

## ON EFFICIENCY CONDITIONS FOR NEW CONSTRAINED MINIMUM PROBLEM

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*Considerăm problema (MP) de minimizare a unui vector de funcționale integrale curbilinii restrictionate cu EDP și/sau IDP. În două lucrări anterioare [5], [6], am studiat condiții necesare de eficiență pentru problema (MP) și am introdus un nou tip de dualitate. Scopul acestei lucrări este de a introduce și studia condiții suficiente de eficiență a unei soluții realizabile a problemei (MP). Rezultatele prezentate în §2 sunt originale.*

*Consider the problem (MP) of minimizing a vector of functionals of curvilinear integrals subject to PDE and/or PDI constraints. In two previous papers [5], [6], we studied efficiency necessary conditions for the problem (MP) and we introduced a new type of duality. The aim of this work is to introduce and study sufficient conditions for the efficiency of a feasible solution of the problem (MP). The results discussed in §2 are new.*

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### 1. Introduction and preliminaries

Let  $(T, h)$  and  $(M, g)$  be Riemannian manifolds of dimensions  $p$  and  $n$ , respectively. Denote by  $t = (t^\alpha)$  and  $x = (x^i)$  the local coordinates on  $T$  and  $M$ , respectively. Let  $J^1(T, M)$  be the first order jet bundle associated to  $T$  and  $M$ .

Using the product order relation on  $\mathbb{R}^p$ , the hyperparallelepiped  $\Omega_{t_0, t_1}$ , in  $\mathbb{R}^p$ , with the diagonal opposite points  $t_0 = (t_0^1, \dots, t_0^p)$  and  $t_1 = (t_1^1, \dots, t_1^p)$ , can be written as being the interval  $[t_0, t_1]$ . Suppose  $\gamma_{t_0, t_1}$  is a piecewise  $C^1$ -class curve joining the points  $t_0$  and  $t_1$ .

The closed Lagrange 1-form densities of  $C^\infty$ -class

$$f_\alpha = (f_\alpha^\ell): J^1(T, M) \rightarrow \mathbb{R}^r, \quad \ell = \overline{1, r}, \quad \alpha = \overline{1, p},$$

determine the following path independent functionals

$$F^\ell(x(\cdot)) = \int_{\gamma_{t_0, t_1}} f_\alpha^\ell(t, x(t), x_\gamma(t)) dt^\alpha$$

where  $x_\gamma(t) = \frac{\partial x}{\partial t^\gamma}(t)$ ,  $\gamma = \overline{1, p}$  are partial velocities.

The closeness conditions (complete integrability conditions) are  $D_\beta f_\alpha^\ell = D_\alpha f_\beta^\ell$ ,  $\alpha, \beta = \overline{1, p}$ ,  $\alpha \neq \beta$ ,  $\ell = \overline{1, r}$ , where  $D_\beta$  is the total derivative.

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We accept that the Lagrange matrices densities

$$\begin{aligned} g &= (g_a^b): J^1(T, M) \rightarrow \mathbb{R}^{ms}, \quad a = \overline{1, s}, \quad b = \overline{1, m}, \quad m < n, \\ h &= (h_a^b): J^1(T, M) \rightarrow \mathbb{R}^{qs}, \quad a = \overline{1, s}, \quad b = \overline{1, q}, \quad q < n, \end{aligned}$$

of  $C^\infty$ -class, define the partial differential inequations (PDI) (of evolution)

$$g(t, x(t), x_\gamma(t)) \leqq 0, \quad t \in \Omega_{t_0, t_1}, \quad (1)$$

and the partial differential equations (PDE) (of evolution)

$$h(t, x(t), x_\gamma(t)) = 0, \quad t \in \Omega_{t_0, t_1}. \quad (2)$$

On the set  $C^\infty(\Omega_{t_0, t_1}, M)$  of all functions  $x: \Omega_{t_0, t_1} \rightarrow M$  of  $C^\infty$ -class, we set the norm

$$\|x\| = \|x\|_\infty + \sum_{\alpha=1}^p \|x_\alpha\|_\infty.$$

We consider the vector of functionals

$$F(x(\cdot)) = \int_{\gamma_{t_0, t_1}} f_\alpha(t, x(t), x_\gamma(t)) dt^\alpha = (F^1(x(\cdot)), \dots, F^r(x(\cdot)))$$

and we would like to obtain sufficient efficiency conditions for problem (MP),

$$(MP) \quad \left\{ \begin{array}{l} \min_{x(\cdot)} F(x(\cdot)) \\ \text{subject to} \\ x(t_0) = x_0, \quad x(t_1) = x_1, \\ g(t, x(t), x_\gamma(t)) \leqq 0, \quad t \in \Omega_{t_0, t_1}, \\ h(t, x(t), x_\gamma(t)) = 0, \quad t \in \Omega_{t_0, t_1}, \end{array} \right.$$

a PDI and/or PDE-constrained minimum problem.

Throughout our work, we denote by

$$\begin{aligned} \mathcal{F}(\Omega_{t_0, t_1}) &= \{x \in C^\infty(\Omega_{t_0, t_1}, M) \mid x(t_0) = x_0, \quad x(t_1) = x_1, \quad g(t, x(t), x_\gamma(t)) \leqq 0, \\ &\quad h(t, x(t), x_\gamma(t)) = 0, \quad t \in \Omega_{t_0, t_1}\} \end{aligned}$$

the *set of all feasible solutions* of the problem (MP).

In their works [5], [6], Ariana Pitea, C. Udriște and Șt. Mititelu introduced and studied the multi-time multi-objective variational problem (MP) of minimizing a vector of path independent curvilinear functionals. More exactly, they gave necessary conditions for the efficiency of a feasible solution of the problem (MP) and studied new types of dualities.

In our work, some sufficient efficiency conditions for the problem (MP) are given. To develop our results, we need the following background [5].

**Definition 1.1.** A feasible solution  $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0, t_1})$  is called efficient point for the program (MP) if and only if for any feasible solution  $x(\cdot) \in \mathcal{F}(\Omega_{t_0, t_1})$ , the inequality  $F(x(\cdot)) \leqq F(x^\circ(\cdot))$  implies the equality  $F(x(\cdot)) = F(x^\circ(\cdot))$ .

**Definition 1.2.** Let  $x^\circ$  be an optimal solution of the problem (MP). Suppose there are in  $\mathbb{R}^r$  the vector  $\lambda^\circ$  having all components nonnegative but at least one positive

and the smooth matrix functions  $\mu: \Omega_{t_0, t_1} \rightarrow \mathbb{R}^{msp}$  and  $\nu: \Omega_{t_0, t_1} \rightarrow \mathbb{R}^{qsp}$  such that

$$\begin{aligned} & <\lambda^\circ, \frac{\partial f}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t))> + <\mu_\alpha^\circ(t), \frac{\partial g}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t))> \\ & + <\nu_\alpha^\circ(t), \frac{\partial h}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t))> - D\gamma \left( <\lambda^\circ, \frac{\partial f}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t))> \right. \\ & \left. + <\mu_\alpha^\circ(t), \frac{\partial g}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t))> + <\nu_\alpha^\circ(t), \frac{\partial h}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t))> \right) = 0, \\ & t \in \Omega_{t_0, t_1}, \alpha = \overline{1, p} \quad (\text{Euler-Lagrange PDEs}). \end{aligned}$$

Then  $x^\circ(\cdot)$  is called a normal optimal solution of problem (MP).

**Theorem 1.1.** *Let  $x^\circ(\cdot)$  be a point from  $\mathcal{F}(\Omega_{t_0, t_1})$ . If  $x^\circ(\cdot)$  is a normal efficient solution of the problem (MP), then there exist a vector  $\lambda^\circ \in \mathbb{R}^r$  and the smooth matrix functions  $\mu(t) = (\mu_\alpha^\circ(t))$ ,  $\nu(t) = (\nu_\alpha^\circ(t))$ , which satisfy the following conditions*

$$(MV) \left\{ \begin{aligned} & <\lambda^\circ, \frac{\partial f}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t))> + <\mu_\alpha^\circ(t), \frac{\partial g}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t))> \\ & + <\nu_\alpha^\circ(t), \frac{\partial h}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t))> - D\gamma \left( <\lambda^\circ, \frac{\partial f}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t))> \right. \\ & \left. + <\mu_\alpha^\circ(t), \frac{\partial g}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t))> + <\nu_\alpha^\circ(t), \frac{\partial h}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t))> \right) = 0, \\ & t \in \Omega_{t_0, t_1}, \alpha = \overline{1, p} \quad (\text{Euler-Lagrange PDEs}) \\ & <\mu_\alpha^\circ(t), g(t, x^\circ(t), x_\gamma^\circ(t))> = 0, \quad t \in \Omega_{t_0, t_1}, \alpha = \overline{1, p}, \\ & \mu_\alpha^\circ(t) \geq 0, \quad t \in \Omega_{t_0, t_1}, \alpha = \overline{1, p}, \\ & \lambda^\circ \geq 0, \\ & <e, \lambda^\circ> = 1, \quad e = (1, \dots, 1) \in \mathbb{R}^r. \end{aligned} \right.$$

## 2. Sufficient conditions of efficiency

In this section, we present certain sufficient conditions of efficiency for the problem (MP), using the  $(\rho, b)$ -quasiinvexity.

Let  $\rho$  be a real number,  $b: C^\infty(\Omega_{t_0, t_1}, M) \times C^\infty(\Omega_{t_0, t_1}, M) \rightarrow [0, \infty)$  a functional, and  $a = (a_\alpha)$ ,  $\alpha = \overline{1, p}$ , a closed 1-form. To  $a$  we associate the curvilinear integral

$$A(x(\cdot)) = \int_{\gamma_{t_0, t_1}} a_\alpha(t, x(t), x_\gamma(t)) dt^\alpha.$$

**Definition 2.1.** The functional  $A$  is called *[strictly]  $(\rho, b)$ -quasiinvex at the point  $x^\circ(\cdot)$*  if there exists a vector function  $\eta: J^1(\Omega_{t_0, t_1}, M) \times J^1(\Omega_{t_0, t_1}, M) \rightarrow \mathbb{R}^n$ , vanishing at the point  $(t, x^\circ(t), x_\gamma^\circ(t), x^\circ(t), x_\gamma^\circ(t))$ , and the functional  $\theta$  defined on the domain  $C^\infty(\Omega_{t_0, t_1}, M) \times C^\infty(\Omega_{t_0, t_1}, M)$  to  $\mathbb{R}^n$ , such that for any  $x(\cdot)$  [ $x(\cdot) \neq x^\circ(\cdot)$ ],

the following implication holds

$$\begin{aligned}
 (A(x(\cdot)) \leqq A(x^\circ(\cdot))) \Rightarrow & \left( b(x(\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0, t_1}} \{ < \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)) , \right. \\
 & \left. \frac{\partial a_\alpha}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) \} dt^\alpha > + < D_\gamma \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), \right. \\
 & \left. \frac{\partial a_\alpha}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \} dt^\alpha [ < \right. \leqq -\rho b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2 \left. \right) .
 \end{aligned}$$

The next theorem is the main result of this work.

**Theorem 2.1.** *Let us consider the feasible solution  $x^\circ(\cdot)$ , the vector  $\lambda^\circ$  and the functions  $\mu^\circ(\cdot)$  and  $\nu^\circ(\cdot)$  from Theorem 1.1, satisfying the relations (MV).*

*Suppose that the following conditions are satisfied:*

- a) *for each  $\ell = \overline{1, r}$ , the functional  $F^\ell(x(\cdot)) = \int_{\gamma_{t_0, t_1}} f_\alpha^\ell(t, x(t), x_\gamma(t)) dt^\alpha$  is  $(\rho_1^\ell, b)$ -quasiinvex at the point  $x^\circ(\cdot)$  with respect to  $\eta$  and  $\theta$ ;*
- b) *the functional  $\int_{\gamma_{t_0, t_1}} < \mu_\alpha^\circ(t), g(t, x(t), x_\gamma(t)) \} dt^\alpha$  is  $(\rho_2, b)$  -quasiinvex at the point  $x^\circ(\cdot)$  with respect to  $\eta$  and  $\theta$ ;*
- c) *the functional  $\int_{\gamma_{t_0, t_1}} < \nu_\alpha^\circ(t), h(t, x(t), x_\gamma(t)) \} dt^\alpha$  is  $(\rho_3, b)$ -quasiinvex at the point  $x^\circ(\cdot)$  with respect to  $\eta$  and  $\theta$ ;*
- d) *one of the integrals of a)-c) is  $(\rho_1^\ell, b)$ ,  $(\rho_2, b)$  or  $(\rho_3, b)$ -strictly quasiinvex at the point  $x^\circ(\cdot)$ ;*
- e)  $\lambda_\ell^\circ \rho_1^\ell + \rho_2 + \rho_3 \geq 0$ .

*Then the point  $x^\circ(\cdot)$  is an efficient solution of the problem (MP).*

*Proof.* Let us suppose that the point  $x^\circ(\cdot)$  is not an efficient solution for the problem (MP). Then, there is a feasible solution  $x(\cdot)$  for the problem (MP), such that for each  $\ell = \overline{1, r}$ ,  $F^\ell(x(\cdot)) \leq F^\ell(x^\circ(\cdot))$ , the case  $x(\cdot) = x^\circ(\cdot)$  being excluded.

According to condition a), it follows

$$\begin{aligned}
 & b(x(\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0, t_1}} \left[ < \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), \frac{\partial f_\alpha^\ell}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) \} \right. \\
 & \left. + < D_\gamma \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), \frac{\partial f_\alpha^\ell}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \} \right] dt^\alpha \\
 & \leqq -\rho_1^\ell b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2, \quad \ell = \overline{1, r}.
 \end{aligned}$$

Multiplying each inequality by  $\lambda_\ell^\circ$ ,  $\ell = \overline{1, r}$  and summing from  $\ell = 1$  to  $r$ , we obtain

$$\begin{aligned}
& b(x(\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0, t_1}} \left[ <\eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), <\lambda^\circ, \frac{\partial f_\alpha}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) >> \right. \\
& \quad \left. + < D_\gamma \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), <\lambda^\circ, \frac{\partial f_\alpha}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) >> \right] dt^\alpha \\
& \leq -\lambda_\ell^\circ \rho_1^\ell b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2. \tag{3}
\end{aligned}$$

By applying the property b), the following relation

$$\int_{\gamma_{t_0, t_1}} <\mu_\alpha^\circ(t), g(t, x(t), x_\gamma(t)) > dt^\alpha \leq \int_{\gamma_{t_0, t_1}} <\mu_\alpha^\circ(t), g(t, x^\circ(t), x_\gamma^\circ(t)) > dt^\alpha$$

leads us to

$$\begin{aligned}
& b(x(\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0, t_1}} \left[ <\eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), <\mu_\alpha^\circ(t), \frac{\partial g}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) >> \right. \\
& \quad \left. + < D_\gamma \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), <\mu_\alpha^\circ(t), \frac{\partial g}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) >> \right] dt^\alpha \\
& \leq -\rho_2 b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2. \tag{4}
\end{aligned}$$

Taking into account the condition c), the equality

$$\int_{\gamma_{t_0, t_1}} <\nu_\alpha^\circ(t), h(t, x(t), x_\gamma(t)) > dt^\alpha = \int_{\gamma_{t_0, t_1}} <\nu_\alpha^\circ(t), h(t, x^\circ(t), x_\gamma^\circ(t)) > dt^\alpha$$

implies

$$\begin{aligned}
& b(x(\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0, t_1}} \left[ <\eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), <\nu_\alpha^\circ(t), \frac{\partial h}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) >> \right. \\
& \quad \left. + < D_\gamma \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), <\nu_\alpha^\circ(t), \frac{\partial h}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) >> \right] dt^\alpha \\
& \leq -\rho_3 b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2. \tag{5}
\end{aligned}$$

Summing side by side the relations (3), (4), (5) and using the condition d), it follows

$$\begin{aligned}
& b(x(\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0, t_1}} \left\langle \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), \left\langle \lambda^\circ, \frac{\partial f_\alpha}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \right\rangle \\
& + \left\langle \mu_\alpha^\circ(t), \frac{\partial g}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle + \left\langle \nu_\alpha^\circ(t), \frac{\partial h}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \gg dt^\alpha \\
& + b(x(\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0, t_1}} \left\langle D_\gamma \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), \right. \\
& \left. \left\langle \lambda^\circ, \frac{\partial f_\alpha}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \right\rangle + \left\langle \mu_\alpha^\circ(t), \frac{\partial g}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \\
& + \left\langle \nu_\alpha^\circ(t), \frac{\partial h}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \gg dt^\alpha \\
& < - \left( \lambda_\ell^\circ \rho_1^\ell + \rho_2 + \rho_3 \right) b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2.
\end{aligned}$$

This inequality implies that  $b(x(\cdot), x^\circ(\cdot)) > 0$ , and we obtain

$$\begin{aligned}
& \int_{\gamma_{t_0, t_1}} \left\langle \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), \left\langle \lambda^\circ, \frac{\partial f_\alpha}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \right\rangle \\
& + \left\langle \mu_\alpha^\circ(t), \frac{\partial g}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle + \left\langle \nu_\alpha^\circ(t), \frac{\partial h}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \gg dt^\alpha \\
& + \int_{\gamma_{t_0, t_1}} \left\langle D_\gamma \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), \left\langle \lambda^\circ, \frac{\partial f_\alpha}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \right\rangle \\
& + \left\langle \mu_\alpha^\circ(t), \frac{\partial g}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle + \left\langle \nu_\alpha^\circ(t), \frac{\partial h}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \gg dt^\alpha \\
& < - \left( \lambda_\ell^\circ \rho_1^\ell + \rho_2 + \rho_3 \right) \|\theta(x(\cdot), x^\circ(\cdot))\|^2,
\end{aligned}$$

that is

$$\begin{aligned}
& \int_{\gamma_{t_0, t_1}} \left\langle \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), \left\langle \lambda^\circ, \frac{\partial f_\alpha}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \right\rangle \\
& + \left\langle \mu_\alpha^\circ(t), \frac{\partial g}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle + \left\langle \nu_\alpha^\circ(t), \frac{\partial h}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle > dt^\alpha \\
& + \int_{\gamma_{t_0, t_1}} D_\gamma \left( \left\langle \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), \left\langle \lambda^\circ, \frac{\partial f_\alpha}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \right\rangle \right. \\
& \left. + \left\langle \mu_\alpha^\circ(t), \frac{\partial g}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle + \left\langle \nu_\alpha^\circ(t), \frac{\partial h}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \right) dt^\alpha \\
& - \int_{\gamma_{t_0, t_1}} \left\langle \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), D_\gamma \left( \left\langle \lambda^\circ, \frac{\partial f_\alpha}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \right. \right. \\
& \left. \left. + \left\langle \mu_\alpha^\circ(t), \frac{\partial g}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle + \left\langle \nu_\alpha^\circ(t), \frac{\partial h}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \right) \right\rangle dt^\alpha \\
& < - \left( \lambda_\ell^\circ \rho_1^\ell + \rho_2 + \rho_3 \right) \|\theta(x(\cdot), x^\circ(\cdot))\|^2.
\end{aligned}$$

Taking into account one of the conditions (MV), the previous inequality becomes

$$\begin{aligned}
& \int_{\gamma_{t_0, t_1}} D_\gamma \left( \left\langle \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)), \left\langle \lambda^\circ, \frac{\partial f_\alpha}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \right\rangle \right. \\
& \left. + \left\langle \mu_\alpha^\circ(t), \frac{\partial g}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle + \left\langle \nu_\alpha^\circ(t), \frac{\partial h}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) \right\rangle \right) dt^\alpha \\
& < - \left( \lambda_\ell^\circ \rho_1^\ell + \rho_2 + \rho_3 \right) \|\theta(x(\cdot), x^\circ(\cdot))\|^2.
\end{aligned}$$

According to [9], §9, we have the following

**Lemma 2.1.** *A total divergence is equal to a total derivative.*

Therefore, the left hand side of the previous inequality is null, and we obtain  $0 < -(\lambda_\ell^\circ \rho_1^\ell + \rho_2 + \rho_3) \|\theta(x(\cdot), x^\circ(\cdot))\|^2$ . Since  $\|\theta(x(\cdot), x^\circ(\cdot))\|$  is positive, it follows a contradiction.

Thus, the point  $x^\circ(\cdot)$  is an efficient solution for the problem (MP).  $\square$

By replacing the integrals from hypotheses b), c) of Theorem 2.1 by the integral

$$\int_{\gamma_{t_0, t_1}} \left[ \left\langle \mu_\alpha^\circ(t), g(t, x(t), x_\gamma^\circ(t)) \right\rangle + \left\langle \nu_\alpha^\circ(t), h(t, x(t), x_\gamma^\circ(t)) \right\rangle \right] dt^\alpha,$$

the following statement is obtained.

**Corollary 2.1.** *Let us consider the vector  $\lambda^\circ$ , a feasible solution  $x^\circ(\cdot)$  of problem (MP) and the functions  $\mu^\circ(\cdot)$ ,  $\nu^\circ(\cdot)$  which satisfy the relations (MV). Suppose that the following conditions are fulfilled:*

a) for each  $\ell = \overline{1, r}$ , the functional  $F^\ell(x(\cdot)) = \int_{\gamma_{t_0, t_1}} f_\alpha^\ell(t, x(t), x_\gamma(t)) dt^\alpha$  is  $(\rho_1^\ell, b)$ -quasiinvex at the point  $x^\circ(\cdot)$  with respect to  $\eta$  and  $\theta$ ;

b) the functional

$$\int_{\gamma_{t_0, t_1}} [ < \mu_\alpha^\circ(t), g(t, x(t), x_\gamma(t)) > + < \nu_\alpha^\circ(t), h(t, x(t), x_\gamma(t)) > ] dt^\alpha$$

is  $(\rho_2, b)$ -quasiinvex at the point  $x^\circ(\cdot)$  with respect to  $\eta$  and  $\theta$ ;

c) one of the integrals of a) or b) is strictly-quasiinvex at the point  $x^\circ(\cdot)$  with respect to  $\eta$  and  $\theta$ ;

d)  $\lambda_\ell^\circ \rho_1^\ell + \rho_2 \geq 0$ .

Then the point  $x^\circ(\cdot)$  is an efficient solution of the problem (MP).

### 3. Conclusions

We considered the problem (MP) of minimizing a vector of functionals of curvilinear integrals subject to PDE and/or PDI constraints. In this work, we introduced and studied sufficient conditions for the efficiency of a feasible solution of the problem (MP). The present study completes our previous results included in papers [4], [5], [6], where we studied efficiency necessary conditions for the problem (MP) and we introduced a new type of duality. For other significant advances related to this subject, the reader is encouraged to study [1]÷[10] and references therein.

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