

## A METHOD FOR STUDYING THE INTERACTION OF WHEEL-TURNOUT MULTI-POINT CONTACT

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*Turnouts are determinant for the running safety of the trains. Therefore, investigations of dynamics and damage in a crossing are of great interest. The basis for the prediction of component life is an adequate determination of stresses and strains. Several numerical tools are available to study rolling contact problems but even with the computer power available today, this remains a challenging task. In the present work, the three dimensional theory elaborated by Kalker is employed for the study of the train–turnout interaction. Following a brief review of the recent studies, the basic principles of Kalker theory are introduced and issues regarding the wheel-switch contact geometry are described. A numerical efficient application example of a wheel-switch multi-point contact study is presented.*

**Key words:** wheel-turnout interaction, contact ellipse, area adhesion/sliding, contact stresses.

### 1. Introduction

Turnouts (switches and crossings points) are determinant for the running safety of the trains. According to maintenance databases, turnouts are the parts where most of the track faults are recorded and therefore, they require high maintenance costs. Turnouts have the most severe service conditions among the track parts and hence intensify vehicle dynamics. Several methods have been developed to simulate the vehicle behavior in such circumstances.

Crossings points and switches are designed to offer a continuous support for the wheels throughout the crossing, giving a smooth transition. However, high impact loads may arise [1,2], leading to damage, fracture and wear of both the crossing and the wheels and to high maintenance costs. Therefore, investigations of dynamics and damage in a crossing are of great interest. The basis for the prediction of component life is an adequate determination of stresses and strains.

The UIC (International Union of Railways) and railway researchers have been accomplished several computer applications for the study of train–turnout interaction [3]. The common feature is the employment of commercial multi-body or finite element (FE) software. Multi-body simulation packages like SIMPACK or GENSYs have been used to calculate the vehicle track interaction due to

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passing over a turnout [4 - 7] and in some cases the results were validated through full-scale field tests [8].

The strip method has been used to determine the wheel–rail contact forces in a turnout with the aid of the Hertzian method to model the wheel–rail contact [9]. The rails have also been modeled in the turn-out zone using several Euler-Bernoulli beams [10].

Flexible track model and real-time wheel-rail penetration contact model in NUCARS was employed for the crossing design [11].

Elastic and elastic–plastic finite element model of the contact are investigated with FE software like ABAQUS. The finite element method is not limited by the half-space assumption or applicable to a linear-elastic material model only [8,12,13].

The models proposed used either an on-line approach or an offline approach in which look-up tables are used to determine the location of contact points. Multibody dynamics schemes usually employ the on-line approach which leads to an accurate prediction of the location of contact points. Constraint contact formulations can be directly implemented using specific techniques, e.g., Lagrange multipliers. Since the contact search is performed on-line, the update of the profiles is required at each integration step.

In the off-line approach, the look-up contact tables are prepared in advance and the position of the contact points is determined by interpolating the tabular data associated with the wheelset lateral and yaw displacements at the current position [6,14-16].

The advantage is that the contact points can be determined without any iteration and there is no need to perform contact search for the second point of contacts. With this procedure, effects of changes in the rail cross-section along the track can be efficiently modeled. Still, even with the computer power available today, this remains a challenging task.

In the present work, the three dimensional theory elaborated by Kalker is employed for the study of the train–turnout interaction. This approach appears to be very well suited because the theory addresses frictional contact problems. Furthermore, a study for the assessment of methods for calculating contact pressure in wheel-rail/switch contact he concluded is that results obtained with CONTACT are similar to those obtained with the FE method although the assumptions seem to be violated, [13]. However, the numerical efficiency is increased. More than that, the computation of the contact points can be performed on-line as well as off-line.

A distinctive feature of the application proposed in this work is occurrence of simultaneous contact patches on a single wheel, i.e., the multi-point contact case. A difficulty in the study of this phenomenon is that the contact areas depend on the deformation of the bodies, whereas these deformations are functions of the

contact loading and the contact area. This is contrary to what is usually the case in finite element analysis, where the load and displacement are prescribed at different (fixed) parts of the boundary. The program CONTACT is intended for concentrated contact problems and may be easily employed in the multi-point contact case.

The paper is organized as follows. In Section 2 the basic principles of Kalker theory are introduced. In Section 3 issues regarding the wheel-switch contact geometry are described. An application example of a wheel-switch contact study for S78 and UIC60 profiles is presented in Section 4. The most important points of the paper are summarized in Section 5.

## 2. The basic principles of Kalker theory

The theory of the linear contact developed by Kalker starts from the De Pater, [17], idea, based on the fact that for small values of the pseudo-slip effect the slip area is very small, which considers that the entire contact surface is assimilated with an adhesion area. According to this hypothesis, the condition at the extremity is given by the following equations:

- within the contact ellipse

$$0 = w_{x,y} = V \cdot (v_x - \varphi \cdot y, v_y + \varphi \cdot x) - V \frac{\partial u(x,y)}{\partial x} \quad (1)$$

- outside the contact ellipse  $0 = p(x,y)$  (2)

By integrating the relation (1) results:

$$g(y) = -V \cdot u(x,y) + V \left( v_x x - \varphi \cdot x \cdot y, v_y x - \frac{1}{2} \varphi \cdot x^2 \right) \quad (3)$$

where  $g(y)$  represents an arbitrary function.

The arbitrary function  $g(y)$  is determined from the condition that the traction force must be constant along the border, where the railway and the wheel come into contact. The final result does not respect though the limit value of the friction force at the end limit of the contact surface [18]. This thing is justified by the fact that in the exterior vicinity of the contact limit the bodies materials are free of internal stress. The traction increases up to the rear edge of the adhesion limit then the normal load is removed and the traction suddenly drops to 0. Thus, on the edge, the friction index shows a discontinuous, infinite fluctuation. Calculating the  $g(y)$  function will lead to a linear expression displaying the interdependence between forces and pseudo-slip:

$$\begin{aligned} F_x &= -a \cdot b \cdot G \cdot C_{11} \cdot v_x, F_y = -a \cdot b \cdot G (C_{22} \cdot v_y + \sqrt{a \cdot b} \cdot C_{23} \cdot \varphi) \\ M_\varphi &= -(\sqrt[3]{a \cdot b} \cdot G \cdot C_{32} + (a \cdot b)^2 G \cdot C_{33} \cdot \varphi) \end{aligned} \quad (4)$$

where:  $a, b$  – semi-axes of the contact ellipse,  $v_x, v_y, \varphi$  – longitudinal, transversal and spinning pseudo-slip movements in the contact points,  $C_{11}, C_{22}, C_{23,32}, C_{33}$  – Kalker parameters which depend on the value of Poisson index and being determined based on the following equations [18]:

$$\begin{aligned} C_{11} &= -\frac{\partial F_x}{\partial v_x} / (a \cdot b \cdot G), \quad C_{22} = -\frac{\partial F_y}{\partial v_y} / (a \cdot b \cdot G) \\ C_{23} &= -\frac{\partial F_y}{\partial \varphi} / (\sqrt[3]{a \cdot b \cdot G})_{v_x=v_y=\varphi=0} \end{aligned} \quad (5)$$

Kalker defines the pseudo-slip parameters as following:

$$\kappa_x = \frac{a \cdot b \cdot G \cdot C_{11} \cdot v_x}{\mu \cdot N}, \quad \kappa_y = \frac{a \cdot b \cdot G \cdot C_{22} \cdot v_y}{\mu \cdot N}, \quad \kappa_\varphi = \frac{\sqrt[3]{a \cdot b \cdot G} \cdot C_{23} \cdot \varphi}{\mu \cdot N} \quad (6)$$

The longitudinal, transversal and spinning pseudo-slip movements are established based on the following equations:

$$v_x = \frac{w_x}{V}, \quad v_y = \frac{w_y}{V}, \quad \varphi = \frac{r \cdot \omega_s}{V}, \quad (7)$$

where:  $\omega_s$  is the spinning angle speed.

The pseudo-slip speeds  $w_{x,y}$  are established based on the (1) relation while their spatial orientation for a forward moving axle presented in the fig. 1:

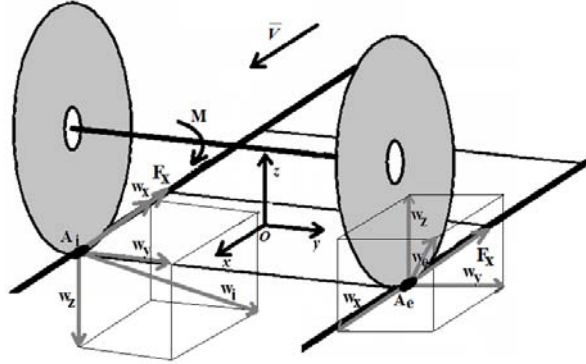


Fig.1. Spatial orientation of the slip speeds of the wheel-railway assembly.

During the past years numerous theories have been developed in regards to the wheel-railway track point of contact, many of them elaborated through software tools based on analytic and proximate approaches, among which the Kalker CONTACT stands out as it assimilates the contact area through half-space dimensions and it divides it through the method of the finite element. Within the contact area the theory of linear elasticity is respected while the bodies are in tangential contact.

Through the help of this algorithm elastic deformations, stick and slip areas, pseudo-slip movements and obviously the contact forces can be determined through a three-dimensional approach. This thing is possible though only if the exact shape of the profiles which come together in the contact area is known.

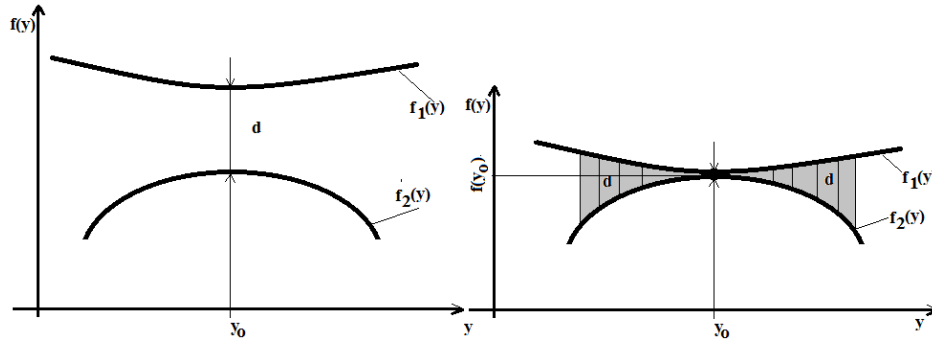
This software allows a three-dimensional approach of the contact issue, simplifying the analysis and the wheel-railway track contact point and of its influence over the safety of transportation as it allows the analysis of the shape and direction of the parameters of the interaction. For a proper analysis of the issues generated by the contact between the wheel and the railway track it is necessary to establish the separation and the dimensions of the contact and slip areas as well as the combined efforts, under the conditions generated by the normal and tangential forces applied from outside.

### 3. The Geometry of the contact between wheel and railway track

The geometric study of the wheel-railway contact consists in determining the contact points between the wheel apparent profile and the normal profile of the railway track. Thus the analysis of the geometric contact between the wheel and the railway track has in the beginning the following steps to follow:

- Tracing the wheel profile;
- Tracing the railway track profile;
- Establishing the position of the wheel in relation to the railway track and of the contact points.

Studies regarding the contact between the wheel and the railway track, focused primarily on determining the tangential profile, have been written down by [19,20,21]. Another method used for representing the rolling profiles consists of displaying them in the form of some circular curves. The method is simple and at the base of it has the premise that the Cartesian coordinates  $(Y, Z)$  for the rolling profiles are known and that the mentioned curves can be made by assembling the circular curves [22]. As a result two successive points from the coordinate profile  $(Y_1, Z_1)$  and  $(Y_2, Z_2)$  considered to be the extremes of the circular curve with the radius  $\rho_2$  and of the flank angles  $\gamma_1$  and  $\gamma_2$ .



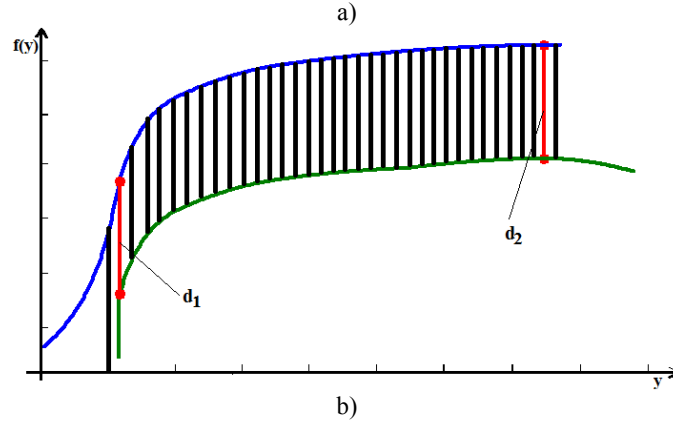


Fig.2. The Method of the Minimum Distance. a) – single point contact; b) – multi-point contact

As a result the parametric equations of the circular curve given by  $\gamma$  can be expressed through:

$$\begin{aligned} Y_2 - Y_1 &= \rho_2 (\sin \gamma_2 - \sin \gamma_1) \\ Z_2 - Z_1 &= \rho_2 (\cos \gamma_2 - \cos \gamma_1) \end{aligned} \quad (8)$$

The point of contact between the wheel and the railway track is located in the place where the inclinations of the two curves are equal, meaning where the first derivative of the function  $d(y)$  is canceled:

$$\begin{aligned} d'(y) &= 0 \\ f_1'(y) - f_2'(y) &= 0 \Rightarrow f_1'(y) = f_2'(y) \end{aligned} \quad (9)$$

where  $d(y) = f_1(y) - f_2(y) = \min$ .

This elementary method is used for determining the contact positions from the rolling track as well as from the edge of the wheel in most of specialized studies – see. Fig. 2 – and it is used in the analysis of the multi-point contact.

#### 4. Numerical application

Within this section we will be analyzing the behavior of an axle equipped with wheels having the S78 profile (the nominal rolling radius is of 460 mm) over a simple railroad siding made of a rail track profile type UIC 60. The speed with which the axle will pass over the railroad siding is of 35 m/s. The contact point is analyzed in a three-dimensional reference frame. The normal load applied on the wheel is considered to be of 10 kN. For an accurate analysis of the contact area between the wheel and the rail the need for differentiating the profiles for the rolling surfaces and for establishing the contact points between these two parts is imposed. The profiles are then drawn through the method of circular curves and the contact points have been established through the method of the minimum

distance – see. Fig. 3. In this image it is easily be observed the presence of the dual-contact composed of a point located on the edge of the wheel and the point from the rolling surface.

In the case of the dual-contact the normal charge on the wheel is distributed on the two contact points. For this case the distribution of the total load on the two contact areas is done based on the hypothesis that the rigidities are equal, the deformation is proportional with the 3/2 rapport, according to the results obtained by Hertz [23] for the contact between two cylindrical bodies. As a result, the distribution of the vertical components within the normal components in the contact areas is expressed through the following equation

$$F_1 / F_2 = (\delta_1 / \delta_2)^{3/2} \quad (10)$$

where  $\delta_1$  and  $\delta_2$  are the components on the vertical direction for the penetrations in the two contact areas and the sum of the normal reactions of the vertical components from within the contact areas, namely  $F_1$  și  $F_2$ , balances the load on the wheel i.e.  $F_1 + F_2 = Q$ . For evaluating the behavior between wheel and the rail tongue an iterative program has been developed within the Kalker CONTACT frame, applied for the dual-contact analysis.

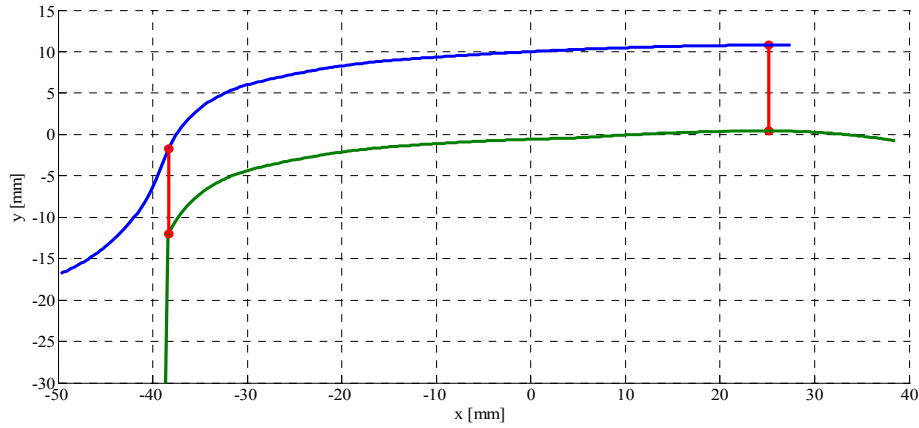


Fig. 3. The contact between the wheel and the railway tongue.

As a result the calculus stages which mention the input data for the CONTACT program are the following:

1. Determining the coordinates of the first point of contact through the method of the minimum height difference.
2. The calculus of the penetration within the hypothesis of a single contact point.
3. Verifying the possibility of the development of a new contact area. If this is possible the previous two stages are re-iterated.
4. Determining the penetrations within the contact areas.

5. Re-compute the values for the vertical components (through the equations shown below). If the values obtained here show differences compared to the values obtained at step 4, differences which are not within the accepted margins, then the calculus at step 4 is re-performed again with vertical components determined at step 5 up until the results obtained are within the imposed limits.

It is analyzed the distribution of the slip and stick areas in the dual-contact areas encountered at the wheel pass over and the rail tongue – see. Fig. 4. It can be observed that in the point from the edge, the distribution of the slip and stick areas is similar with rolling under the conditions in which the spin pseudo-slip is null, meaning that the wheel behaves as a spinning top – see Fig. 4, left.

In point from the wheel flange the slip area prevails which is in fact highlighted through the increase of the density of the friction strength – see Fig. 4, right, 5, right. The distribution of the normal pressures in the contact area with the railway tongue can be seen in Fig. 6. It can easily be observed that the normal pressure is higher in the top of the railway tongue; this explains the increased signs of wear and tear within this area – see Fig. 6, left.

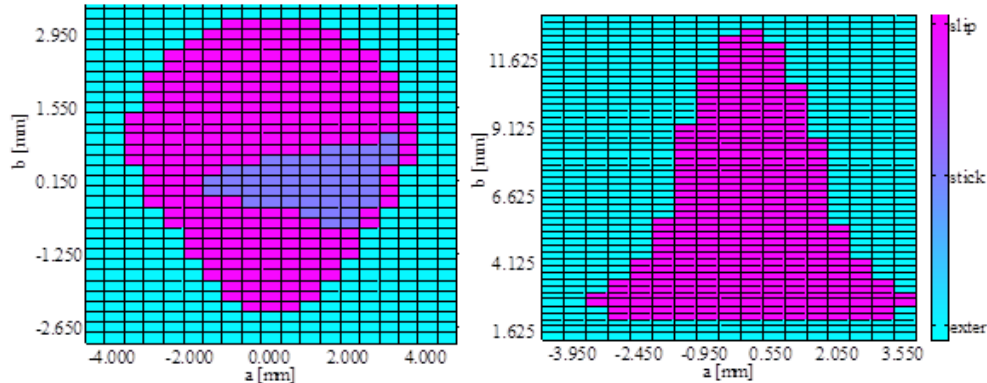


Fig. 4. The distribution of the slip and stick areas.

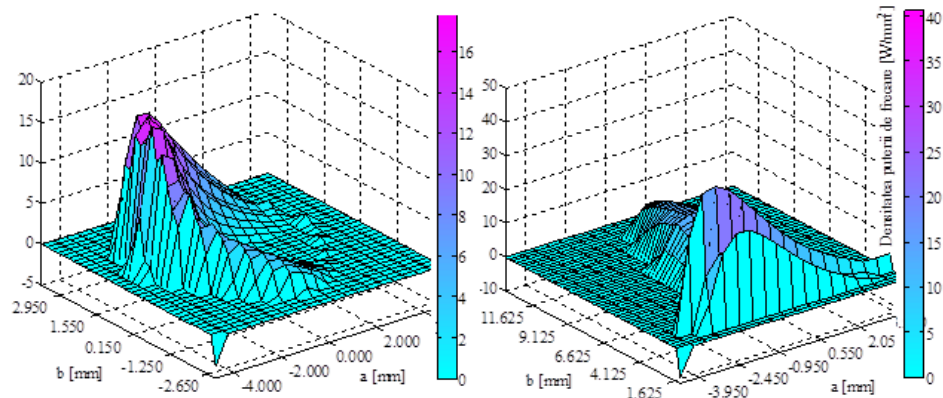


Fig. 5. The density of the friction power.



The tangential forces are transmitted through the adherence appeared between the two bodies found in interaction, which is determined by the elastic deformations and the slip from within the contact area. The density of the friction power corresponds to the distribution of the tangential reactions, to the rolling speed and the relative slip movement from within the contact area – see Fig. 5.

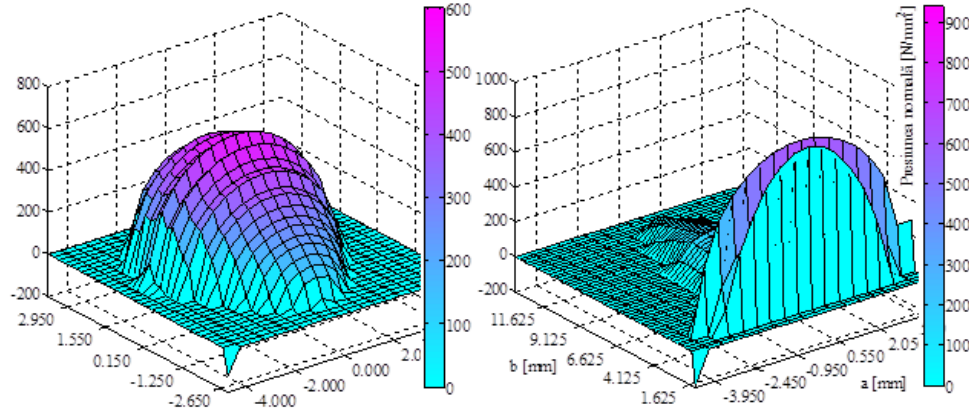


Fig. 6. The distribution of the normal pressure

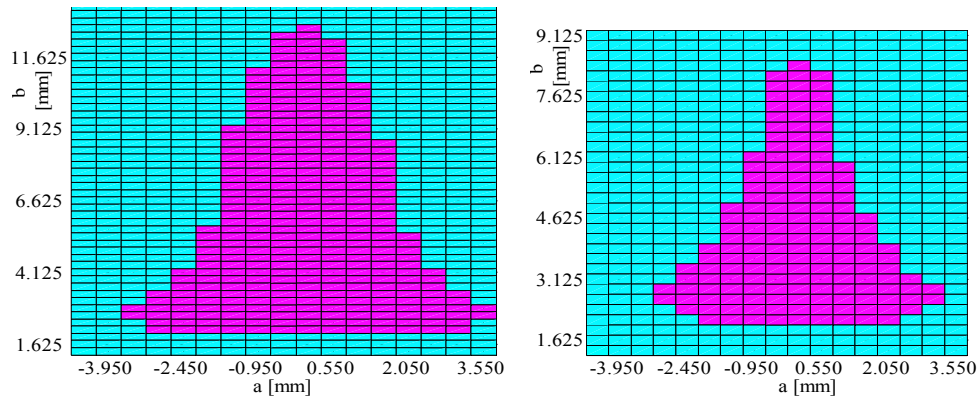


Fig. 7a. The distribution of the slip and stick areas depending on the normal load

In figure 7, as mentioned above, it can be seen that the dimension of the contact areas is influenced by the normal load applied on the wheel. A wheelset with a normal load on the wheel of  $5\text{ kN}$  (right figure) and  $10\text{ kN}$  (left figure) was analyzed. In the point from the rolling surface the shape of the contact area related to the normal load on the wheel is not significantly modified, as seen in Fig. 7a. Its dimensions vary proportional to the increase of the load on the wheel. Also, decreasing the load on the charging wheel lowers the guiding capabilities of the axle. The limit situation of unloading the charging wheel can occur when the

wheel is rolling at low speeds over curves with maximum cant deficiency and high torsions. In Fig. 7b it can easily be observed that the size of the normal load on the wheel modifies significantly the shape as well as the size of the adherence area on transversal direction, situation also shown by the increase of the tangential force generated on this direction (from 51  $kN$  up to 158  $kN$ ).

Regarding the vectorial distribution of the traction tangential force related to the normal load on the wheel this gets enhanced through its vectorial field increase along with the load – see Fig. 8a.

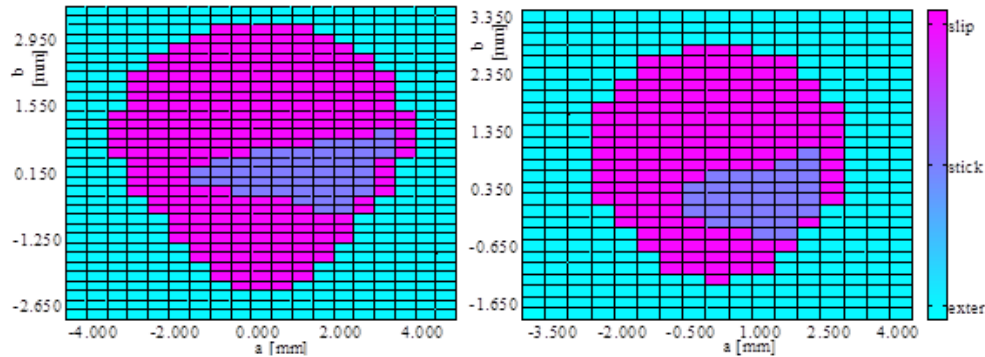


Fig. 7b. The distribution of the slip and stick areas depending on the normal load

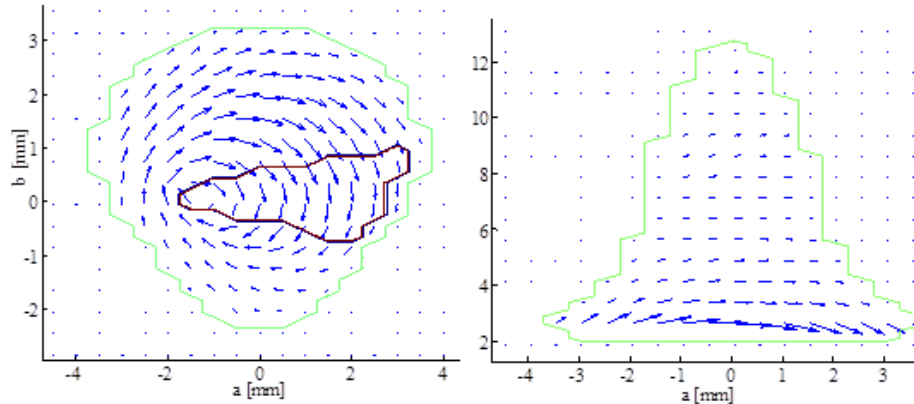


Fig. 8a. The vectorial distribution of the traction tangential force related to the normal load

In the contact point from the edge of the wheel, according to the distribution of the field of the traction tangential force, the wheel is rolling on the railway tongue in a similar situation as the rolling of it without spin slip. In this case the intensity of the traction field increases along with the load – see figure 8.a,b. No major changes appear in the contact point from the surface of the rolling wheel.

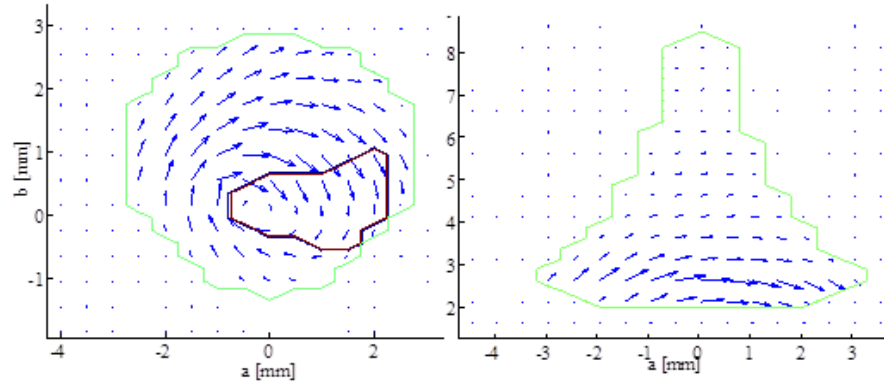


Fig. 8b. The vectorial distribution of the tangential traction force related to the normal load at the contact point between the wheel and the railway tongue

According to Fig. 9a,b the loss of power resulted through the friction occurred in the contact surface is intensified by increasing the normal load on the wheel. Thus, on the edge of the wheel it is encountered an increase, from  $64.4 \text{ W/mm}^2$  to  $133.7 \text{ W/mm}^2$ , situation justified by the prevalence of the slip area.

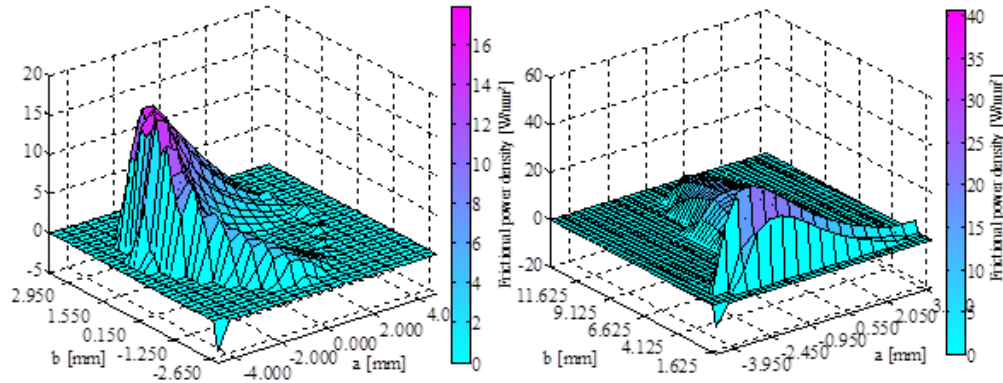


Fig. 9a. The distribution of loss of power through friction depending on the normal load.

In Fig. 10 it can be seen the distribution of the loss of friction power from the contact surface depending on the rolling speed. Two different speed values were taken into consideration –  $35 \text{ m/s}$  (Fig 10a) and  $45 \text{ m/s}$  (Fig. 10b). Thus it can be seen that increasing the rolling speed results in amplifying the loss of power through friction between the two contact points belonging to the wheel and the railway tongue. It can also be seen a significant increase of  $122.3 \text{ W/mm}^2$  in the contact point of the surface of the rolling wheel given by the prepotency of the slip area.

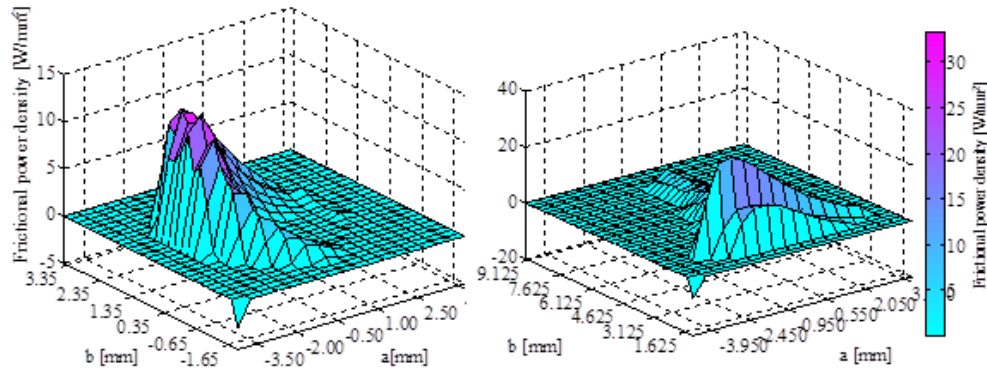


Fig. 9b. The distribution of power loss through friction in regard to the normal load.

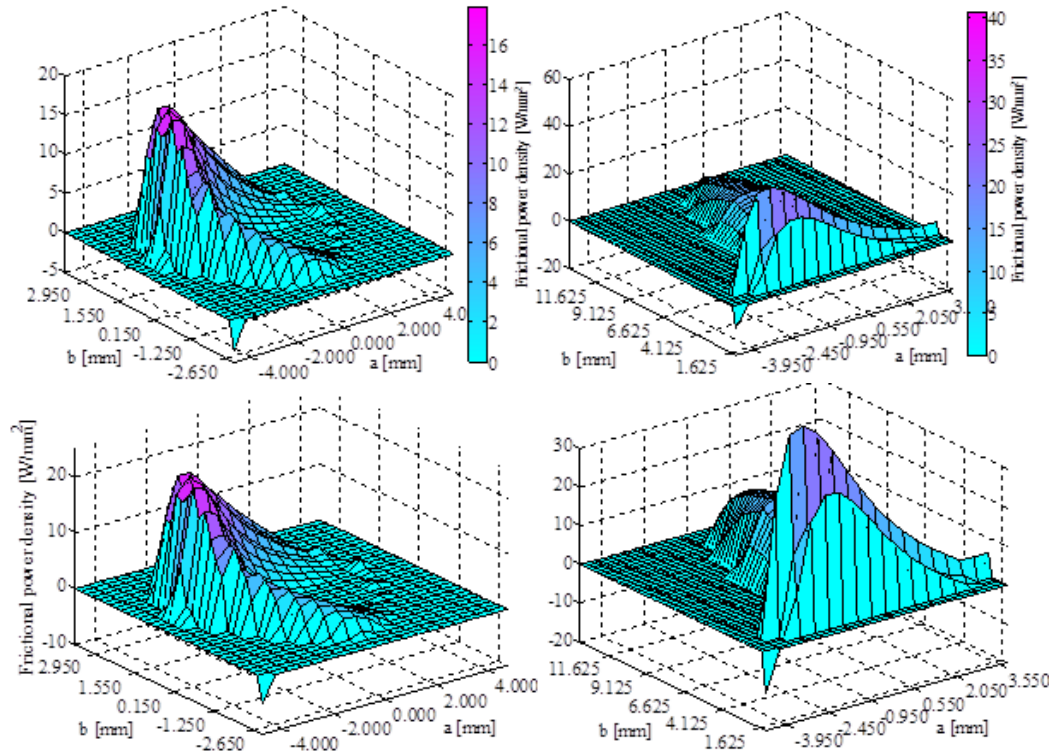


Fig. 10. The distribution of power loss through friction in regard to the rolling speed.

The elastic energy stored in the contact surface related to the normal load on the wheel will commensurately increase with it. Thus, in the contact point from the edge of the wheel the energy will increase from  $29.6 \text{ J/mm}^2$  to  $92.3 \text{ J/mm}^2$  and in the abutment point from  $31.5 \text{ J/mm}^2$  to  $85.7 \text{ J/mm}^2$ . It can be considered that the elastic energy is intensified by the dimensions and the meshing of the contact area.

The parameter which influences the transfer of the tangential forces towards pseudo-slip is defined as a proportion between the slip velocity and the forward rolling speed of the axle. This ratio actually marks the difference between the friction encountered in the static contact and the one that produces the relative movement.

The need of the study made in this direction is explained through the fact that for studying the phenomena found in the contact between the wheel and the railway track, essentially for determining the tangential forces, determining the slip velocities as well as the distribution of pressures in the contact area is compulsory for the behavior of the rolling vehicle.

### 5. Conclusions

For an accurate approach of the problems raised by the wheel-railway track contact it is necessary to establish from the beginning the delimitation and the sizes of the gliding and grip areas as well as the unitary efforts resulted from them under normal forces as well under tangential forces applied from outside.

A method for the wheel-turnout interaction study is developed in the present work. The tridimensional Kalker theory is employed using the computer program CONTACT. The application is designed for frictional concentrated contact problems. The solution is numerically efficient and it has been previously validated for the general case of the wheel-rail contact.

Multi-point contact is a common situation for the studied system. In the approach proposed, the bodies deformations and the contact areas can be determined from the contact loading.

Further, the number and position of the contact points can be calculated either on-line or off-line. The scheme proposed also allows the computation of the loading and stress state starting from the local deformations.

The influence of the wheel load and velocity upon contact parameters is analyzed. Contact stresses and deformations are computed. The distribution of the tangential forces in the contact patch and frictional power density are also studied.

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