

WEIGHTING MATRICES DETERMINATION USING POLE PLACEMENT FOR TRACKING MANEUVERS

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In the following paper is presented an algorithm for determining the weighting matrices from the cost function so that the solution of the linear quadratic problem assures the desired poles placement of the resulted dynamics. The theoretical developments are illustrated by a case study of a satellite attitude control.

Keywords: linear quadratic regulator, pole placement, weighting matrices

1. Introduction

In designing an attitude-tracking controller, a compromise has to be made between control torques and maneuver time. For example, a smaller control input implies a longer period of time in order to acquire the desired attitude, but in this case the control actuators used, in this case reaction wheels, are less overloaded.

Leaving from the fact that the closed-loop system eigenvalues can be arbitrarily placed anywhere in the left side of the complex plane and the larger the distance of the poles location to the imaginary axis, the more demanding the physical control output, the present paper use the method developed in [1] for constructing a linear quadratic regulator that achieves the desired pole placement while satisfying the optimality. A weighting matrix is determined in such a way that the desired pole location is achieved by the optimal feedback gain corresponding to the weighting matrix of the performance criterion.

In literature, several methods of determining the weighting matrices has been developed, where the closed-loop poles may be shifted only along the real axis [2] or to relocate a single eigenvalue (or a pair of complex conjugate eigenvalues) to a desired position [3]. Another method of weights determination is by using Multi-objective Evolution Algorithm, when a Pareto-optimal solution is obtained [4].

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In the field of aerospace, for the applications of the linear quadratic regulator (LQR) method, the Q and R parameters arising in the cost function are usually determined by trial and error [5]. To optimize Q and R might seem as a repetition since the linear quadratic regulator methodology already gives optimal values of the controller gains with the lowest cost. However, this is actually obtained for a specified value of the weighting matrices. For each choice of weighting matrices, the LQR would give an optimal gain with the lowest possible cost, but this does not necessarily imply a good time domain performance [6]. For the attitude tracking of a satellite it is preferable to increase the settling time in order to prevent the control saturation. This implies a pole placement of the satellite dynamics closer to the imaginary axis.

2. Mathematical model

The success of the satellite mission depends on the spacecraft orientation in a known reference frame. The attitude motion of the satellite is described by the kinematics and dynamics equations.

For the representation of the kinematics, quaternion based parametrization will be used. The quaternion elements depend on the coordinates of Euler axis and on its rotation corresponding to a rigid body [7] [8]. Leaving from the fundamental equation of motion, which relates to the time derivative of the angular momentum due to applied torques, the attitude dynamics is obtained. The equations of motions derived from the model are [7]:

$$\dot{q} = \frac{1}{2}\Omega(\omega) \times q \quad (1)$$

$$J\dot{\omega}_b(t) = -\omega_b \times J\omega_b(t) + N_{ctr}(t) + N_d(t) \quad (2)$$

where $J\omega_b$ is the angular momentum of the spacecraft, $\Omega(\omega)$ is skew-symmetric matrix with respect to Spacecraft angular velocity ω_b , q denotes the quaternion, and N_{ctr} , N_d are the control torques and all the disturbance torques respectively. The Equation (2) then has the form

$$\dot{\omega}_b(t) = J^{-1} \left(-\omega_b \times J\omega_b(t) + N_{ctr}(t) + N_g(t) + N_d(t) \right). \quad (3)$$

As it can be seen the resulted equation has a nonlinear form, therefore to apply a linear control strategy on the spacecraft dynamics, the system must be linearized. The linearization will be performed at the operating point $q_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$ and $\omega_0 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$ so that the dynamics becomes as simple as possible.

The linearized form of the system (1), (2) is:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}\tag{4}$$

It follows that

$$A = \begin{bmatrix} 0_{3 \times 3} & I_{3 \times 3} \cdot 0.5 \\ 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} \quad B = \begin{bmatrix} 0_{3 \times 3} \\ J^{-1} \end{bmatrix}\tag{5}$$

$$C = \begin{bmatrix} 1 & 0 & \dots \\ 0 & 1 & \\ \vdots & & \ddots \end{bmatrix}; D = 0.\tag{6}$$

where J represents the inertial matrix, I the identity matrix and 0 the zero matrix of appropriate dimensions. Since the quaternion elements are related by the constraint $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$, it follows that

$$q_4 = \sqrt{1 - q_1^2 - q_2^2 - q_3^2}\tag{7}$$

and therefore the state vector x can be reduced to six independent variables, namely

$$x = \begin{bmatrix} q_1 & q_2 & q_3 & \omega_1 & \omega_2 & \omega_3 \end{bmatrix}.\tag{8}$$

3. Optimal control theory and weighting matrix determination

The next paragraph, representing the design of the weighting matrix is based on some developments derived in [1], which enable us to construct an optimal regulator providing specified closed-loop poles.

In the following we will briefly present the main stages of this design method. Considering the time-invariant system model from equation (3) the optimal stabilizing control law which minimizes the quadratic cost function:

$$J(x(t), u(t)) = \frac{1}{2} \int_0^\infty (x^T(t) Q x(t) + u^T(t) R u(t)) dt\tag{9}$$

with $Q \geq 0$ and $R > 0$, is given by

$$u(t) = -Kx(t)\tag{10}$$

where the gain matrix has the form [9]

$$K = R^{-1} B^T P\tag{11}$$

with P being the stabilizing solution of the algebraic Riccati equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0.\tag{12}$$

It is assumed that the pairs (A, B) and (\sqrt{Q}, A) are controllable and observable, respectively. These assumptions guarantee the existence of the unique

stabilizing solution of the Riccati equation (12) and the solvability of pole placement problem. Under these assumptions we will extract a specified real mode or two complex conjugate modes applying a nonsingular transformation M such that

$$M^{-1}AM = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \quad (13)$$

$$M^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad (14)$$

where A_{11} is either 1-by-1, specifying the real modes, or 2-by-2 matrix

$$A_{11} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}. \quad (15)$$

if the selected modes are $\lambda_{1,2} = \alpha \pm j\beta$.

In order to determine the matrix M , we used the linear transformation relative to a basis of real eigenvectors associated with complex eigenvalues. Therefore, we calculated first the complex vector v associated with the complex eigenvalue λ . Decomposing λ and v in real and imaginary components we get $\lambda = \alpha \pm j\beta$ and $u = x \pm jy$. By using the two vectors x and y as basis, $B = \{y, x\}$, associated with the complex conjugated eigenvalues, one obtains a 2-by-2 Jordan block of the form [13]:

$$[A]_B = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}. \quad (16)$$

Further, we have to determine the weighting matrix Q that is to be constructed according to the pole assignment requirements. Let Q_{11} be a positive semi-definite matrix with the same size as A_{11} , and set the weighting matrix Q as

$$Q_i = (M^{-1})^T \begin{bmatrix} Q_{11} & 0 \\ 0 & 0 \end{bmatrix} M^{-1}. \quad (17)$$

where " i " denotes the current step of the repetitive proposed procedure.

The eigenvalues of A_{11} can be shifted while keeping all other eigenvalues of A unchanged. Thus, appropriate selection of the weighting matrix Q through Q_{11} is crucial in the design of optimal regulators with prescribed closed-loop poles. The selection of weighting matrix $R > 0$ is arbitrary from this point of view, and R could be used as a scaling factor for the input channels. Scaling R for single-input systems has no effect, since it will only result in the same amount of scaling on Q .

Therefore, for the transformation matrix M defined above, one obtains:

$$M^{-1}BR^{-1}B^T(M^{-1})^T = \begin{bmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{bmatrix} \quad (18)$$

where V_{II} has the same size as A_{II} and Q_{II} and

$$V_{11} = v_0 \begin{bmatrix} 1 & 0 \\ 0 & v \end{bmatrix} \quad (19)$$

with $v_0 > 0$, $0 \leq v \leq 1$. Note that, when $v_0 \neq 1$, Q_{11} from (15) becomes Q_{11}/v_0 .

When a real pole is to be shifted, the matrices A_{II} , Q_{II} , V_{II} reduces to scalars. It is known that a real pole, either stable or unstable can only be shifted along the real axis within the left-half plane, and that the absolute value of the closed-loop pole is larger than that of the open-loop pole. When a complex conjugate is to be shifted, the matrices A_{II} , Q_{II} and V_{II} are 2-by-2 matrices. In this case, one can introduce the Hamiltonian matrix:

$$H = \begin{bmatrix} A_{11} & V_{11} \\ -Q_{11} & -A_{11}^T \end{bmatrix} \quad (20)$$

associated with the regulator problem of the second-order system, with Q_{II} having the partition

$$Q_{11} = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix}. \quad (21)$$

Direct algebraic computations show that the characteristic equation of H has the form:

$$s^4 + C_2 s^2 + C_0 = 0. \quad (22)$$

where coefficients C_2 and C_0 are given by:

$$\begin{aligned} C_2 &= 2(\beta^2 - \alpha^2) - \bar{C}_2 \\ C_0 &= (\beta^2 + \alpha^2)^2 - \bar{C}_0 \end{aligned} \quad (23)$$

where the following notations have been introduced:

$$\begin{aligned} \bar{C}_2 &= q_1 + vq_2 \\ \bar{C}_0 &= (\alpha^2 + v\beta^2)q_1 + 2(1-v)\alpha\beta q_2 + (v\alpha^2 + \beta^2)q_3 + v(q_1 q_3 - q_2^2). \end{aligned} \quad (24)$$

Let weighting matrix Q_{II} , correspond to the optimal closed-loop poles $\alpha_d \pm j\beta_d$ with the coefficients C_2 and C_0 described by:

$$\begin{aligned} C_2 &= 2(\beta_d^2 - \alpha_d^2) \\ C_0 &= (\beta_d^2 + \alpha_d^2)^2. \end{aligned} \quad (25)$$

Determining \bar{C}_2 and \bar{C}_0 from (23), with C_0 and C_2 given by (25), the equations (23), together with the condition $Q_{11} \geq 0$ give q_1 , q_2 and q_3 . Replacing Q_{11} in (17), the eigenvalues of A_{11} can be shifted while keeping all other eigenvalues of A unchanged. To shift all poles together, the weighting matrix Q and the corresponding optimal feedback gain matrix K are, respectively,

$$Q = \sum_{i=0}^n Q_i \text{ and } K = \sum_{i=0}^n K_i. \quad (26)$$

where n represent the number of poles to be relocated.

The described procedure is illustrated in Fig. 1.

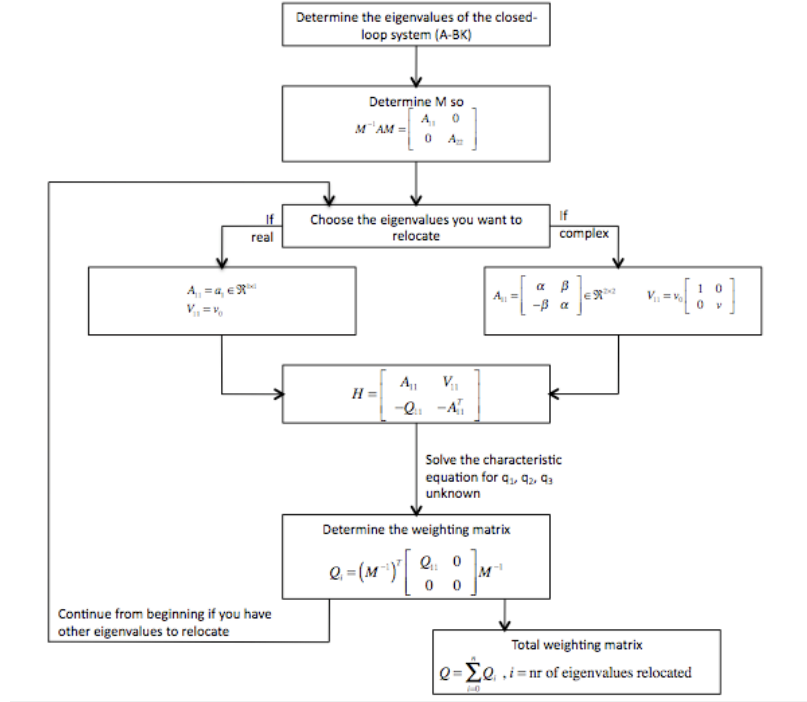


Fig. 1. Algorithm flow-chart

4. Results and discussions

The control law is implemented in a simulator that had the aim to illustrate the system behavior in acquiring a desired attitude, in our case corresponds with Moon orbit, leaving from random initial attitude $q_0 = \begin{bmatrix} 0.1585 & 0.5915 & 0.3425 & -0.7125 \end{bmatrix}$ and zero initial angular velocity. For the simulations, the quaternion error represents the difference from the body

frame (at the start of a rotational tracking manoeuvre) and the reference (desired) frame, and it can be written as:

$$\begin{bmatrix} q_{1e} \\ q_{2e} \\ q_{3e} \\ q_{4e} \end{bmatrix} = \begin{bmatrix} q_{4c} & q_{3c} & -q_{2c} & -q_{1c} \\ -q_{3c} & q_{4c} & q_{1c} & -q_{2c} \\ q_{2c} & -q_{1c} & q_{4c} & -q_{3c} \\ q_{1c} & q_{2c} & q_{3c} & q_{4c} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \quad (26)$$

$q_{1c...4c}$ represents the quaternion in the reference frame and $q_{1...4}$ the measured quaternion [7].

The resulted control gains are applied to the nonlinear satellite dynamics where the spacecraft is considered as a rigid body [8]. The inertial matrix that characterizes the spacecraft, has the form [11]

$$J = \begin{bmatrix} 32.49 & 0.01 & -0.37 \\ 0.01 & 34.92 & 0.04 \\ -0.37 & 0.04 & 12.85 \end{bmatrix}.$$

After stabilizing the system by using first the weighting matrices $Q_I = \text{diag}(1, 1, 1, 0, 0, 0)$ and $R = I_{3 \times 3}$ the resulted eigenvalues are:

$$\lambda_{init} = \{-0.0892 \pm 0.0892j; -0.0856 \pm 0.085j; -0.138 \pm -0.138j\}.$$

By choosing to move closer to the imaginary axis each eigenvalue so that the final eigenvalues to be:

$$\lambda_f = \{-0.0181 \pm 0.0201j; -0.0146 \pm 0.0166j; -0.0597 \pm -0.0207j\}$$

and following the steps explained in chapter 4 the weighting matrix Q and the gain matrix K becomes:

$$Q_f = \begin{bmatrix} 0.0018 & 0 & 0 & 0.1698 & 0 & 0 \\ 0 & 8.01e-04 & 0 & 0 & 0.1601 & 0 \\ 0 & 0 & 4.25e-04 & 0 & 0 & 0.0306 \\ 0.1698 & 0 & 0 & -0.1511 & 0 & 0 \\ 0 & 0.1601 & 0 & 0 & -0.1452 & 0 \\ 0 & 0 & 0.0306 & 0 & 0 & -0.0271 \end{bmatrix}$$

$$K_f = \begin{bmatrix} 0.0427 & 0.0283 & 0.0206 & : & 1.0912 & 0.9058 & 0.4936 \end{bmatrix}.$$

Although the inertial matrix contains the product of inertia, the corresponding gains are nearly zero and would not be used in the control law. In order to highlight the effectiveness of the proposed method a comparison is made with the nominal control, where the nominal gain vector is calculated using the transient response, $t_s = 120 \text{ sec}$, the damping ration $\gamma = 0.7$ and consider [12]

$$\begin{aligned}\omega_n^2 &= \frac{k_q}{J} \\ 2\gamma\omega_n &= \frac{k_\omega}{J}\end{aligned}\tag{27}$$

where J denotes the inertia matrix. In this case, the gain matrix is:

$$K_n = \begin{bmatrix} 0.5848 & 0.6286 & 0.2322 & \vdots & 4.32 & 4.644 & 1.709 \end{bmatrix}.$$

The Figs. 2, 3 and 4 shows the system behaviour in terms of the quaternion convergence and the generated control torques for the nominal case. Even if we have a quick response, the torques exceed the maximum control torques. All simulation presented below have been performed using the nonlinear model of the spacecraft.

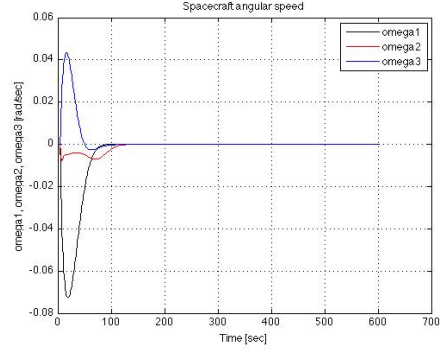
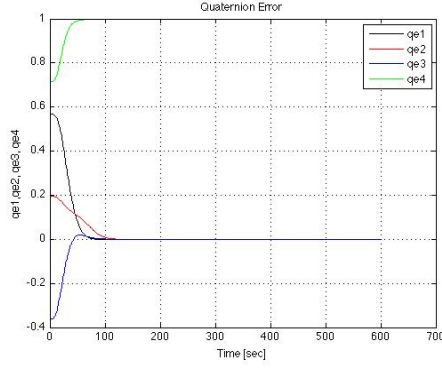


Fig. 2. Quaternion error (Nominal case) Fig. 3. Spacecraft angular speed (Nominal case)

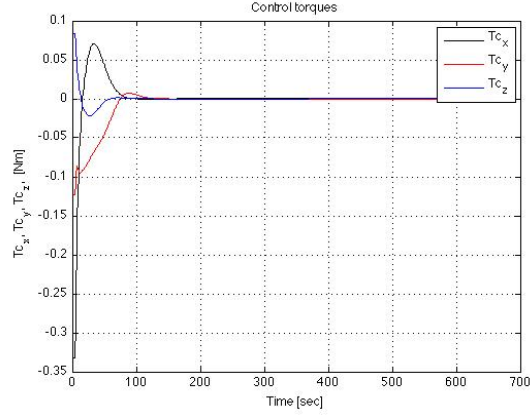


Fig. 4. Spacecraft control torques (Nominal case)

We can see that the proposed method with the pole placement has a good convergence characteristic of the quaternion without causing the overloads of the wheels. By changing the pole location closer to the imaginary axis, the less demanding the control output will be, which can be seen in the Fig. 7.

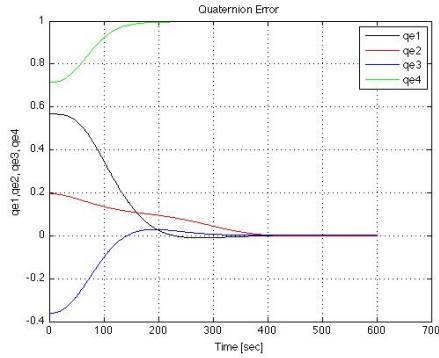


Fig. 5. Quaternion error (Optimal case)

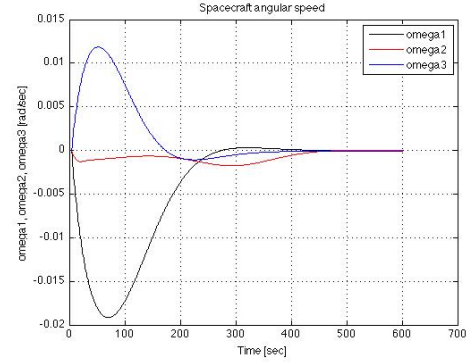


Fig. 6. Spacecraft angular speed (Optimal case)

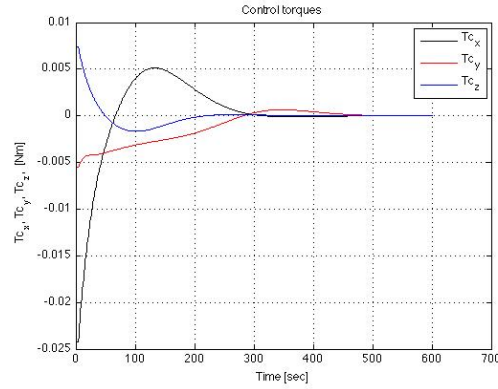


Fig. 7. Spacecraft control torques (Optimal case)

5. Conclusions

In the present paper, the determination of the weighting matrix using pole placement has been described. Considering the hardware constraints in terms of little control torques, which implies a small distance of the poles location to the imaginary axis, the purpose was to find the weighting matrix that satisfies that. For the comparison, a nominal controller, considered the settling time and damping ratio, was used.

The method has proven to be less demanding in finding the suitable solution for the system by shifting the undesired eigenvalues. Moreover, the weighting matrices were determined without using trial and error.

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