

PARA-BLASCHKE ISOPARAMETRIC HYPERSURFACES IN THE UNIT SPHERE $S^{n+1}(1)$ (II)

Shichang Shu¹, Bianping Su²

Let $\mathbf{D} = \mathbf{A} + \lambda\mathbf{B}$ be the para-Blaschke tensor of the immersion \mathbf{x} , where λ is a constant, \mathbf{A} and \mathbf{B} are the Blaschke tensor and the Möbius second fundamental form of \mathbf{x} . A hypersurface $\mathbf{x} : M \rightarrow S^{n+1}(1)$ in the unit sphere $S^{n+1}(1)$ without umbilical points is called a para-Blaschke isoparametric hypersurface if the Möbius form Φ vanishes identically and all of its para-Blaschke eigenvalues are constants. In [11], we classified the para-Blaschke isoparametric hypersurfaces with three distinct Blaschke eigenvalues, one of which is simple or with three distinct Möbius principal curvatures, one of which is simple. In this article, we continue to study the topic of para-Blaschke isoparametric hypersurfaces and obtain the classification of para-Blaschke isoparametric hypersurfaces with three distinct para-Blaschke eigenvalues, one of which is simple.

Keywords: Möbius metric, para-Blaschke tensor, para-Blaschke eigenvalue, Para-Blaschke isoparametric

MSC2010: 53A30, 53B25.

1. Introduction

In Möbius differential geometry, Wang [13] studied invariants of hypersurfaces in the unit sphere $S^{n+1}(1)$ under the Möbius transformation group. Let $\mathbf{x} : M \rightarrow S^{n+1}(1)$ be an n -dimensional immersed hypersurface without umbilical points in $S^{n+1}(1)$. We choose a local orthonormal basis $\{e_i\}$ for the induced metric $I = d\mathbf{x} \cdot d\mathbf{x}$ with dual basis $\{\theta_i\}$. Let $II = \sum_{i,j} h_{ij} \theta_i \otimes \theta_j$ be the second fundamental form and $H = \frac{1}{n} \sum_i h_{ii}$ the mean curvature of the immersion \mathbf{x} . By putting $\rho^2 = \frac{n}{n-1} \{ \sum_{i,j} h_{ij}^2 - nH^2 \}$, Wang [13] defined the Möbius metric, the Möbius form, the Blaschke tensor and the Möbius second fundamental form of the immersion \mathbf{x} by $g = \rho^2 d\mathbf{x} \cdot d\mathbf{x}$, $\Phi = \rho \sum_i C_i \theta_i$, $\mathbf{A} = \rho^2 \sum_{i,j} A_{ij} \theta_i \otimes \theta_j$ and $\mathbf{B} = \rho^2 \sum_{i,j} B_{ij} \theta_i \otimes \theta_j$, respectively, where C_i , A_{ij} , B_{ij} are defined by (2.7)–(2.9). It was proved that g , Φ , \mathbf{A} and \mathbf{B} are Möbius invariants. We should notice that it is one of the important aims to characterize submanifolds in terms of Möbius invariants. Concerning this topic, there are many important results, one can see [2]–[12]. Recently, by making use of the two important Möbius invariants, the Blaschke tensor \mathbf{A} and the Möbius second

¹Professor, School of Mathematics and Information Science, Xianyang Normal University, Xianyang, 712000, Shaanxi, P.R. China, e-mail: shushichang@126.com

²Professor, Department of Science, Xi'an University of Architecture and Technology, Xi'an, 710055, Shaanxi, P.R. China, e-mail: subianping@126.com

fundamental form \mathbf{B} of the immersion \mathbf{x} , Cheng, Li and Qi [3], Zhong and Sun [14] defined a symmetric $(0, 2)$ tensor $\mathbf{D} = \mathbf{A} + \lambda\mathbf{B}$ which is so called *the para-Blaschke tensor* of \mathbf{x} , where λ is a constant. An eigenvalue of the Blaschke tensor is called *a Blaschke eigenvalue* of \mathbf{x} , an eigenvalue of the Möbius second fundamental form is called *a Möbius principal curvature* of \mathbf{x} and an eigenvalue of the para-Blaschke tensor is called *a para-Blaschke eigenvalue* of \mathbf{x} . It is reasonable to introduce the definition: a hypersurface $\mathbf{x} : M \mapsto S^{n+1}(1)$ without umbilical points is called *a Blaschke isoparametric hypersurface*, (resp., *a Möbius isoparametric hypersurface*), (resp., *a para-Blaschke isoparametric hypersurface*), if the Möbius form $\Phi \equiv 0$ and the Blaschke eigenvalues, (resp., the Möbius principal curvatures), (resp., the para-Blaschke eigenvalues) of the immersion x are constants.

If \mathbf{x} has one distinct constant para-Blaschke eigenvalue, that is, $\mathbf{A} + \lambda\mathbf{B} + \mu g = 0$, Li and Wang [8] completely classified these hypersurfaces without umbilical points and vanishing Möbius form. If \mathbf{x} has two distinct constant para-Blaschke eigenvalues, the classification was obtained by Zhong and Sun [14]. If \mathbf{x} has three distinct constant para-Blaschke eigenvalues, what classification can we obtain? In this article, we obtain the classification of para-Blaschke isoparametric hypersurfaces with three distinct para-Blaschke eigenvalues, one of which is simple. We should notice that, in [11], the authors classified the para-Blaschke isoparametric hypersurfaces with three distinct Blaschke eigenvalues, one of which is simple or with three distinct Möbius principal curvatures, one of which is simple, thus, this article may be considered as a continuing study to the topic of para-Blaschke isoparametric hypersurfaces in $S^{n+1}(1)$. We shall prove the following:

Theorem 1.1 (Main Theorem). *Let \mathbf{x} be an $n(n \geq 3)$ -dimensional immersed para-Blaschke isoparametric hypersurface in the unit sphere $S^{n+1}(1)$ with three distinct para-Blaschke eigenvalues, one of which is simple. Then \mathbf{x} is locally Möbius equivalent to*

- (1) $CSS(p, q, r)$ for some constants p, q, r given by Example 2.1, or
- (2) a Euclidean isoparametric hypersurface with three or four distinct Euclidean principal curvatures.

Remark 1.1. *In the second case of Main Theorem, if $n = 3$, by Cartan's result in [1], we know that it is in fact a tube of constant radius over a standard Veronese embedding of \mathbf{RP}^2 . If $n = 4$, from [6], we know that it is either the image of σ of the cone $\tilde{x} : N^3 \times \mathbf{R}^+ \mapsto \mathbf{R}^5$ defined by $\tilde{x}(x, t) = tx$, where $t \in \mathbf{R}^+$ and $x : N^3 \mapsto S^4 \hookrightarrow \mathbf{R}^5$ is the Cartan isoparametric immersion in S^4 with three Euclidean principal curvatures, or the Euclidean isoparametric hypersurfaces in S^5 with four distinct Euclidean principal curvatures. Therefore, we see that Main Theorem reduces to Theorem 4.3 of [15] and Theorem 4.2 of [16].*

Remark 1.2. *For the Möbius isoparametric hypersurfaces and Blaschke isoparametric hypersurfaces, we should notice that Hu and Li [5] obtained the immersed Möbius isoparametric hypersurfaces with three distinct Möbius principal curvatures,*

one of which is simple, Li and Peng [9] obtained the immersed Blaschke isoparametric hypersurfaces with three distinct Blaschke eigenvalues one of which is simple.

2. Möbius fundamental formulas

In this section, we review the fundamental formulas on Möbius geometry in $S^{n+1}(1)$ (see [13]).

Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n -dimensional hypersurface of $S^{n+1}(1)$ without umbilical points. For an immersed hypersurface $\mathbf{x} : M \mapsto S^{n+1}(1) \hookrightarrow \mathbf{R}^{n+2}$ of $S^{n+1}(1)$ without umbilical points, we define its Möbius position vector $Y : M \mapsto \mathbf{L}^{n+3}$ by $Y = \rho(1, \mathbf{x})$, where $\rho^2 = \frac{n}{n-1}\{\sum_{i,j} h_{ij}^2 - nH^2\}$. Let Δ be the Laplace-Beltrami operator of Möbius metric $g = \rho^2 d\mathbf{x} \cdot d\mathbf{x}$. We define $N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle\Delta Y, \Delta Y\rangle Y$, the structure equations on M with respect to the Möbius metric g can be written as follows:

$$dY = \sum_i \omega_i Y_i, \quad (2.1)$$

$$dN = \sum_i \psi_i Y_i + \phi E_{n+1}, \quad (2.2)$$

$$dY_i = -\psi_i Y - \omega_i N + \sum_j \omega_{ij} Y_j + \omega_{in+1} E_{n+1}, \quad (2.3)$$

$$dE_{n+1} = -\phi Y - \sum_i \omega_{in+1} Y_i, \quad (2.4)$$

where $\{\psi_i, \omega_{ij}, \omega_{in+1}, \phi\}$ are 1-forms on M with

$$\omega_{ij} + \omega_{ji} = 0, \quad d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad (2.5)$$

$$\psi_i = \sum_j A_{ij} \omega_j, \quad A_{ij} = A_{ji}, \quad \omega_{in+1} = \sum_j B_{ij} \omega_j, \quad B_{ij} = B_{ji}, \quad \phi = \sum_i C_i \omega_i, \quad (2.6)$$

and A_{ij} , B_{ij} and C_i are locally defined functions:

$$C_i = -\rho^{-2}\{H_{,i} + \sum_j (h_{ij} - H\delta_{ij})e_j(\log \rho)\}, \quad (2.7)$$

$$A_{ij} = -\rho^{-2}\{\text{Hess}_{ij}(\log \rho) - e_i(\log \rho)e_j(\log \rho) - Hh_{ij}\} \quad (2.8)$$

$$- \frac{1}{2}\rho^{-2}(|\nabla(\log \rho)|^2 - 1 + H^2)\delta_{ij},$$

$$B_{ij} = \rho^{-1}(h_{ij} - H\delta_{ij}), \quad (2.9)$$

here Hess_{ij} , ∇ are the Hessian matrix and the gradient with respect to the induced metric $d\mathbf{x} \cdot d\mathbf{x}$. From [13]

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \quad R_{ijkl} = -R_{jikl}, \quad (2.10)$$

$$\sum_i B_{ii} = 0, \quad \sum_{i,j} B_{ij}^2 = \frac{n-1}{n}, \quad \text{tr}\mathbf{A} = \frac{1}{2n}(1 + n^2 R). \quad (2.11)$$

Let $C_{i,j}, A_{ij,k}, B_{ij,k}$ be the covariant derivative of C_i, A_{ij}, B_{ij} . From (2.1)-(2.4), we infer that

$$A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k, \quad (2.12)$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}A_{kj} - B_{kj}A_{ki}), \quad (2.13)$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \quad (2.14)$$

$$R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk} + \delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}, \quad (2.15)$$

where R_{ijkl} denotes the curvature tensor with respect to the Möbius metric g and R is the normalized Möbius scalar curvature of the immersion \mathbf{x} . Assume that the Möbius form $\Phi \equiv 0$, we have for all indices i, j, k that

$$A_{ij,k} = A_{ik,j}, \quad B_{ij,k} = B_{ik,j}, \quad \sum_k B_{ik}A_{kj} = \sum_k B_{kj}A_{ki}. \quad (2.16)$$

Denote by $\mathbf{D} = \sum_{i,j} D_{ij}\omega_i \otimes \omega_j$ the $(0, 2)$ para-Blaschke tensor,

$$D_{ij} = A_{ij} + \lambda B_{ij}, \quad 1 \leq i, j \leq n, \quad (2.17)$$

where λ is a constant. The covariant derivative of D_{ij} is defined by

$$\sum_k D_{ij,k}\omega_k = dD_{ij} + \sum_k D_{ik}\omega_{kj} + \sum_k D_{kj}\omega_{ki}. \quad (2.18)$$

From (2.16) and (2.17), we have for all indices i, j, k that

$$D_{ij,k} = D_{ik,j}. \quad (2.19)$$

Defining the second covariant derivative of D_{ij} by

$$\sum_l D_{ij,kl}\omega_l = dD_{ij,k} + \sum_l D_{lj,k}\omega_{li} + \sum_l D_{il,k}\omega_{lj} + \sum_l D_{ij,l}\omega_{lk}, \quad (2.20)$$

we have the Ricci identity

$$D_{ij,kl} - D_{ij,lk} = \sum_m D_{mj}R_{mikl} + \sum_m D_{im}R_{mjkl}. \quad (2.21)$$

We recall the following example given by [4].

Example 2.1. For any natural number p, q , $p + q < n$ and real number $r \in (0, 1)$, consider the immersed hypersurface $u : S^p(r) \times S^q(\sqrt{1-r^2}) \times \mathbf{R}^+ \times \mathbf{R}^{n-p-q-1} \mapsto \mathbf{R}^{n+1}$

$$\begin{aligned} u &= (tu', tu'', u'''), \\ u' &\in S^p(r) \subset \mathbf{R}^{p+1}, \quad u'' \in S^q(\sqrt{1-r^2}) \subset \mathbf{R}^{q+1}, \quad u''' \in \mathbf{R}^{n-p-q-1}. \end{aligned}$$

then $\mathbf{x} = \sigma \circ u : S^p(r) \times S^q(\sqrt{1-r^2}) \times \mathbf{R}^+ \times \mathbf{R}^{n-p-q-1} \mapsto S^{n+1}(1)$ is a hypersurface in $S^{n+1}(1)$ without umbilical points and with vanishing Möbius form, which is denoted by $CSS(p, q, r)$. From [4] and [9], by a direct calculation, we know that $CSS(p, q, r)$ has three distinct Möbius principal curvatures and three distinct para-Blaschke eigenvalues.

3. Proof of Main Theorem

Throughout this section, we shall make the following convention on the ranges of indices:

$$\begin{aligned} 1 \leq a, b \leq m_1, \quad m_1 + 1 \leq p, q \leq m_1 + m_2, \\ m_1 + m_2 + 1 \leq \alpha, \beta \leq m_1 + m_2 + m_3 = n, \quad 1 \leq i, j, k \leq n. \end{aligned}$$

Let A , B and D denote the $n \times n$ -symmetric matrices (A_{ij}) , (B_{ij}) and (D_{ij}) , respectively, where A_{ij} , B_{ij} and D_{ij} are defined by (2.8), (2.9) and (2.17). From (2.16) and (2.17), we know that $BA = AB$, $DA = AD$ and $BD = DB$. Thus, we may always choose a local orthonormal basis $\{E_1, E_2, \dots, E_n\}$ such that

$$A_{ij} = A_i \delta_{ij}, \quad B_{ij} = B_i \delta_{ij}, \quad D_{ij} = D_i \delta_{ij}, \quad (3.1)$$

where A_i , B_i and D_i are the Blaschke eigenvalues, the Möbius principal curvatures and the para-Blaschke eigenvalues of the immersion \mathbf{x} . We may prove the following:

Proposition 3.1. *Let \mathbf{x} be an n ($n \geq 3$)-dimensional immersed para-Blaschke isoparametric hypersurface in $S^{n+1}(1)$ with three distinct para-Blaschke eigenvalues, one of which is simple. If the para-Blaschke tensor is not parallel, then \mathbf{x} is a Möbius isoparametric hypersurface with three or four distinct Möbius principal curvatures.*

Proof of Proposition 3.1. Let D_1 , D_2 and D_3 be the three distinct constant para-Blaschke eigenvalues of \mathbf{x} with multiplicities m_1 , m_2 and m_3 . From (2.18), we have

$$D_{ij,k} = E_k(D_i)\delta_{ij} + \Gamma_{ik}^j(D_i - D_j), \quad (3.2)$$

where Γ_{ik}^j is the Levi-Civita connection for the Möbius metric g given by

$$\omega_{ij} = \sum_k \Gamma_{ik}^j \omega_k, \quad \Gamma_{ik}^j = -\Gamma_{jk}^i. \quad (3.3)$$

Thus, by (3.2), we have

$$D_{ij,k} = \Gamma_{ik}^j(D_i - D_j). \quad (3.4)$$

It follows that

$$D_{ab,k} = D_{pq,k} = D_{\alpha\beta,k} = 0 \quad \text{for any } a, b, p, q, \alpha, \beta, k. \quad (3.5)$$

Since the para-Blaschke tensor is not parallel, we see that the only possible nonzero elements in $\{D_{ij,k}\}$ are of the form $\{D_{ap,\alpha}\}$. Since $n \geq 3$, without loss of generality, we may assume that $m_3 = 1$, $m_1 \geq 1$ and $m_2 \geq 1$.

From (2.10), (2.5) and (3.3), the curvature tensor of \mathbf{x} may be given by (see [10])

$$\begin{aligned} R_{ijkl} = & E_l(\Gamma_{ik}^j) - E_k(\Gamma_{il}^j) + \sum_m \Gamma_{im}^j \Gamma_{lk}^m \\ & - \sum_m \Gamma_{im}^j \Gamma_{kl}^m + \sum_m \Gamma_{ik}^m \Gamma_{ml}^j - \sum_m \Gamma_{il}^m \Gamma_{mk}^j. \end{aligned} \quad (3.6)$$

Thus, from (3.4) and (3.5), we have

$$\Gamma_{ab}^p = \Gamma_{ab}^\alpha = 0, \quad \Gamma_{pq}^a = \Gamma_{pq}^\alpha = 0, \quad \Gamma_{\alpha\beta}^a = \Gamma_{\alpha\beta}^p = 0, \quad (3.7)$$

$$\Gamma_{a\alpha}^p = \frac{D_{ap,\alpha}}{D_1 - D_2}, \quad \Gamma_{\alpha p}^a = \frac{D_{\alpha a,p}}{D_3 - D_1}, \quad \Gamma_{pa}^\alpha = \frac{D_{p\alpha,a}}{D_2 - D_3}. \quad (3.8)$$

From (3.7) and (3.8), we have

$$\Gamma_{nn}^a = \Gamma_{nn}^p = 0, \quad \Gamma_{aa}^n = \Gamma_{pp}^n = 0. \quad (3.9)$$

$$\Gamma_{an}^p = \frac{D_{ap,n}}{D_1 - D_2}, \quad \Gamma_{nb}^p = \frac{D_{bp,n}}{D_3 - D_2}, \quad \Gamma_{bq}^n = \frac{D_{bq,n}}{D_1 - D_3}, \quad (3.10)$$

$$\Gamma_{qb}^n = \frac{D_{bq,n}}{D_2 - D_3}.$$

Thus, we have

$$R_{apbq} = \Gamma_{an}^p \Gamma_{qb}^n - \Gamma_{an}^p \Gamma_{bq}^n - \Gamma_{aq}^n \Gamma_{nb}^p = \frac{D_{ap,n} D_{bq,n} + D_{aq,n} D_{bp,n}}{(D_1 - D_3)(D_2 - D_3)}. \quad (3.11)$$

On the other hand, from (2.15), we have

$$\begin{aligned} R_{apbq} &= (B_a B_p + A_a + A_p) \delta_{ab} \delta_{pq} \\ &= \{(B_a - \lambda)(B_p - \lambda) + D_1 + D_2 - \lambda^2\} \delta_{ab} \delta_{pq}. \end{aligned} \quad (3.12)$$

It follows from (3.11) and (3.12) that

$$\frac{D_{ap,n} D_{bq,n} + D_{aq,n} D_{bp,n}}{(D_1 - D_3)(D_2 - D_3)} = \{(B_a - \lambda)(B_p - \lambda) + D_1 + D_2 - \lambda^2\} \delta_{ab} \delta_{pq}. \quad (3.13)$$

If $a = b$, we have

$$\frac{2D_{ap,n} D_{aq,n}}{(D_1 - D_3)(D_2 - D_3)} = \{(B_a - \lambda)(B_p - \lambda) + D_1 + D_2 - \lambda^2\} \delta_{pq}. \quad (3.14)$$

If $p = q$, we have

$$\frac{2D_{ap,n} D_{bp,n}}{(D_1 - D_3)(D_2 - D_3)} = \{(B_a - \lambda)(B_p - \lambda) + D_1 + D_2 - \lambda^2\} \delta_{ab}. \quad (3.15)$$

If $m_1 = 1$, it follows that

$$\frac{2D_{1p,n} D_{1q,n}}{(D_1 - D_3)(D_2 - D_3)} = \{(B_1 - \lambda)(B_p - \lambda) + D_1 + D_2 - \lambda^2\} \delta_{pq}. \quad (3.16)$$

Since the para-Blaschke tensor is not parallel, we may prove that there exists exactly one p , such that $D_{1p,n} \neq 0$. In fact, if there exists at least two p_1, p_2 , ($p_1 \neq p_2$) such that $D_{1p_1,n} \neq 0$, $D_{1p_2,n} \neq 0$. But from (3.16), we have $D_{1p_1,n} D_{1p_2,n} = 0$, this is a contradiction.

If $m_2 = 1$, it follows that

$$\frac{2D_{am_1+1,n} D_{bm_1+1,n}}{(D_1 - D_3)(D_2 - D_3)} = \{(B_a - \lambda)(B_{m_1+1} - \lambda) + D_1 + D_2 - \lambda^2\} \delta_{ab}.$$

The same reason implies that there exists exactly one a , such that $D_{am_1+1,n} \neq 0$.

If $m_1 \geq 2$ and $m_2 \geq 2$, we may prove that there exists exactly one a and exactly one p such that $D_{ap,n} \neq 0$. In fact, if there exists at least two a_1, a_2 , ($a_1 \neq a_2$) such that $D_{a_1p,n} \neq 0$, $D_{a_2p,n} \neq 0$. From (3.15), we see that $D_{a_1p,n} D_{a_2p,n} = 0$, this is a

contradiction. The same reason implies that there exists exactly one p , such that $D_{ap,n} \neq 0$.

Combining the above three cases, we see that if $m_1 \geq 1$ and $m_2 \geq 1$, there exists exactly one a and exactly one p , say a_1 and p_1 , such that

$$D_{a_1 p_1, n} \neq 0, \quad D_{ap,n} = 0, \quad \text{for } a \neq a_1, \forall p, \quad \text{or for } \forall a, p \neq p_1. \quad (3.17)$$

By (3.14) and (3.17), we get

$$(B_{a_1} - \lambda)(B_{p_1} - \lambda) + D_1 + D_2 - \lambda^2 = \frac{2D_{a_1 p_1, n}^2}{(D_1 - D_3)(D_2 - D_3)}, \quad (3.18)$$

$$(B_a - \lambda)(B_p - \lambda) + D_1 + D_2 - \lambda^2 = 0, \quad a \neq a_1, \quad p \neq p_1, \quad (3.19)$$

$$(B_a - \lambda)(B_{p_1} - \lambda) + D_1 + D_2 - \lambda^2 = 0, \quad a \neq a_1, \quad (3.20)$$

$$(B_{a_1} - \lambda)(B_p - \lambda) + D_1 + D_2 - \lambda^2 = 0, \quad p \neq p_1. \quad (3.21)$$

From (3.4)–(3.6), (2.15) and for the reason above, we get

$$(B_{a_1} - \lambda)(B_n - \lambda) + D_1 + D_3 - \lambda^2 = \frac{2D_{a_1 p_1, n}^2}{(D_1 - D_2)(D_3 - D_2)}, \quad (3.22)$$

$$(B_a - \lambda)(B_n - \lambda) + D_1 + D_3 - \lambda^2 = 0, \quad a \neq a_1 \quad (3.23)$$

$$(B_{p_1} - \lambda)(B_n - \lambda) + D_2 + D_3 - \lambda^2 = \frac{2D_{a_1 p_1, n}^2}{(D_2 - D_1)(D_3 - D_1)}, \quad (3.24)$$

$$(B_p - \lambda)(B_n - \lambda) + D_2 + D_3 - \lambda^2 = 0, \quad p \neq p_1. \quad (3.25)$$

We may prove that $D_{a_1 p_1, n}$ is a constant. In fact, from (2.20), (3.3), (3.4) and (3.5), we have

$$\sum_l D_{ab,pl} \omega_l = \sum_l \frac{D_{nb,p} D_{na,l} + D_{na,p} D_{nb,l}}{D_3 - D_1} \omega_l.$$

Thus

$$D_{ab,pl} = \frac{D_{nb,p} D_{na,l} + D_{na,p} D_{nb,l}}{D_3 - D_1}, \quad \forall a, b, p, l. \quad (3.26)$$

By reasoning as above, we also have

$$D_{pq,al} = \frac{D_{na,p} D_{nl,q} + D_{na,q} D_{nl,p}}{D_3 - D_2}, \quad \forall a, p, q, l, \quad (3.27)$$

$$D_{nn,ap} = 0, \quad \forall a, p. \quad (3.28)$$

From (2.21), we have $D_{ij,kl} - D_{ij,lk} = (D_i - D_j)R_{ijkl}$. By (2.15) and $B_{ij} = B_i \delta_{ij}$, we know that if three of $\{i, j, k, l\}$ are distinct, then $R_{ijkl} = 0$. Thus, if three of $\{i, j, k, l\}$ are distinct, we have

$$D_{ij,kl} = D_{ij,lk}. \quad (3.29)$$

From (2.20), (3.4) and (3.5), we have

$$\begin{aligned} dD_{a_1p_1,n} = & D_{a_1p_1,na_1}\omega_{a_1} + \sum_{l=a, a \neq a_1} D_{a_1p_1, nl}\omega_l + D_{a_1p_1, np_1}\omega_{p_1} \\ & + \sum_{l=p, p \neq p_1} D_{a_1p_1, nl}\omega_l + D_{a_1p_1, nn}\omega_n. \end{aligned}$$

Thus, it follows from (3.26)–(3.29) that $dD_{a_1p_1,n} = 0$, that is, $D_{a_1p_1,n}$ is constant.

It follows from (3.23) and (3.25) that $B_a = B_b$ for any $a \neq b, a \neq a_1, b \neq a_1$ and $B_p = B_q$ for any $p \neq q, p \neq p_1, q \neq p_1$. Therefore, we know that \mathbf{x} has at most five distinct Möbius principal curvatures: $B_a, B_{a_1}, B_p, B_{p_1}, B_n$.

From (3.20) and (3.23), (3.21) and (3.25), (3.22) and (3.23), we have

$$(B_a - \lambda)(B_{p_1} - B_n) + D_2 - D_3 = 0, \quad a \neq a_1, \quad (3.30)$$

$$(B_p - \lambda)(B_{a_1} - B_n) + D_1 - D_3 = 0, \quad p \neq p_1, \quad (3.31)$$

$$(B_{a_1} - B_a)(B_n - \lambda) = \frac{2D_{a_1p_1,n}^2}{(D_1 - D_2)(D_3 - D_2)}, \quad a \neq a_1. \quad (3.32)$$

Thus, we obtain that $B_a \neq \lambda (a \neq a_1)$, $B_p \neq \lambda (p \neq p_1)$, $B_n \neq \lambda$. From (3.18), (3.22) and (3.24), we see that

$$\Delta_1(B_n - \lambda)^2 = \Delta_2\Delta_3, \quad (3.33)$$

where

$$\Delta_1 = \frac{2D_{a_1p_1,n}^2}{(D_1 - D_3)(D_2 - D_3)} + \lambda^2 - D_1 - D_2,$$

$$\Delta_2 = \frac{2D_{a_1p_1,n}^2}{(D_1 - D_2)(D_3 - D_2)} + \lambda^2 - D_1 - D_3,$$

$$\Delta_3 = \frac{2D_{a_1p_1,n}^2}{(D_2 - D_1)(D_3 - D_1)} + \lambda^2 - D_2 - D_3,$$

are constants. If $\Delta_1 = 0$, from (3.18), we get $B_{a_1} = \lambda$ or $B_{p_1} = \lambda$. It follows from (3.20) and (3.21) that $D_1 + D_2 - \lambda^2 = 0$. Thus (3.18) implies that $D_{a_1p_1,n} = 0$, a contradiction. Therefore, we know that $\Delta_1 \neq 0$. From (3.33) and $B_n \neq \lambda$, we get $\Delta_2 \neq 0$ and $\Delta_3 \neq 0$. Thus $(B_n - \lambda)^2 = \frac{\Delta_2\Delta_3}{\Delta_1}$ is constant, that is, B_n is constant. From (3.22)–(3.25), we see that $B_{a_1}, B_a (a \neq a_1), B_{p_1}, B_p (p \neq p_1)$ are constants.

We may prove that $m_2 = 1$. In fact, if $m_2 \geq 2$, then there exists some $p \neq p_1$ such that (3.18)–(3.25) hold. It follows from (3.19), (3.23) and (3.25) that $D_1 + D_2 - \lambda^2 \neq 0, D_1 + D_3 - \lambda^2 \neq 0, D_2 + D_3 - \lambda^2 \neq 0$. From (3.20) and (3.23), (3.21) and (3.25), we get

$$B_{p_1} - \lambda = \frac{\lambda^2 - D_1 - D_2}{\lambda^2 - D_1 - D_3}(B_n - \lambda), \quad B_{a_1} - \lambda = \frac{\lambda^2 - D_1 - D_2}{\lambda^2 - D_2 - D_3}(B_n - \lambda). \quad (3.34)$$

From (3.22) and (3.24), we have

$$\frac{(B_{a_1} - \lambda)(B_n - \lambda) + D_1 + D_3 - \lambda^2}{D_3 - D_1} + \frac{(B_{p_1} - \lambda)(B_n - \lambda) + D_2 + D_3 - \lambda^2}{D_3 - D_2} = 0.$$

From (3.34) and by a direct calculation, we see that

$$\begin{aligned} & \frac{(\lambda^2 - D_1 - D_2)\{\lambda^2(D_1 + D_2 - 2D_3) + 2(D_3^2 - D_1D_2)\}}{(\lambda^2 - D_2 - D_3)(\lambda^2 - D_1 - D_3)(D_1 - D_3)(D_2 - D_3)} \\ & \times \left\{ (B_n - \lambda)^2 - \frac{(\lambda^2 - D_1 - D_3)(\lambda^2 - D_2 - D_3)}{\lambda^2 - D_1 - D_2} \right\} = 0. \end{aligned} \quad (3.35)$$

If $(B_n - \lambda)^2 = \frac{(\lambda^2 - D_1 - D_3)(\lambda^2 - D_2 - D_3)}{\lambda^2 - D_1 - D_2}$, by (3.22) and (3.34), we see that $D_{a_1 p_1, n} = 0$, this is a contradiction. Thus from (3.35), we get

$$\lambda^2(D_1 + D_2 - 2D_3) + 2(D_3^2 - D_1D_2) = 0. \quad (3.36)$$

Similarly, from (3.18) and (3.22), we get

$$\frac{(B_{a_1} - \lambda)(B_{p_1} - \lambda) + D_1 + D_2 - \lambda^2}{D_1 - D_2} + \frac{(B_{a_1} - \lambda)(B_n - \lambda) + D_1 + D_3 - \lambda^2}{D_1 - D_3} = 0.$$

From (3.34) and by a direct calculation, we see that

$$\begin{aligned} & \frac{\lambda^2(2D_1 - D_2 - D_3) - 2(D_1^2 - D_2D_3)}{(\lambda^2 - D_1 - D_3)(D_1 - D_2)(D_1 - D_3)} \\ & \times \left\{ (B_{a_1} - \lambda)(B_n - \lambda) + D_1 + D_3 - \lambda^2 \right\} = 0. \end{aligned}$$

From (3.22), we know that $(B_{a_1} - \lambda)(B_n - \lambda) + D_1 + D_3 - \lambda^2 \neq 0$. Thus

$$\lambda^2(2D_1 - D_2 - D_3) - 2(D_1^2 - D_2D_3) = 0. \quad (3.37)$$

If $\lambda = 0$, we see from (3.36) and (3.37) that $D_1 = D_3$, a contradiction. If $\lambda \neq 0$, by a direct calculation, we see from (3.36) and (3.37) that $D_1^2 + D_2^2 + D_3^2 = D_1D_2 + D_2D_3 + D_1D_3$. Thus $(D_1 - D_2)^2 + (D_2 - D_3)^2 + (D_3 - D_1)^2 = 2(D_1^2 + D_2^2 + D_3^2 - D_1D_2 - D_2D_3 - D_1D_3) = 0$. It follows that $D_1 = D_2 = D_3$, also a contradiction. Therefore, we know that $m_2 = 1$ and \mathbf{x} has at most four distinct constant Möbius principal curvatures: $B_a, B_{a_1}, B_{m_1+1}, B_n$. From (2.11), we know that the number of the distinct Möbius principal curvatures is not one. We may also see that the number of the distinct Möbius principal curvatures is not two. In fact, if it is two, from Li et al [7], we know that the Möbius second fundamental form is parallel, this indicates that, from Lemma 4.6 of [15], the Blaschke tensor is parallel. Thus the para-Blaschke tensor is parallel, a contradiction. Therefore, we see that the number of the distinct Möbius principal curvatures is only three or four. This completes the proof of Proposition 3.1. \square

Proof of Theorem 1.1 (Main Theorem). If the para-Blaschke tensor is parallel, since \mathbf{x} has three distinct para-Blaschke eigenvalues, from the result of Cheng, Li and Qi [3], we know that \mathbf{x} is locally Möbius equivalent to $CSS(p, q, r)$ for some constants p, q, r given by Example 2.1.

If the para-Blaschke tensor is not parallel, from Proposition 3.1, we know that \mathbf{x} is a Möbius isoparametric hypersurface with three or four distinct Möbius principal curvatures. Since the para-Blaschke eigenvalues and the Möbius principal curvatures are constant, we see that \mathbf{x} has constant Blaschke eigenvalues and the normalized Möbius scalar curvature R is constant by (2.11). From $g = \rho^2 d\mathbf{x} \cdot d\mathbf{x}$, we can choose

a local orthonormal basis for $d\mathbf{x} \cdot d\mathbf{x}$ such that $h_{ij} = \lambda_i \delta_{ij}$, and $B_{ij} = B_i \delta_{ij}$, where λ_i and B_i are the Euclidean principal curvatures and the Möbius principal curvatures, respectively. From (2.9), we have

$$B_i = \rho^{-1}(\lambda_i - H). \quad (3.38)$$

Thus $B_i - B_j = \rho^{-1}(\lambda_i - \lambda_j)$, this implies that the number of the distinct Euclidean principal curvatures is also three or four. Let r be the normalized Euclidean scalar curvature. We know that $r = R$. From (2.15), we have

$$R_{ijij} = B_i B_j + A_i + A_j = (B_i - \lambda)(B_j - \lambda) + D_i + D_j - \lambda^2, \quad i \neq j. \quad (3.39)$$

By (3.18)–(3.25) and (3.39), we have

$$\begin{aligned} R_{a_1 p_1 a_1 p_1} &= \frac{2D_{a_1 p_1, n}^2}{(D_1 - D_3)(D_2 - D_3)}, \quad R_{apap} = 0, \quad \text{for } a \neq a_1, \quad p \neq p_1, \\ R_{ap_1 a_1 p_1} &= 0, \quad \text{for } a \neq a_1, \quad R_{a_1 p_1 a_1 p} = 0, \quad \text{for } p \neq p_1, \\ R_{a_1 n a_1 n} &= \frac{2D_{a_1 p_1, n}^2}{(D_1 - D_2)(D_3 - D_2)}, \quad R_{anan} = 0, \quad \text{for } a \neq a_1, \\ R_{p_1 n p_1 n} &= \frac{2D_{a_1 p_1, n}^2}{(D_2 - D_1)(D_3 - D_1)}, \quad R_{pnpn} = 0, \quad \text{for } p \neq p_1. \end{aligned}$$

Thus

$$\begin{aligned} R &= \frac{1}{n(n-1)} \left\{ R_{a_1 p_1 a_1 p_1} + \sum_{a \neq a_1, p \neq p_1} R_{apap} + \sum_{a \neq a_1} R_{ap_1 a_1 p_1} + \sum_{p \neq p_1} R_{a_1 p_1 a_1 p} \right. \\ &\quad \left. + R_{a_1 n a_1 n} + \sum_{a \neq a_1} R_{anan} + R_{p_1 n p_1 n} + \sum_{p \neq p_1} R_{pnpn} \right\} = 0. \end{aligned}$$

It follows that $r = 0$. From the Gaussian equation $n(n-1)(r-1) = n^2 H^2 - \sum_{i,j} h_{ij}^2$, we have $\rho^2 = n^2(1 + H^2)$. From (2.7), we have

$$0 = H_{,i} + \rho B_i e_i (\log n \sqrt{1 + H^2}) = \frac{\sqrt{1 + H^2} + n B_i H}{\sqrt{1 + H^2}} H_{,i}. \quad (3.40)$$

We may prove that H must be constant. In fact, if H is not constant, then there is some i such that $H_{,i} \neq 0$. Thus, for such i $\sqrt{1 + H^2} + n B_i H = 0$, that is, $1 + (1 - n^2 B_i^2)H^2 = 0$. Thus, we see that $1 - n^2 B_i^2 \neq 0$ and $H^2 = \frac{1}{n^2 B_i^2 - 1}$ is constant, a contradiction. Since H is constant, we see that ρ is also constant. It follows from (3.38) that λ_i are constants for any i . Thus, we see that \mathbf{x} is locally Möbius equivalent to a Euclidean isoparametric hypersurface with three or four distinct Euclidean principal curvatures. This completes the proof of Theorem 1.1 (Main Theorem). \square

4. Conclusions

To sum up, in this article, we study the topic of para-Blaschke isoparametric hypersurfaces and obtain the classification of para-Blaschke isoparametric hypersurfaces with three distinct para-Blaschke eigenvalues, one of which is simple. The

obtained result is the sequential study of the topic in [11], where we classified the para-Blaschke isoparametric hypersurfaces with three distinct Blaschke eigenvalues, one of which is simple or with three distinct Möbius principal curvatures, one of which is simple. We should notice that our result generalizes the results of [15] and [16] (Theorem 4.3 of [15] and Theorem 4.2 of [16]) to high dimensional para-Blaschke isoparametric hypersurfaces in the unit sphere $S^{n+1}(1)$.

Acknowledgement Project supported by NSF of Shaanxi Educational Department (11JK0479, 2010JK642).

REFERENCES

- [1] E. Cartan, Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques, *Math. Z.*, **45** (1939), 335–367.
- [2] Q-M. Cheng, S.C. Shu, A Möbius characterization of submanifolds, *J. Math. Soc. Japan*, **58** (2006), 903–925.
- [3] Q-M. Cheng, X.X. Li, X.R. Qi, A classification of hypersurfaces with parallel para-Blaschke tensor in S^{m+1} , *International J. Math.*, **21** (2010), 297–316.
- [4] Z.J. Hu, H. Li, Classification of hypersurfaces with parallel Möbius second fundamental form in $(n+1)$ -dimensional sphere, *Science in China Ser.A Math.*, **47** (2004), 417–430.
- [5] Z.J. Hu, D.Y. Li, Möbius isoparametric hypersurfaces with three distinct principal curvatures, *Pacific Math J.*, **232** (2007), 289–311.
- [6] Z.J. Hu, H. Li, C.P. Wang, Classification of Möbius isoparametric hypersurfaces in S^5 , *Monatsh. Math.*, **151** (2007), 202–222.
- [7] H. Li, H.L. Liu, C.P. Wang and G.S. Zhao, Möbius isoparametric hypersurface in S^{n+1} with two distinct principal curvatures, *Acta Math. Sinica, English Series*, **18** (2002), 437–446.
- [8] H. Li and C.P. Wang, Möbius geometry of hypersurfaces with constant mean curvature and constant scalar curvature, *Manuscripta Math.*, **112** (2003), 1–13.
- [9] X.X. Li and Y.J. Peng, Classification of Blaschke isoparametric hypersurfaces with three distinct Blaschke eigenvalues, *Results Math.*, **58** (2010), 145–172.
- [10] G.H. Li, Möbius hypersurfaces in S^{n+1} with three distinct principal curvatures, *J. Geom.*, **80** (2004), 154–165.
- [11] S.C. Shu and B.P. Su, Para-Blaschke isoparametric hypersurfaces in a unit sphere $S^{n+1}(1)$, *Glasgow Math. J.*, **54** (2012), 579–597.
- [12] C.P. Wang, Möbius geometry for hypersurfaces in S^4 , *Nagoya Math. J.*, **139** (1995), 1–20.
- [13] C.P. Wang, Möbius geometry of submanifolds in S^n , *Manuscripta Math.*, **96** (1998), 517–534.

- [14] D.X. Zhong and H.A. Sun, The hypersurfaces in a unit sphere with constant para-Blaschke eigenvalues, *Acta Math. Sinica, Chinese Series*, **51** (2008), 579–592.
- [15] D.X. Zhong and H.A. Sun, The hypersurfaces in S^4 with constant para-Blaschke eigenvalues, *Advances in Math., Chinese Series*, **37** (2008), 657–669.
- [16] D.X. Zhong, H.A. Sun and T.F. Zhang, The hypersurfaces in S^5 with constant para-Blaschke eigenvalues, *Acta Math. Sinica, Chinese Series*, **53** (2010), 263–278.