	1.0749	-0.4393	-2	-12
1	0.9258	0.8341	-0.4063	19

From the above data, we can see that the error is not decreasing as the improvement of the calculation accuracy of the left integral term. It is dangerous to perform numerical calculations at this time. As we stated before, the error of the measure data is used to be inevitable, and we can't ignore the rounding errors about the computer. So it is difficult to obtain stable numerical solutions for such problems. Based on the above reasons, a stabled method must be adopted—regularization method.

## 2.2. Tikhonov method and Landweber iteration method

The traditional Tikhonov regularization [8] solves the following minimization problem

$$\min_x J_\alpha^\delta(x) := \|Kx - y^\delta\|^2 + \alpha\|x\|^2, \quad x \in X. \quad (3)$$

If the operator  $K : X \rightarrow Y$  is linear and bounded, the regularization  $\alpha > 0$ , then the unique minimum  $x^{\alpha,\delta}$  of  $J_\alpha^\delta$  is also the unique solution of the normal equation

$$(\alpha I + K^* K)x^{\alpha,\delta} = K^* y. \quad (4)$$

Let  $(\mu_j, x_j, y_j)$  be a singular system for  $K$ , then the solution of  $Kx = y$  is presented by

$$x = \sum_{j=1}^{\infty} \frac{1}{\mu_j} (y, y_j) x_j. \quad (5)$$

By the way, Tikhonov method gave a strategy

$$q(\alpha, \mu) = \frac{\mu^2}{\alpha + \mu^2} \quad (6)$$

to damp the factor  $\frac{1}{\mu_j}$  of (5). The function  $q(\alpha, \mu) : (0, \infty) \times (0, \|K\|] \rightarrow \mathbb{R}$  is called as a regularizing filter function. Based on these information, Tikhonov regularization strategy  $R_\alpha : Y \rightarrow X, \alpha > 0$  is defined by

$$R_\alpha y = \sum_{j=1}^{\infty} \frac{1}{\mu_j} \frac{\mu_j^2}{\alpha + \mu_j^2} (y, y_j) x_j, \quad y \in Y. \quad (7)$$

Another methodology to give a regularizing filter function is

$$q(\alpha, \mu) = 1 - (1 - a\mu^2)^{\frac{1}{\alpha}}, \quad (8)$$

if  $\frac{1}{\alpha} = m$ , and  $m$  represents the iterations, then (8) is the filter function of the Landweber iteration

$$x^0 := 0, \quad x^m = (I - aK^* K)x^{m-1} + aK^* y. \quad (9)$$

(6) and (8) are regularization filter functions, both of them satisfy the following definition.

**Definition 2.1.** [3] Let  $K : X \rightarrow Y$  be compact with singular system  $(\mu_j, x_j, y_j)$ ,  $\mu(K)$  be the closure of  $\bigcup_{j=1}^{\infty} \{\mu_j\}$ , and  $q : \mu(K) \subset (0, \mu_1) \rightarrow \mathbb{R}$  be a function with the following properties:

$$\sup_{\mu_j > 0} \left| \frac{q(\alpha, \mu_j)}{\mu_j} \right| = c(\alpha) < \infty, \quad (10a)$$

$$|q(\alpha, \mu_j)| \leq c < \infty, \quad c \text{ is independent of } \alpha, j, \quad (10b)$$

$$\lim_{\alpha \rightarrow 0} q(\alpha, \mu_j) = 1 \text{ pointwise in } \mu_j. \quad (10c)$$

Let  $R_\alpha : Y \rightarrow X$  be a family operators,  $\alpha > 0$ , which is defined by

$$R_\alpha y = \sum_{j=1}^{\infty} \frac{q(\alpha, \mu_j)}{\mu_j} (y, y_j) x_j, \quad y \in Y, \quad (11)$$

then it is a regularization strategy or a filter based regularization method with  $\|R_\alpha\| = c(\alpha)$ , and  $q(\alpha, \mu)$  is called a filter function.

In addition, regularization operators corresponding to (6) and (8) are optimal order under an a-priori assumption [4], or are optimal strategies in the sense of the worst-case error [8]. For the integrity of the article, we give the definition. Next to the definition, there is a sufficient theorem to realize the optimal convergence rate.

**Definition 2.2.** For given  $\sigma, E > 0$ , let

$$X_{\sigma,E} := \{x \in X \mid \exists z \in X, \|z\| \leq E, x = (K^*K)^{\frac{\sigma}{2}}z\} \subset X.$$

Define

$$\mathcal{F}(\delta, \sigma, R_\alpha) := \sup\{\|x - x^{\alpha, \delta}\| : x \in X_1, \|y - y^\delta\| \leq \delta\},$$

for any  $X_1 \subset X$  a subspace,  $\delta > 0$ , and for a regularization method  $R_\alpha$ , if

$$\mathcal{F}(\delta, \sigma, R_\alpha) \leq c\delta^{\frac{\sigma}{\sigma+1}}E^{\frac{1}{\sigma+1}}$$

holds, then a regularization method  $R_\alpha$  is called of optimal order under the a-priori assumption  $x \in X_{\sigma,E}$ . If  $E$  is unknown, then redefine a set

$$X_\sigma := \bigcup_{\sigma > 0} X_{\sigma,E},$$

and if

$$\mathcal{F}(\delta, \sigma, R_\alpha) \leq c\delta^{\frac{\sigma}{\sigma+1}},$$

holds, then we call a regularization method  $R_\alpha$  is of optimal order under the a-priori assumption  $x \in X_\sigma$ .

**Theorem 2.1.** [12] Let  $K : X \rightarrow Y$  be a linear compact operator,  $R_\alpha : Y \rightarrow X$  is a filter based regularization method, it will be of optimal order under the a-priori assumption  $x \in X_{\sigma,E}$ ,  $\sigma, E > 0$ ,

$$\sup_{0 < \mu \leq \mu_1} \left| \frac{q(\alpha, \mu)}{\mu} \right| \leq c\alpha^{-\gamma}, \quad (12a)$$

$$\sup_{0 < \mu \leq \mu_1} |(1 - q(\alpha, \mu))\mu^\sigma| \leq c_\sigma \alpha^{\gamma\sigma}, \quad (12b)$$

with the regularization parameter  $\alpha = \hat{c} \left( \frac{\delta}{E} \right)^{\frac{1}{\gamma(\sigma+1)}}, \hat{c} = \left( \frac{c}{\sigma c_\sigma} \right)^{\frac{1}{\gamma(\sigma+1)}} > 0$ .

### 3. Iterated Tikhonov regularization method

In this section, we first look back the standard Tikhonov iteration method, then introduce a new iterated Tikhonov regularization method, it is a generalization of the classical Tikhonov method.

The standard Tikhonov iteration method is

$$x^{0, \alpha, \delta} = 0, (\alpha I + K^*K)x^{m+1, \alpha, \delta} = K^*y^\delta + \alpha x^{m, \alpha, \delta}. \quad (13)$$

It can be shown that the corresponding regularization filter function is

$$q^m(\alpha, \mu) = 1 - \left( \frac{\alpha}{\alpha + \mu^2} \right)^m, m = 1, 2, \dots \quad (14)$$

In [4, 7], the authors introduced a Weighted-II Tikhonov method as the filter based method with the filter function

$$q_l(\alpha, \mu) = \frac{\mu^2}{\mu^2 + \alpha \left( 1 - \left( \frac{\mu}{\mu_1} \right)^2 \right)^l}, \quad (15)$$

for  $\alpha > 0$  and  $l \in \mathbb{N}$ . Here, we also recall a filter based method—the fractional Tikhonov method [3, 9] with filter function

$$q^r(\alpha, \mu) = \frac{\mu^{2r}}{(\alpha + \mu^2)^r}, \quad (16)$$

for  $\alpha > 0$  and  $r \geq \frac{1}{2}$ . Now, we can introduce a mixed method which combines the filter function of the fractional Tikhonov method and weighted-II Tikhonov method.

**Definition 3.1.** Fixing  $q_l^r(\alpha, \mu)$  such that

$$q_l^r(\alpha, \mu) := \frac{\mu^{2r}}{\left( \alpha \left( 1 - \left( \frac{\mu}{\mu_1} \right)^2 \right)^l + \mu^2 \right)^r}, \quad (17)$$

we define the mixed method (fractional weighted Tikhonov method) as the filter based method

$$R_{\alpha,l}^r y := \sum_{j=1}^{\infty} \frac{q_l^r(\alpha, \mu)}{\mu_j} (y, y_j) x_j. \quad (18)$$

It is clear that for  $l = 0$  and  $r = 1$ , it becomes the classical Tikhonov method.

**Theorem 3.1.** Let  $K : X \rightarrow Y$  be a linear compact operator with infinite dimensional range and  $R_{\alpha,l}^r$  be the corresponding family mixed method operator. Then for every given  $r \geq \frac{1}{2}$ ,  $l \in \mathbb{N}$ ,  $R_{\alpha,l}^r$  is a regularization method of optimal order under the a-priori assumption  $x \in X_{\sigma,E}$  with  $0 < \sigma \leq 2$ . Further, if the regularization parameter satisfies  $\alpha = (\frac{\delta}{E})^{\frac{2}{\sigma+1}}$ , then the best possible rate of convergence with respect to  $\delta$  is  $\|x - x^{\alpha, \delta, r}\| = \mathcal{O}(\delta^{\frac{2}{3}})$  with  $\sigma = 2$ . Moreover, if  $\|x - x^{\alpha, \delta}\| = \mathcal{O}(\alpha)$ , then  $x \in X_2$ .

*Proof.* It has been proved  $q_l(\alpha, \mu)$  is a filter function in [4], so  $q_l(\alpha, \mu)$  satisfies the filter function conditions (10a-10c). By Proposition 12 of [3], for  $r \geq \frac{1}{2}$ , the function  $q_l^r(\alpha, \mu)$  which meets the condition (10a-10c) can be verified easily. The proof of  $Q_l^r(\alpha, \mu)$  can meet the requirements (12a-12b) combining the proof of Proposition 12 in [3]. The difference is that  $Q_l^1(\alpha, \mu)$  is the weighted-II Tikhonov method, and it also has the optimal order  $\mathcal{O}(\delta^{\frac{2}{3}})$  with  $\gamma = \frac{1}{2}$  in (12b) for every  $0 < \sigma \leq 2$ .  $\square$

In the following, we will discuss the saturation for the mixed Tikhonov regularization.

**Theorem 3.2.** Let  $K : X \rightarrow Y$  be a linear compact operator with infinite dimensional range and let  $R_{\alpha,l}^r$  be the corresponding family of fractional Tikhonov regularization operators in Definition 3.1 with  $r \geq \frac{1}{2}$ ,  $l \in \mathbb{N}$ . Let  $\alpha = \alpha(\delta, y^\delta)$  be any parameter choice rule, and if

$$\sup \{ \|x - x^{\alpha, \delta, r}\| : \|P(y - y^\delta)\| \leq \delta \} = o\left(\delta^{\frac{2}{3}}\right), \quad (19)$$

then  $x = 0$  with  $P$  is the orthogonal projector onto  $\overline{R(K)}$ .

*Proof.* For  $r = 1$ , it is clear that the saturation result follows from weight-II Tikhonov regularization [3]. For  $r \neq 1$ , we have

$$x - x_r^{\alpha, \delta} = \sum_{j=1}^{\infty} \frac{1}{\mu_j} (1 - q_l^r(\alpha, \mu_j)) (y, y_j) x_j, \quad (20)$$

and

$$1 - q_l^r(\alpha, \mu) = \frac{\left( 1 + \frac{\mu^2}{\alpha \left( 1 - \left( \frac{\mu}{\mu_1} \right)^2 \right)^l} \right)^r - \left( \frac{\mu^2}{\alpha \left( 1 - \left( \frac{\mu}{\mu_1} \right)^2 \right)^l} \right)^r}{\left( 1 + \frac{\mu^2}{\alpha \left( 1 - \left( \frac{\mu}{\mu_1} \right)^2 \right)^l} \right)^{r-1}} \cdot (1 - q_l^1(\alpha, \mu)).$$

We notice that the above equality will be

$$1 - q_l^r(\alpha, \mu) = f\left(\frac{\mu^2}{a}\right) \cdot (1 - q_l^1(\alpha, \mu)),$$

where  $f(x) = \frac{(x+1)^r - x^r}{(x+1)^{r-1}}$ ,  $a = \alpha \left(1 - \left(\frac{\mu}{\mu_1}\right)^2\right)^l$  and  $q_l^1(\alpha, \mu) = q_l(\alpha, \mu)$  is the weighted-II Tikhonov.  $f(x)$  is a monotone function, it satisfies  $f(0) = 1$  and  $\lim_{x \rightarrow \infty} f(x) = r$ . Hence, we can get

$$\min\{1, r\} (1 - q_l^1(\alpha, \mu)) \leq (1 - q_l^r(\alpha, \mu)) \leq \max\{1, r\} (1 - q_l^1(\alpha, \mu)).$$

Naturally,

$$\sup\{\|x - x^{\alpha, \delta, r}\| : \|P(y - y^\delta)\| \leq \delta\} \geq \min\{1, r\} \cdot \sup\{\|x - x^{\alpha, \delta, 1}\| : \|P(y - y^\delta)\| \leq \delta\},$$

for every  $y^\delta$  satisfies  $\|y - y^\delta\| \leq \delta$ . From Proposition 3.6 in [3], we have

$$\sup\{\|x - x^{\alpha, \delta, 1}\| : \|P(y - y^\delta)\| \leq \delta\} = o(\delta^{\frac{2}{3}}),$$

Hence, the conclusion follows from the saturation result for Weighted-II Tikhonov (see Corollary 5.3 [3]).  $\square$

From now on, we propose a new iterated regularization method based on the above mixed Tikhonov. By iterations, we find that a large  $m$  will provide a faster convergence rate (see Theorem 3.3).

**Definition 3.2.** Define the iterated fractional weighted Tikhonov method as

$$\left(K^*K + \alpha \left(I - \frac{K^*K}{\|K^*K\|}\right)^l\right)^r x^{m, \alpha} = (K^*K)^{r-1} K^* y + \left[\left(K^*K + \alpha \left(I - \frac{K^*K}{\|K^*K\|}\right)^l\right)^r - (K^*K)^r\right] x^{m-1, \alpha} \quad (21)$$

with  $x^{0, \alpha} := 0$ ,  $r \geq \frac{1}{2}$ ,  $\alpha > 0$  and  $l \in \mathbb{N}$ . We can define  $x^{m, \alpha, \delta}$  as the  $m$ -th iteration of (21) whenever  $y$  is replaced by the noise data  $y^\delta$ .

In whole paper, for convenience, (21) will be called the new iterated Tikhonov method.

**Theorem 3.3.** The new iterated Tikhonov in (21) is a filter based regularization method with filter function

$$Q_l^{m, r}(\alpha, \mu) = 1 - (1 - Q_l^r(\alpha, \mu))^m, \quad (22)$$

with  $Q_l^r(\alpha, \mu) = q_l^r(\alpha, \mu) = \left(\frac{\mu^2}{\mu^2 + \alpha(1 - (\frac{\mu}{\mu_1})^2)^l}\right)^r$ . Moreover, this method is of optimal order under the a-priori assumption  $x \in X_{\sigma, E}$ , for  $l \in \mathbb{N}$  and  $0 \leq \sigma \leq 2m$ . Further, regularization parameter  $\alpha = \left(\frac{\delta}{E}\right)^{\frac{2m}{1+\sigma}}$ , yields the best convergence rate  $\|x - x^{m, \alpha, \delta}\| \leq \mathcal{O}(\delta^{\frac{2m}{2m+1}})$  with  $\sigma = 2m$ .

*Proof.* Denote  $C = \left(K^*K + \alpha \left(I - \frac{K^*K}{\|K^*K\|}\right)^l\right)^r$ ,  $B = C^{-1}(K^*K)^{r-1} K^* y$ , and  $A = C^{-1} [C - (K^*K)^r]$ .

By the iteration formulas (21), we have

$$\begin{aligned} x^{m, \alpha, \delta} &= Ax^{m-1, \alpha, \delta} + B = A^2 x^{m-2, \alpha, \delta} + (A^1 + A^0)B \\ &= \dots \\ &= \sum_{k=0}^{m-1} A^k B = \sum_{k=0}^{m-1} C^{-k} [C - (K^*K)^r]^k C^{-1} (K^*K)^{r-1} K^* y. \end{aligned}$$

Let  $R_\alpha^m = \sum_{k=0}^{m-1} C^{-k} [C - (K^*K)^r]^k C^{-1} (K^*K)^{r-1} K^*$ , and the singular system be  $\{\mu_j, x_j, y_j\}$ , then

$$R_\alpha^m y = \sum_{j=1}^{\infty} \frac{1}{\mu_j} Q_l^{m, r}(\alpha, \mu_j) (y, y_j) x_j,$$

and

$$Q_l^{m,r}(\alpha, \mu_j) = \sum_{k=0}^{m-1} \left( \frac{\left( \mu_j^2 + \alpha \left( 1 - \left( \frac{\mu_j}{\mu_1} \right)^2 \right)^l \right)^r - \mu_j^{2r}}{\left( \mu_j^2 + \alpha \left( 1 - \left( \frac{\mu_j}{\mu_1} \right)^2 \right)^l \right)^r} \right)^k \left( \frac{\mu_j^2}{\left( \mu_j^2 + \alpha \left( 1 - \left( \frac{\mu_j}{\mu_1} \right)^2 \right)^l \right)^r} \right)^r.$$

By the definition of  $q_l(\alpha, \mu)$ , then

$$Q_l^{m,r}(\alpha, \mu) = \sum_{k=0}^{m-1} (1 - (q_l(\alpha, \mu))^r)^k (q_l(\alpha, \mu))^r.$$

It is easily to get  $Q_l^{m,r}(\alpha, \mu) = 1 - (1 - (q_l(\alpha, \mu))^r)^m$ , that is the conclusion as we stated.

From the relationship between  $Q_l^{m,r}(\alpha, \mu)$  and  $Q_l^r(\alpha, \mu)$ , we can deduce

$$Q_l^r(\alpha, \mu) \leq Q_l^{m,r}(\alpha, \mu) \leq m Q_l^r(\alpha, \mu).$$

Clearly,  $q_l(\alpha, \mu)$  is weighted-II Tikhonov and it is a regularization filter method. Hence,  $Q_l^{m,r}(\alpha, \mu)$  satisfies the conditions (10a-10b) and (12a). At the same time, (10c) is easy to check, so it is a filter function naturally.  $Q_l^{m,r}(\alpha, \mu)$  adapt to the filter based regularization conditions. Finally, we make sure  $Q_l^{m,r}(\alpha, \mu)$  satisfies condition (12b) for the order optimality.

$$\begin{aligned} 1 - Q_l^{m,r}(\alpha, \mu) &= (1 - Q_l^r(\alpha, \mu))^m \leq 1 - Q_l^r(\alpha, \mu) \\ &\leq \max\{1, r\}^m (1 - Q_l^1(\alpha, \mu))^m = c \left( 1 - Q_l^{m,1}(\alpha, \mu) \right), \end{aligned}$$

and notice that  $Q_l^1(\alpha, \mu) = q_l(\alpha, \mu) = \frac{\mu^2}{\mu^2 + \alpha \left( 1 - \left( \frac{\mu}{\mu_1} \right)^2 \right)^l}$  is the weighted-II filter function and  $Q_l^{m,1}(\alpha, \mu) = 1 - \left( 1 - \frac{\mu^2}{\alpha \left( 1 - \left( \frac{\mu}{\mu_1} \right)^2 \right)^l + \mu^2} \right)^m$  is the filter function of the stationary iterated Tikhonov. So that condition (12b) follows from the properties of stationary Weighted-II iterated Tikhonov, and  $\gamma = \frac{1}{2}$ ,  $0 \leq \sigma \leq 2m$ , therefore, we get the best convergence rate  $\mathcal{O}(\delta^{\frac{2m}{2m+1}})$ .  $\square$

#### 4. Numerical experiments

The purpose of this section is to illustrate the validity from the previous sections with the following example. The classical iterative Tikhonov regularization method, Landweber method and the new iterated Tikhonov method are adopted to get the iterative numerical solutions.

Consider the following integral equation of the first kind:

$$\int_0^\infty e^{-st} x(t) dt = h(s), \quad 0 \leq t < \infty. \quad (23)$$

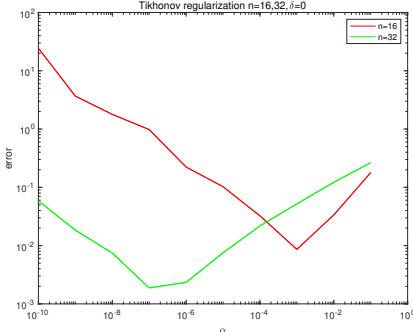
The kernel operator is given by  $(Kx)(t) = \int_0^\infty e^{-st} x(s) ds$ . For numerical computation, we use Gauss-Laguerre quadrature rule with  $n$  points to get the matrix  $A$  corresponding to the kernel. The measure data about the right-hand side function is denoted by  $y^\delta = y + \delta \|\eta\|$ , where  $\eta$  obeys the standard normal distribution, and the perturbation magnitude is  $\delta$ .

In this example, the right-hand side function  $h(s) = \frac{2}{2s+1}$ , hence (23) has the unique solution  $x(t) = e^{-\frac{t}{2}}$ . As mentioned above, we can use regularization method to solve the numerical solution. The operator  $K$  is self-adjoint, so discrete Tikhonov equation takes the following form

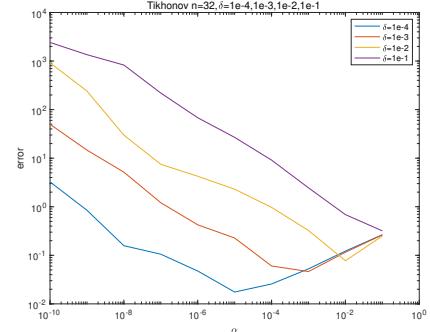
$$(\alpha I + A^2) x^{\alpha, \delta} = Ay^\delta. \quad (24)$$

#### 4.1. Classical method implementation

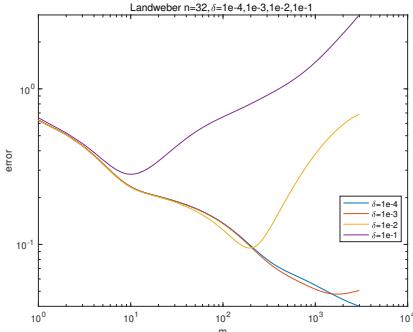
First, let the perturbation  $\delta = 0$ , that is only the discrete error by quadrature rule will be generated, choose different regularization parameter  $\alpha = 10^{-i}, i = 1, 2, \dots, 10$  by priori, and the quadrature points number  $n = 16, 32$ . The numerical discrete errors variation diagram  $|x - x^{\alpha, \delta}|_{l_2}$  are showed in Figure 1(a). From Figure 1(a), if  $\alpha$  is small, the error has a big difference between  $n = 16$  and  $n = 32$  as  $\alpha < 10^{-4}$  especially.



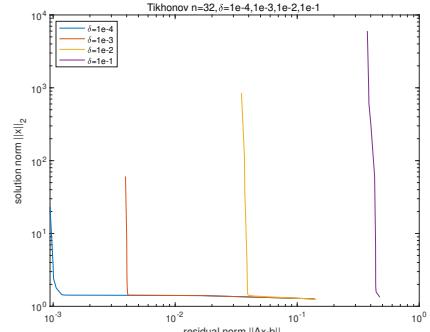
(a) Numerical error for different regularization parameter  $n$  in Tikhonov



(b) Numerical error for different perturbation parameter  $\delta$  in Tikhonov



(c) Numerical error for different perturbation parameter  $\delta$  in Landweber



(d) The relationship between residual norm and solution norm for different perturbation parameter  $\delta$  in Tikhonov

FIGURE 1. Numerical error for  $n$  and  $\delta$  in Tikhonov and Landweber method

Next, we consider the numerical error for different perturbation  $\delta$  in Tikhonov method (see Figure 1(b)), and in Landweber method with  $a = 0.5$  to solve (24) and iteration steps  $m = 1, 2, \dots, 3000$  (see Figure 1(c)). From the trends of the figures, they show that the numerical error first decrease then increase as  $\alpha$  or  $m$  increase, this coincides with the theory. Besides, we observe that both methods are comparable where precision is concerned.

Figure 1(d) presents the relationship of residual norm and solution norm, when the magnitude of perturbation  $\delta = 10^{-j}, j = 1, 2, 3, 4$  in Tikhonov method. As we can see, the small perturbation will have a small error with the same  $\alpha$  basically. Besides, it looks like a  $L$  curve, that is to say, there is a optimal  $\alpha$  keeping the solution norm and residual norm balance.

#### 4.2. New iterated Tikhonov implementation

Now we use the new iterated Tikhonov regularization method to solve (24). Let  $\alpha = 1e-3$ ,  $a = 0.5$ ,  $\delta = 1e-4$ ,  $l = 4$ , and  $n = 32$ , then we compare the total numerical error by using

classical Tikhonov method (13), Landweber iteration method (9) and the new iterated Tikhonov method (21) when iteration steps changes, see Figure 2. From Figure 2, we find that the new iterated

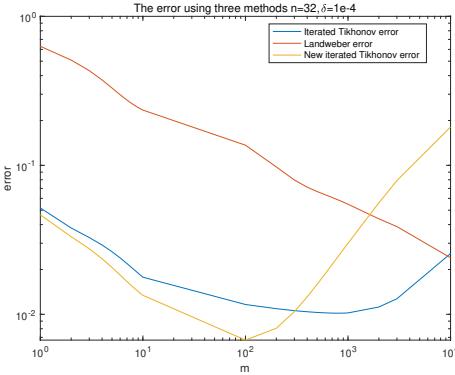


FIGURE 2. Iterated Tikhonov error, Landweber iteration error and New iterated Tikhonov error for different  $m$

Tikhonov method only need less iteration steps to get a smaller error than the other two methods for this problem under these parameters setting, which proves the validity of the proposed method. Finally, let  $l = 2, m = 100, r = 0.8, \alpha = 1e - 0.9 * 1e - 1, 1e - 3, 1e - 3$  for different perturbation  $\delta$ , the following Table 2 gives the auxiliary specification to prove the efficiency of the new iterated Tikhonov method.

TABLE 2. The numerical error for different  $\delta$  by three methods

	$\delta = 1e - 4$	$\delta = 1e - 3$	$\delta = 1e - 2$	$\delta = 1e - 1$
Iterated Tikhonov	0.2692	0.1035	0.0648	0.0051
Landweber	0.8654	0.1598	0.1336	0.1370
New iterated Tikhonov	0.2418	0.0734	0.0222	0.0046

## 5. Conclusion

This paper has shown the iterated fractional weight regularization method is an efficient method to solve the Fredholm integral equation of the first kind. The numerical experiments conducted have validated the accuracy of the proposed method and shown that the comparability with the classical Tikhonov method.

## Acknowledgements

We are grateful to the referee for his (or her) careful reviewing which substantially improve the paper. The work was funded by China Scholarship Council (Nos.202308130195, 202308130193), Science and Technology Project of Hebei Education Department No. QN2023030, Natural Science Foundation of Hebei Province No.A2023205045, and Science Foundation of Hebei Normal University (Nos. L2022B30, L2022B02, L2025ZD04).

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