

## KINEMATICS OF THE EPICYCLICAL MINUTEMAN COVER DRIVE

Stefan STAICU<sup>1</sup>

*Articolul stabilește relații matriceale pentru analiza cinematică a unui tren de roți dințate cu un grad de libertate. Mecanismul epiciclic Minuteman este un sistem cu patru elemente mobile și trei cuplaje prin roți dințate controlate de un motor electric. Cunoscând mișcarea de rotație a efectorului, problema de cinematică inversă este rezolvată printr-un procedeu bazat pe relații de conectivitate. În final, se obțin câteva grafice pentru unghiurile de rotație, vitezele unghiulare și accelerările unghiulare de la intrarea și ieșirea din sistemul mecanic.*

*Matrix relations for the kinematics analysis of a 1-DOF planetary gear train are established in the paper. The epicyclic mechanism of the Minuteman cover drive is a system with four moving links and three gear pairs controlled by one electric motor. Knowing the rotation motion of the effector, the inverse kinematics problem is solved based of the connectivity relations. Finally, some simulation graphs for the input and output angles of rotation, angular velocities and angular accelerations are obtained.*

**Keywords:** Connectivity relations; Gear train; Kinematics

### List of symbols

- $q_{k,k-1}$ : orthogonal relative transformation matrices from the frame  $x_{k-1}y_{k-1}z_{k-1}$  to following frame  $x_ky_kz_k$
- $\vec{u}_1, \vec{u}_2, \vec{u}_3$ : three orthogonal unit vectors
- $\alpha$ : rotation angle of the output link ;  $\varphi_{k,k-1}$ : relative rotation angle of  $T_k$  rigid body
- $\vec{\omega}_{k,k-1}$ : relative angular velocity of  $T_k$  ;  $\vec{\omega}_{k0}$ : absolute angular velocity of  $T_k$
- $\tilde{\omega}_{k,k-1}$ : skew symmetric matrix associated to the angular velocity  $\vec{\omega}_{k,k-1}$
- $\vec{\varepsilon}_{k,k-1}$ : relative angular acceleration of  $T_k$  ;  $\tilde{\varepsilon}_{k0}$ : absolute angular acceleration of  $T_k$
- $\tilde{\varepsilon}_{k,k-1}$ : skew symmetric matrix associated to the angular acceleration  $\vec{\varepsilon}_{k,k-1}$
- $\vec{r}_k^C$ : position vector of the mass centre of  $T_k$  rigid body

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<sup>1</sup> Professor, Department of Mechanics, University POLITEHNICA of Bucharest, ROMANIA, e-mail: [staicunstefan@yahoo.com](mailto:staicunstefan@yahoo.com)

## 1. Introduction

The industrial robots with orienting gear trains can perform several operations such as welding, flame cutting, spray painting, milling or assembling. Being comparatively simple and compact in size, the bevel-gear wrist mechanisms can be sealed in a metallic box that keeps the device out of contamination. Furthermore, using bevel gear trains for power transmission, the actuators can be mounted remotely on the forearm, thereby reducing the weight and inertia of a robot manipulator.

The epicyclic gear trains are incorporated in the structure of industrial robots and have two or three output rotations. Generally, these mechanical systems have conical and cylindrical toothed elements in their structure, while the input axes are parallel and the output axes are orthogonal. The three rotary orientation movements are usually performed around the axes of a Cartesian orthogonal frame, having its axes linked to the last arm of the robot's positioning mechanism.

Numerous methods for kinematics analysis of epicyclic gear trains have been proposed by several researchers. Planetary gear trains with three degrees of freedom are adopted as the design concept for robotic wrist (Hsieh and Sheu [1]; Paul and Stevenson [2]; Willis [3]; Ma and Gupta [4]; White [5]). The gear drives are commonly used for speed reduction and torque amplification in mechanical systems. If the advantage of the reduced bi-coupled transmissions, namely their reduced number of gears, is to be fully utilized, then their component transmissions must contain the smallest possible number of gears.

## 2. Inverse kinematics model

Recursive relations for kinematics of a 1-DOF orienting gear train are developed in this paper. The mechanism topology of the Minuteman cover drive consists of four moving links, four turning pairs and three gear pairs (Fig.1).

First, we wish to find the overall speed reduction ratio of this mechanism. A matrix methodology for the kinematics analysis based on the concept of fundamental circuit of an open-loop chain is presented. This method involves the identification of all open-loop chains and the derivation of the geometric relationships between the orientation of the output link and the joint angles of the chains, including the input actuator displacements [6], [7], [8].

Let  $O_0x_0y_0z_0(T_0)$  be a fixed Cartesian orthogonal frame, about which the mechanism moves. In the Minuteman cover drive, the first ring gear 0 is fixed at the ground, the sun gear 1a of radius  $r_1$  is the input link connected to link 0, while the moving ring gear 1b serves as output member.

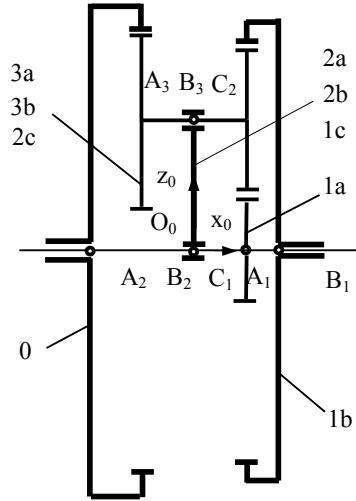


Fig. 1 The Minuteman cover drive

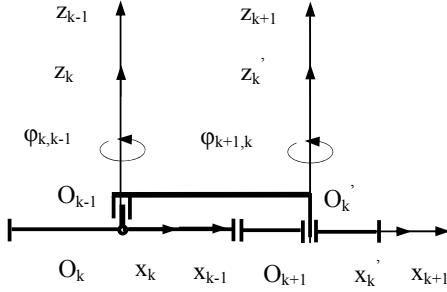


Fig. 2 Gear fundamental circuit

The compound planet gear  $3a = 3b = 2c$  meshes with sun gear  $1a$  as well as the two ring gears and is supported with a revolute joint by the carrier  $2a = 2b = 1c$  as a connected coupling shaft of  $h$  in height. The central body  $3a$  is adjacent to carrier  $2a$  and consists of two cylindrical gears of radius  $r_2$ ,  $r_3$ , respectively. Otherwise, the reduced bi-coupled transmission becomes a simply one-DOF compounding planetary gear train.

In what follows, we introduce a matrix approach which utilizes the *theory of fundamental circuits* developed by Tsai [6]. There exists a *real* or *fictitious* carrier for every gear pair in a planetary gear train and a fundamental matrix equation for each loop can be written:

$$q_{k+1,k-1} = q_{k+1,k}^\varphi q_{k,k-1}, \quad \varphi_{k,k-1} = n_{k+1,k-1} \varphi_{k+1,k}, \quad (q = a, b, c), \quad (1)$$

where  $\varphi_{k,k-1}$  and  $\varphi_{k+1,k}$  denote two successive relative angles of rotation of the carrier  $T_k$  and the planet gear  $T_{k+1}$ , respectively. The gear ratios of a gear pair is defined as

$$n_{k+1,k-1} = r_{k+1} / r_{k-1} = z_{k+1} / z_{k-1}, \quad (2)$$

where  $r_{k-1}$ ,  $r_{k+1}$  and  $z_{k-1}$ ,  $z_{k+1}$  are the radius and the number of teeth of two gears, respectively (Fig. 2).

We consider the rotation angles  $\varphi_{10}^A$  of the actuator  $A_1$  as single variable giving the instantaneous position of the mechanism (Fig. 3). Pursuing three serial kinematical circuits  $0-1a-2a-3a$ ,  $0-1b-2b-3b$ ,  $0-1c-2c$ , we obtain some successive matrices of transformation for relative parallel rotations [9], [10]:

$$\begin{aligned}
a_{10} &= a_{10}^\varphi \theta_1, a_{21} = a_{21}^\varphi \theta_2, a_{32} = a_{32}^\varphi \\
b_{10} &= b_{10}^\varphi \theta_1, b_{21} = b_{21}^\varphi, b_{32} = b_{32}^\varphi \theta_2 \\
c_{10} &= c_{10}^\varphi \theta_1, c_{21} = c_{21}^\varphi \theta_2,
\end{aligned} \tag{3}$$

where one denoted

$$\begin{aligned}
\theta_1 &= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \theta_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
q_{k,k-1}^\varphi &= \begin{bmatrix} \cos \varphi_{k,k-1}^i & \sin \varphi_{k,k-1}^i & 0 \\ -\sin \varphi_{k,k-1}^i & \cos \varphi_{k,k-1}^i & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (k = 1, 2, 3) \\
q_{k0} &= \prod_{s=1}^k q_{k-s+1,k-j}, \quad (q = a, b, c), \quad (i = A, B, C).
\end{aligned} \tag{4}$$

The upper index  $i$  corresponds successively to three circuits  $A, B, C$ . We note that the element  $3a = 3b = 2c$  constitute the common body of three concurrent kinematical chains.

Let us suppose that the absolute motion of the end-effector attached at the output link  $1b$  is a rotation expressed by the analytical function

$$\varphi_{10}^B = \alpha = \alpha_0 [1 - \cos(\frac{\pi}{6}t)]. \tag{5}$$

The value  $2\alpha_0$  is a parameter characterizing the final position of the end-effector.

Starting from absolute matrices  $a_{30} = a_{32}a_{21}a_{10}$ ,  $b_{30} = b_{32}b_{21}b_{10}$ ,  $c_{20} = c_{21}c_{10}$ , some *constraint rotation conditions* for the central planet gear  $3a = 3b = 2c$  are given by the following identities

$$a_{30} = b_{30} = c_{20}. \tag{6}$$

On the other hand, the constraint geometric conditions established along above three kinematical chains are expressed by matrix equations

$$a_{20}^T \vec{r}_{32}^A = b_{30}^T \vec{r}_{32}^B = c_{20}^T \vec{r}_{21}^C, \tag{7}$$

where, for example, one denoted

$$\begin{aligned}
\vec{u}_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\vec{r}_{32}^A &= -\vec{r}_{32}^B = -\vec{r}_{21}^C = h \vec{u}_1, \quad h = r_1 + r_2.
\end{aligned} \tag{8}$$

From the equations (6), (7), we obtain easily the real-time evolution of all characteristic joint angles, as follows:

$$\begin{aligned}\varphi_{10}^B &= \alpha, \quad \varphi_{32}^A = \varphi_{32}^B = \varphi_{21}^C = \frac{\alpha}{n_3 - n_2}, \quad \varphi_{21}^B = n_2 \varphi_{21}^C \\ \varphi_{10}^A &= (n_1 + n_3) \varphi_{21}^C, \quad \varphi_{21}^A = n_1 \varphi_{21}^C, \quad \varphi_{10}^C = n_3 \varphi_{21}^C \\ n_1 &= \frac{r_2}{r_1}, \quad n_2 = \frac{r_2}{r_1 + 2r_2}, \quad n_3 = \frac{r_3}{r_1 + r_2 + r_3}.\end{aligned}\quad (9)$$

In the design of power transmission mechanisms such as speed reducers or automotive transmissions, it is necessary to analyze the speed ratios between their input and output members and, sometimes, angular velocities or angular accelerations of all intermediate members.

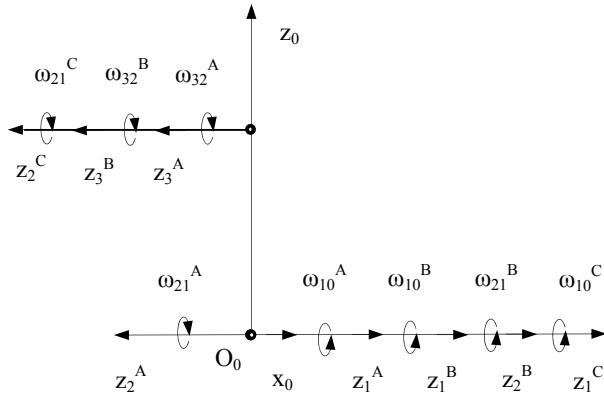


Fig. 3 Kinematical scheme of the mechanism

The analysis of the kinematics of bevel-gear wrist mechanisms of gyroscopic structure, for example, is very complex due to the fact that the carriers and planet gears may possess simultaneous angular velocities about nonparallel axes. The conventional tabular or analytical method, which concentrates on planar epicyclic gear trains, is no longer applicable. To overcome this difficulty, Freudenstein, Longman and Chen [11] applied the dual relative velocity and dual matrix of transformation for the analysis of epicyclic bevel-gear trains. The most straightforward approaches make use of the theory of fundamental circuits introduced by Freudenstein and Yang [12]. Tsai, Chen and Lin [13], Chang and Tsai [14] and Hedman [15] showed that the kinematical analysis of geared robotic mechanisms can be accomplished by applying these systematic methods [16], [17], [18].

Since a kinematical chain is an assemblage of links and joints, these can be symbolized in a more abstract form known as equivalent graph representation (Fig. 4). So, we use the *associated graph* to represent the topology of the

mechanism. From this equivalent graph, the fundamental circuits can be easily identified.

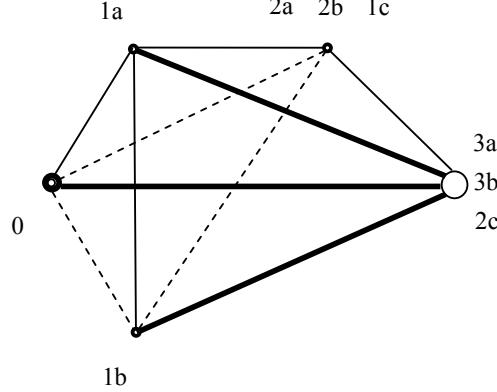


Fig. 4 Associated graph of the mechanism

In the kinematical graph representation we denote the links by vertices and the joints by edges (Yan and Hsieh [19], [20]). Two small concentric circles label the vertex denoting the fixed link 0. To distinguish the difference between the pairs connections, the gear pairs  $1a - 3a$ ,  $1b - 3b$ ,  $0 - 2c$  are designed by thick edges and the revolute joints  $0 - 1a$ ,  $1a - 2a$ ,  $2a - 3a$ ,  $1a - 1b$  by thin edges. Four edged paths, which start from the base link 0 and end at the central link  $3a = 3b = 2c$  consist of following vertices  $1a, 2a, 3a, 1b, 2b, 3b$  and  $1c, 2c$ . There are three independent loops, three fundamental circuits  $(1a, 2a, 3a), (1b, 2b, 3b), (0, 1c, 2c)$  and we identify two *fictitious* carriers.

The kinematics of an element for each circuit is characterized by skew-symmetric matrices given by the recursive relations [21], [22], [23]:

$$\tilde{\omega}_{k0}^i = q_{k,k-1} \tilde{\omega}_{k-1,0}^i q_{k,k-1}^T + \tilde{\omega}_{k,k-1}^i, \quad \tilde{\omega}_{k,k-1}^i = \dot{\phi}_{k,k-1}^i \tilde{u}_3, \quad (10)$$

where  $\tilde{u}_3$  is a skew-symmetric matrix associated with the unit vector  $\vec{u}_3$ . These matrices are *associated* to the angular velocities

$$\bar{\omega}_{k0}^i = q_{k,k-1} \bar{\omega}_{k-1,0}^i + \bar{\omega}_{k,k-1}^i, \quad \bar{\omega}_{k,k-1}^i = \dot{\phi}_{k,k-1}^i \vec{u}_3. \quad (11)$$

Knowing the rotation motion of the output link  $1b$  by the relations (5), one develops the inverse kinematical problem and determines the velocities  $\vec{v}_{k0}^i, \bar{\omega}_{k0}^i$  and the accelerations  $\vec{\gamma}_{k0}^i, \vec{\varepsilon}_{k0}^i$  of each of the moving links.

Based on the important remark characterising the gear ratio

$$\omega_{k,k-1} = n_{k+1,k-1} \omega_{k+1,k}, \quad (12)$$

the derivatives with respect to time of the relations (9) lead to the relative angular velocities of all links as function of the angular velocity  $\dot{\phi}_{10}^B = \dot{\alpha}$  of the output gear:

$$\omega_{k,k-1}^i = \dot{\phi}_{k,k-1}^i, \quad (i = A, B, C) \quad (k = 1, 2, 3). \quad (13)$$

Concerning the relative angular accelerations of the compounding elements of the mechanism, these are immediately given by deriving the relations on the velocities (13):  $\varepsilon_{k,k-1}^i = \dot{\omega}_{k,k-1}^i$ .

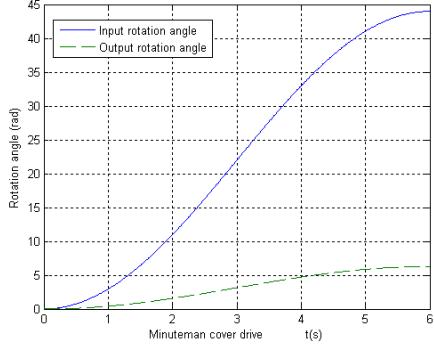


Fig. 5 Input and output rotation angles  $\phi_{10}^A, \phi_{10}^B$

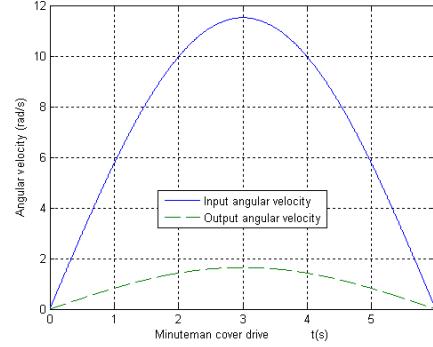


Fig. 6 Input and output angular velocities  $\omega_{10}^A, \omega_{10}^B$

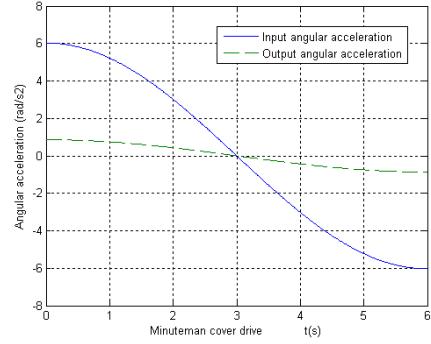


Fig. 7 Input and output angular accelerations  $\varepsilon_{10}^A, \varepsilon_{10}^B$

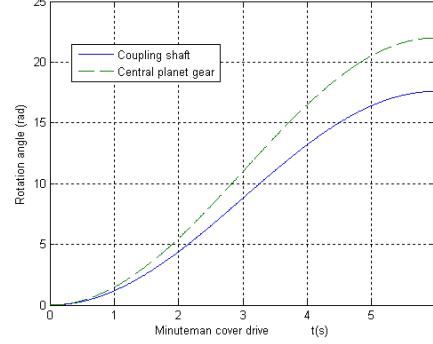


Fig. 8 Absolute angles of rotation  $\phi_{10}^C, \phi_{20}^C$

The angular accelerations  $\tilde{\varepsilon}_{k0}^i$  and the useful square matrices  $\tilde{\omega}_{k0}^i \tilde{\omega}_{k0}^i + \tilde{\varepsilon}_{k0}^i$  are calculated with the following recursive formulae [24], [25], [26]:

$$\begin{aligned} \tilde{\varepsilon}_{k0}^i &= q_{k,k-1} \tilde{\varepsilon}_{k-1,0}^i + \varepsilon_{k,k-1}^i \tilde{u}_3 + \omega_{k,k-1}^i q_{k,k-1} \tilde{\omega}_{k-1,0}^i q_{k,k-1}^T \tilde{u}_3 \\ \tilde{\omega}_{k0}^i \tilde{\omega}_{k0}^i + \tilde{\varepsilon}_{k0}^i &= q_{k,k-1} \left( \tilde{\omega}_{k-1,0}^i \tilde{\omega}_{k-1,0}^i + \tilde{\varepsilon}_{k-1,0}^i \right) q_{k,k-1}^T + \\ &+ \omega_{k,k-1}^i \omega_{k,k-1}^i \tilde{u}_3 \tilde{u}_3 + \varepsilon_{k,k-1}^i \tilde{u}_3 + 2\omega_{k,k-1}^i q_{k,k-1} \tilde{\omega}_{k-1,0}^i q_{k,k-1}^T \tilde{u}_3. \end{aligned} \quad (14)$$

For simulation purposes let us consider a mechanism which has the following characteristics

$$r_1 = 0.03 \text{ m}, \quad r_2 = 0.02 \text{ m}, \quad r_3 = 0.04 \text{ m}, \quad \alpha_0 = \pi, \quad h = d = 0.05 \text{ m}, \quad \Delta t = 6 \text{ s}.$$

To solve the inverse kinematics of the epicyclic gear train, a program which implements the suggested algorithm is developed in MATLAB. For illustration, it is assumed that for a period of six second the end-effector starts at rest from its initial position and is moving in a known rotation motion.

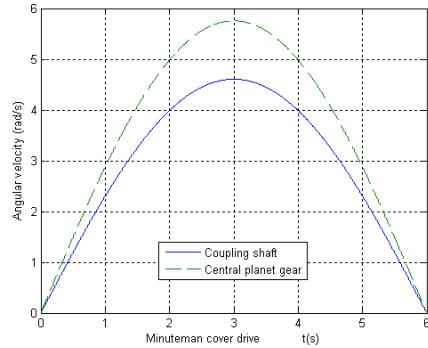


Fig. 9 Absolute angular velocities  $\omega_{10}^C, \omega_{20}^C$

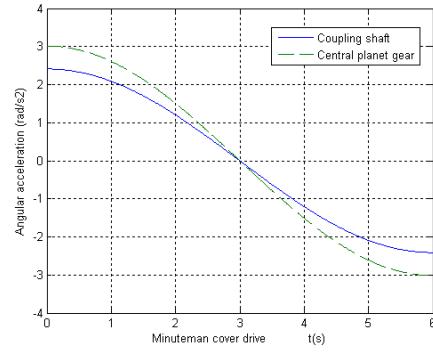


Fig. 10 Absolute angular accelerations  $\varepsilon_{10}^C, \varepsilon_{20}^C$

A numerical study of the kinematics is carried out by computation of the input and output angles of rotation  $\varphi_{10}^A, \varphi_{10}^B$  (Fig. 5), angular velocities  $\omega_{10}^A, \omega_{10}^B$  (Fig. 6) and angular accelerations  $\varepsilon_{10}^A, \varepsilon_{10}^B$  (Fig. 7). We add, also, the time-history evolution of absolute angles of rotation  $\varphi_{10}^C, \varphi_{20}^C$  (Fig. 8), angular velocities  $\omega_{10}^C, \omega_{20}^C$  (Fig. 9) and angular accelerations  $\varepsilon_{10}^C, \varepsilon_{20}^C$  (Fig. 10) of the coupling shaft 1c and the central gear 2c, respectively.

## 1 Conclusions

Within the inverse kinematics analysis, some exact matrix relations giving the position, velocity and acceleration of each link for a 1-DOF epicyclic gear train have been established.

Based on the matrix relations of connectivity, the new approach described above is very efficient and establishes a direct recursive determination of the variation in real-time of input and output angles of rotation, angular velocities and angular accelerations.

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