

## A TWO-STEP DECOUPLED FINITE ELEMENT ALGORITHM FOR A NONLINEAR FLUID-FLUID INTERACTION PROBLEM

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*In this work we design a two-step decoupled finite element algorithm for a nonlinear fluid-fluid interaction model. In the first step we apply the geometric averaging method and use the implicit-explicit method in the second step. Besides, in order to enlarge the time step, we introduce some diffusion terms in both steps of the algorithm. Finally, several numerical tests prove that the algorithm has a good accuracy, which shows that the algorithm can simulate this nonlinear fluid-fluid interaction model well.*

**Keywords:** fluid-fluid interaction problem, two-step decoupled algorithm, large time step, small viscosity.

**MSC2010:** 65M60, 65M12.

### 1. Introduction

In this article, we mainly design a two-step decoupled finite element algorithm for a nonlinear fluid-fluid interaction model, which is studied in [13]. The bounded domain  $\Omega$  that we will consider for the model is comprised by two subdomains  $\Omega_1$  and  $\Omega_2$ , which are coupled across their shared interface  $I$ . Both the subdomains are subsets of  $\mathbb{R}^2$  and have outward unit normal vectors  $n_i$ . Besides,  $\tau$  is any vector such that  $\tau \cdot n_i = 0$ . Note that nonlinear interface conditions on  $I$  are applied in this fluid-fluid model and the interface  $I$  is assumed a line segment. Given kinematic viscosities  $\nu_i > 0$  ( $i = 1, 2$ ), friction coefficient  $\kappa \in \mathbb{R}$ , body force  $f_i : [0, T] \rightarrow H^1(\Omega_i)^2$  and initial velocity  $u_i^0 \in H^1(\Omega_i)^2$ , find the velocity  $u_i : [0, T] \times \Omega_i \rightarrow \mathbb{R}^2$  and pressure  $p_i : [0, T] \times \Omega_i \rightarrow \mathbb{R}$  satisfying:

$$\begin{aligned}
 & u_{i,t} - \nu_i \Delta u_i + u_i \cdot \nabla u_i + \nabla p_i = f_i && \text{in } \Omega_i, \\
 & -\nu_i n_i \cdot \nabla u_i \cdot \tau = \kappa |u_i - u_j| (u_i - u_j) \cdot \tau && \text{on } I, \ i, j = 1, 2, \ i \neq j, \\
 & u_i \cdot n_i = 0 && \text{on } I, \\
 & \nabla \cdot u_i = 0 && \text{in } \Omega_i, \\
 & u_i(0, x) = u_i^0(x) && \text{in } \Omega_i, \\
 & u_i = 0 && \text{on } \Gamma_i = \partial\Omega_i \setminus I,
 \end{aligned} \tag{1}$$

where  $|\cdot|$  represents the Euclidean norm.

As it is known, many important applications require an accurate solution of multi-domain and multi-physics coupling of one fluid with another fluid [3, 4, 10, 35]. In fact, the

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model (1) that is one of them arises in many important scientific, engineering and industrial applications, such as heterogeneous of blood flow [14] and atmosphere-ocean interaction [24]. Although this model can reduce the dynamic core of the atmosphere-ocean problem to its simplest form, it still retains some essential difficulties of the problem. Besides, in terms of numerical methods, there remains some important challenging problems. The main difficulties are the coupling of the pressure, the incompressibility conditions, the nonlinearity term and the complicated coupled system with some nonlinear interaction conditions. In fact, a review of numerical coupling strategies is provided in [23], where an alternative coupling method is proposed and analyzed. But with very large problems, decoupled algorithms are preferred. Therefore, much effort has been developed to the development of efficient decoupled numerical methods for investigating the model (1).

Using an operator-splitting method and some optimization-based nonoverlapping domain decomposition methods, Bresch and Koko [7] have presented a numerical simulation of the considered model by solving one coupled degenerated Stokes problem and one uncoupled linear advection-diffusion problem. However, uncoupling can only occur in the residual calculation and in the preconditioning step. Hence, Connors et al. [13] have presented two decoupled time stepping methods, i.e., the implicit-explicit (IMEX) method and geometric averaging (GA) method. The GA method is unconditionally stable while the IMEX method is only conditionally stable, which is proved by Zhang et al. [34], although it is the simplest and most natural decoupled method [12]. Another unconditionally stable method is given by Connors and Howell [11]. They have applied decoupled subproblems and differing time steps strategies to solve the fluid-fluid interaction model. Recently, a local projection stabilization and characteristic decoupled time stepping scheme is proposed by Qian et al. [26] for the fluid-fluid interaction problem. They apply a GA method to deal with the nonlinear interface condition, which yields an unconditionally stable partitioned method. Besides, Connors [9] proposes a statistical turbulence model for ensemble calculations with two fluids coupled across a flat interface. Aggul et al. [1] develop a predictor-corrector-type method that is also an unconditionally stable scheme but has a second-order time accuracy.

Different from the above methods for the fluid-fluid interaction model (1), a two-step decoupled finite element algorithm will be designed, which will solve numerically the considered model with small viscosity and large time step. As is known, there is an abundance of literature regarding two-step finite element methods. Chorin [8] and Temam [32] have designed a projection algorithm, which undergoes some evolution and is well further developed, such as the pressure correction methods [16, 28] and the matrix factorization methods [27]. Chorin/Temam's algorithm, which is based on the projection of an intermediate velocity field onto the space of vector fields and changes the difficult linear solve of a saddle point system into two easier linear solves, can be improved by making the pressure explicit in the viscous step and by correcting it in the projection step. With the help of a pressure correction term added to the projection step, based on some regularity hypotheses of the exact solution, Shen [29, 30] has improved the estimates of the Chorin/Temam projection method for the intermediate velocity and the pressure. However, the end-of-step velocity of the projection method does not satisfy the exact boundary conditions. Hence, according to the idea of Chorin/Temam projection algorithm, the fractional step methods [17, 31] are proposed, where the pressure is determined by just solving one Poisson equation per time step, which greatly reduces the computational cost.

The viscosity-splitting method is a class of fractional step methods and the convergence and stability of the method have been proved in [15]. More recently, a first-order time-discrete splitting scheme using decomposition of the viscosity (called viscosity-splitting method [6]) is proposed for solving the incompressible time-dependent Navier-Stokes equations in three-dimensional domains. Under mild regularity assumptions on the continuous solution, an error analysis is provided [5]. Further, optimal error estimates are obtained by Guillén-González and Redondo-Nebale [18], where a weight at the initial time step must be included to deduce the optimal error estimates for the pressure. Besides, they have used a time-discrete scheme as an auxiliary problem to study a fully discrete finite element scheme and obtained optimal first order approximation for velocity and pressure [19]. In addition, the viscosity-splitting method is applied to the primitive equations of the ocean [20] and Boussinesq problem [37]. Zhang et al. [36] have introduced a large time stepping viscosity-splitting finite element method for the viscoelastic flow problem under stability condition.

In this paper, inspired by [18] and [36], we will design a two-step decoupled finite element algorithm which can solve the considered model with small viscosity and large time step. Besides, compared with the IMEX algorithm, this new scheme is unconditionally stable, which will be proved in Section 3. The rest of the paper is arranged as follows: In the next section, we introduce some mathematical preliminaries and provide the corresponding variational form for the problem (1). In Section 3, we show the two-step decoupled finite element algorithm for the nonlinear fluid-fluid interaction model. Besides, the stability of the two-step decoupled method is given. Then in Section 4, numerical experiment is implemented to verify the theoretical results. We end with a short conclusion in the last section.

## 2. Notation and preliminaries

In this section, we denote the usual  $L^2(\Omega_i)$  norm and its inner product by  $\|\cdot\|_0$  and  $(\cdot, \cdot)$ , respectively. The  $L^p(\Omega_i)$  norms and the Sobolev  $W_p^m(\Omega_i)$  norms are denoted by  $\|\cdot\|_{L^p(\Omega_i)}$  and  $\|\cdot\|_{W_p^m(\Omega_i)}$  for  $m \in \mathbb{N}^+$ ,  $1 \leq p \leq \infty$ . In particular,  $H^m(\Omega_i)$  is used for the Sobolev space  $W_2^m(\Omega_i)$ .  $\|\cdot\|_m$  denotes the norm in  $H^m(\Omega_i)$ .

For the mathematical setting of the fluid-fluid interaction problem (1), we introduce the following Hilbert spaces:

$$X_i = \{v_i \in H^1(\Omega_i)^2; v_i|_{\partial\Omega_i \setminus I} = 0; v_i \cdot n_i = 0 \text{ on } I\}, \quad i = 1, 2,$$

and

$$M_i = \{q_i \in L^2(\Omega_i); (q_i, 1) = 0\}, \quad i = 1, 2.$$

For  $f_i$  an element in the dual space of  $X_i$ , its norm is defined by

$$\|f_i\|_{-1} = \sup_{v_i \in X_i} \frac{(f_i, v_i)}{\|\nabla v_i\|_0}.$$

Based on the above definitions of functional spaces, the corresponding variational formulation of problem (1) is given as follows: find  $u_i : [0, T] \rightarrow X_i$  and  $p_i : [0, T] \rightarrow M_i$  for  $i, j = 1, 2, i \neq j$  such that

$$\begin{aligned} (u_{i,t}, v_i) + v_i a(u_i, v_i) - d(v_i, p_i) + b(u_i, u_i, v_i) + \int_I \kappa |u_i - u_j| (u_i - u_j) v_i ds &= (f_i, v_i), \\ d(u_i, q_i) &= 0, \quad \forall (v_i, q_i) \in (X_i, M_i), \end{aligned} \quad (2)$$

where  $(u_{i,t}, v_i) = \int_{\Omega_i} \frac{\partial u_i}{\partial t} v_i d\Omega_i$ ,  $i = 1, 2$ , and the continuous bilinear forms  $a(\cdot, \cdot)$  and  $d(\cdot, \cdot)$  are defined on  $X_i \times X_i$  and  $X_i \times M_i$ , respectively, by

$$\begin{aligned} a(u_i, v_i) &= (\nabla u_i, \nabla v_i), \quad u_i, v_i \in X_i, \\ d(v_i, q_i) &= -(v_i, \nabla q_i) = (\operatorname{div} v_i, q_i), \quad v_i \in X_i, \quad q_i \in M_i. \end{aligned}$$

For functions  $u_i, v_i, w_i \in X_i$ , the trilinear term  $b(\cdot, \cdot, \cdot)$  [25, 33] is defined as

$$\begin{aligned} b(u_i, v_i, w_i) &= ((u_i \cdot \nabla) v_i, w_i) + \frac{1}{2}((\operatorname{div} u_i) v_i, w_i) \\ &= \frac{1}{2}((u_i \cdot \nabla) v_i, w_i) - \frac{1}{2}((u_i \cdot \nabla) w_i, v_i). \end{aligned}$$

### 3. A two-step decoupled finite element algorithm for the nonlinear fluid-fluid interaction problem

Firstly, we denote by  $X_i^h \subset X_i$  and  $M_i^h \subset M_i$  the finite element spaces for velocity and pressure under a regular partition  $\pi_i^h$  of the subdomain  $\Omega_i$ , with the largest diameter  $h_i$  for  $\pi_i^h$ . Furthermore, the finite element space pair  $X_i^h \times M_i^h$  is assumed to satisfy the usual discrete inf-sup condition or LBB<sup>h</sup> condition for stability of the discrete pressure:

$$\inf_{q_i^h \in M_i^h} \sup_{v_i^h \in X_i^h} \frac{d(q_i^h, v_i^h)}{\|\nabla v_i^h\|_0 \|q_i^h\|_0} = \beta_i^h > 0,$$

where  $\beta_i^h$  is a constant that is independent of  $h_i$ . For the subdomain  $\Omega_j$ , we set the same definition and then we choose  $h = \max\{h_i, h_j\}$  as the maximum triangle diameter in  $\Omega$ . In this paper, the spatial discretization is accomplished using the conforming P1+bubble/P1 element pair (the MINI-element [2]).

Secondly, let  $\{t_n\}_{n=0}^N$  be a uniform partition of  $[0, T]$  and  $t_n = n\Delta t$  with the time step  $\Delta t > 0$ . For a function  $f$  defined on  $[0, T]$ , we set  $f^n = f(t_n)$ . Besides, we define  $(u_i^n, p_i^n)$  to be an approximation of the solution of (1) at  $t = t_n$ .

In the following algorithm, we split the nonlinear term  $(u_i \cdot \nabla) u_i$  and the incompressibility condition  $\nabla \cdot u_i = 0$  in two steps. Meanwhile, we enforce the same homogeneous Dirichlet boundary conditions on  $\partial\Omega_i \setminus I$ .

**Algorithm 3.1.** The first step: Given  $u_i^{n-1}, u_i^n \in X_i^h$  with  $1 \leq n \leq N$ , find  $u_i^{n+\frac{1}{2}} \in X_i^h$  satisfying

$$\begin{aligned} &\left( \frac{u_i^{n+\frac{1}{2}} - u_i^n}{\Delta t}, v_i \right) + (\theta_i + \nu_i) a(u_i^{n+\frac{1}{2}}, v_i) - \theta_i a(u_i^n, v_i) + b(u_i^n, u_i^{n+\frac{1}{2}}, v_i) \\ &+ \int_I \kappa |u_i^n - u_j^n| u_i^{n+\frac{1}{2}} v_i ds - \int_I \kappa |u_i^{n-1} - u_j^{n-1}|^{\frac{1}{2}} |u_i^n - u_j^n|^{\frac{1}{2}} u_j^n v_i ds = (f_i^{n+1}, v_i), \end{aligned} \quad (3)$$

for all  $v_i \in X_i^h$  and where  $\theta_i > 0$  are bounded parameters.

The second step: Based on  $u_i^{n+\frac{1}{2}}$  from (3), find  $u_i^{n+1} \in X_i^h$  and  $p_i^{n+1} \in M_i^h$  satisfying

$$\begin{aligned} &\left( \frac{u_i^{n+1} - u_i^{n+\frac{1}{2}}}{\Delta t}, v_i \right) + (\theta_i + \nu_i) a(u_i^{n+1} - u_i^{n+\frac{1}{2}}, v_i) - d(p_i^{n+1}, v_i) \\ &+ \int_I \kappa |u_i^n - u_j^n| u_i^{n+1} v_i ds - \int_I \kappa |u_i^n - u_j^n| u_j^{n+\frac{1}{2}} v_i ds = 0, \quad \forall v_i \in X_i^h, \\ &d(u_i^{n+1}, q_i) = 0, \quad \forall q_i \in M_i^h. \end{aligned} \quad (4)$$

Obviously, if we take  $\theta_i = 0$ , then Algorithm 3.1 becomes the viscosity-splitting method (the reader can refer to [18] for more details). Note that  $\theta_i$  are used to enlarge the time step and enhance numerical stability for the small viscosity problem. The technique has been adopted to consider the Cahn-Hilliard equation [22] and the viscoelastic flow problem [36].

In the above algorithm, for the trilinear term  $b(\cdot, \cdot, \cdot)$  we apply the semi-implicit scheme [21] which yields a linear system with a variable coefficient matrix and can save a lot of computational cost. Moreover, for the nonlinear fluid-fluid interaction, in the first step we apply the GA method while in the second step we use the IMEX method. The advantage of using the two-step decoupled scheme (3) and (4) rather than a coupled method is the decoupling of the convective effects from incompressibility.

In the following part of this section, we hope to gain stability results concerning the unconditional stability of Algorithm 3.1.

**Theorem 3.1.** *Let  $u_i^{n+1}$ ,  $i = 1, 2$ ,  $n = 0, 1, 2, \dots, m$  ( $m \leq N - 1$ ), be the solution of Algorithm 3.1. Then we have*

$$\begin{aligned}
& \sum_{i=1}^2 \|u_i^{m+1}\|_0^2 + \sum_{i=1}^2 \sum_{n=1}^m (\|u_i^{n+\frac{1}{2}} - u_i^n\|_0^2 + \|u_i^{n+1} - u_i^{n+\frac{1}{2}}\|_0^2) \\
& + \sum_{i=1}^2 \sum_{n=1}^m [\theta_i \Delta t \|\nabla(u_i^{n+\frac{1}{2}} - u_i^n)\|_0^2 + \Delta t (\theta_i + v_i) \|\nabla(u_i^{n+1} - u_i^{n+\frac{1}{2}})\|_0^2] \\
& + \sum_{i=1}^2 \theta_i \Delta t \|\nabla u_i^{m+1}\|_0^2 + \sum_{i=1}^2 \sum_{n=1}^m v_i \Delta t \|\nabla u_i^{n+1}\|_0^2 + \sum_{i=1}^2 \Delta t \int_I \kappa |u_i^n - u_j^m| (u_i^{m+1})^2 ds \\
& + \sum_{i=1}^2 \sum_{n=1}^m [\Delta t \int_I \kappa (|u_i^n - u_j^n|^{\frac{1}{2}} u_i^{n+\frac{1}{2}} - |u_i^{n-1} - u_j^{n-1}|^{\frac{1}{2}} u_j^{n+\frac{1}{2}})^2 ds \\
& + \Delta t \int_I \kappa (|u_i^n - u_j^n|^{\frac{1}{2}} u_i^{n+1} - |u_i^n - u_j^n|^{\frac{1}{2}} u_j^{n+\frac{1}{2}})^2 ds] \\
& \leq \sum_{i=1}^2 \|u_i^1\|_0^2 + \sum_{i=1}^2 \theta_i \Delta t \|\nabla u_i^1\|_0^2 + \sum_{i=1}^2 \Delta t \int_I \kappa |u_i^0 - u_j^0| (u_j^1)^2 ds + \sum_{i=1}^2 \sum_{n=1}^m \left( \frac{\Delta t}{v_i} \|f_i^{n+1}\|_{-1}^2 \right).
\end{aligned} \tag{5}$$

**Proof.** Firstly, taking  $v_i = 2\Delta t u_i^{n+\frac{1}{2}}$  in (3), it follows that

$$\begin{aligned}
& \|u_i^{n+\frac{1}{2}}\|_0^2 - \|u_i^n\|_0^2 + \|u_i^{n+\frac{1}{2}} - u_i^n\|_0^2 + 2\Delta t v_i \|\nabla u_i^{n+\frac{1}{2}}\|_0^2 \\
& + \theta_i \Delta t (\|\nabla u_i^{n+\frac{1}{2}}\|_0^2 - \|\nabla u_i^n\|_0^2 + \|\nabla(u_i^{n+\frac{1}{2}} - u_i^n)\|_0^2) + 2\Delta t \int_I \kappa |u_i^n - u_j^n| (u_i^{n+\frac{1}{2}})^2 ds \\
& - 2\Delta t \int_I \kappa |u_i^{n-1} - u_j^{n-1}|^{\frac{1}{2}} |u_i^n - u_j^n|^{\frac{1}{2}} u_i^{n+\frac{1}{2}} ds \\
& = 2\Delta t (f_i^{n+1}, u_i^{n+\frac{1}{2}}) \leq \Delta t v_i \|\nabla u_i^{n+\frac{1}{2}}\|_0^2 + \frac{\Delta t}{v_i} \|f_i^{n+1}\|_{-1}^2.
\end{aligned} \tag{6}$$

Then, for the interface terms in the above equation, by simple calculation, there holds

$$\begin{aligned}
& 2\Delta t \int_I \kappa |u_i^n - u_j^n| (u_i^{n+\frac{1}{2}})^2 ds - 2\Delta t \int_I \kappa |u_i^{n-1} - u_j^{n-1}|^{\frac{1}{2}} |u_i^n - u_j^n|^{\frac{1}{2}} u_j^n u_i^{n+\frac{1}{2}} ds \\
& = \Delta t \int_I \kappa |u_i^n - u_j^n| (u_i^{n+\frac{1}{2}})^2 ds - \Delta t \int_I \kappa |u_i^{n-1} - u_j^{n-1}| (u_j^n)^2 ds \\
& \quad + \Delta t \int_I \kappa (|u_i^n - u_j^n|^{\frac{1}{2}} u_i^{n+\frac{1}{2}} - |u_i^{n-1} - u_j^{n-1}|^{\frac{1}{2}} u_j^n)^2 ds.
\end{aligned} \tag{7}$$

Further, combining (6) and (7), we have

$$\begin{aligned}
& \|u_i^{n+\frac{1}{2}}\|_0^2 - \|u_i^n\|_0^2 + \|u_i^{n+\frac{1}{2}} - u_i^n\|_0^2 + \Delta t v_i \|\nabla u_i^{n+\frac{1}{2}}\|_0^2 \\
& \quad + \theta_i \Delta t (\|\nabla u_i^{n+\frac{1}{2}}\|_0^2 - \|\nabla u_i^n\|_0^2 + \|\nabla(u_i^{n+\frac{1}{2}} - u_i^n)\|_0^2) + \Delta t \int_I \kappa |u_i^n - u_j^n| (u_i^{n+\frac{1}{2}})^2 ds \\
& \quad - \Delta t \int_I \kappa |u_i^{n-1} - u_j^{n-1}| (u_j^n)^2 ds + \Delta t \int_I \kappa (|u_i^n - u_j^n|^{\frac{1}{2}} u_i^{n+\frac{1}{2}} - |u_i^{n-1} - u_j^{n-1}|^{\frac{1}{2}} u_j^n)^2 ds \\
& \leq \frac{\Delta t}{v_i} \|f_i^{n+1}\|_{-1}^2.
\end{aligned} \tag{8}$$

Secondly, choosing  $v_i = 2\Delta t u_i^{n+1}$  and  $q_i = 2\Delta t p_i^{n+1}$  in (4), we obtain

$$\begin{aligned}
& \|u_i^{n+1}\|_0^2 - \|u_i^{n+\frac{1}{2}}\|_0^2 + \|u_i^{n+1} - u_i^{n+\frac{1}{2}}\|_0^2 \\
& \quad + (\theta_i + v_i) \Delta t (\|\nabla u_i^{n+1}\|_0^2 - \|\nabla u_i^{n+\frac{1}{2}}\|_0^2 + \|\nabla(u_i^{n+1} - u_i^{n+\frac{1}{2}})\|_0^2) \\
& \quad + 2\Delta t \int_I \kappa |u_i^n - u_j^n| (u_i^{n+1})^2 ds - 2\Delta t \int_I \kappa |u_i^n - u_j^n| u_j^{n+\frac{1}{2}} u_i^{n+1} ds = 0.
\end{aligned} \tag{9}$$

As for the interface terms, we have

$$\begin{aligned}
& 2\Delta t \int_I \kappa |u_i^n - u_j^n| (u_i^{n+1})^2 ds - 2\Delta t \int_I \kappa |u_i^n - u_j^n| u_j^{n+\frac{1}{2}} u_i^{n+1} ds \\
& = \Delta t \int_I \kappa |u_i^n - u_j^n| (u_i^{n+1})^2 ds - \Delta t \int_I \kappa |u_i^n - u_j^n| (u_j^{n+\frac{1}{2}})^2 ds \\
& \quad + \Delta t \int_I \kappa (|u_i^n - u_j^n|^{\frac{1}{2}} u_i^{n+1} - |u_i^n - u_j^n|^{\frac{1}{2}} u_j^{n+\frac{1}{2}})^2 ds.
\end{aligned} \tag{10}$$

Then, combining (9) with (10), we arrive at

$$\begin{aligned}
& \|u_i^{n+1}\|_0^2 - \|u_i^{n+\frac{1}{2}}\|_0^2 + \|u_i^{n+1} - u_i^{n+\frac{1}{2}}\|_0^2 \\
& \quad + (\theta_i + v_i) \Delta t (\|\nabla u_i^{n+1}\|_0^2 - \|\nabla u_i^{n+\frac{1}{2}}\|_0^2 + \|\nabla(u_i^{n+1} - u_i^{n+\frac{1}{2}})\|_0^2) \\
& \quad + \Delta t \int_I \kappa |u_i^n - u_j^n| (u_i^{n+1})^2 ds - \Delta t \int_I \kappa |u_i^n - u_j^n| (u_j^{n+\frac{1}{2}})^2 ds \\
& \quad + \Delta t \int_I \kappa (|u_i^n - u_j^n|^{\frac{1}{2}} u_i^{n+1} - |u_i^n - u_j^n|^{\frac{1}{2}} u_j^{n+\frac{1}{2}})^2 ds = 0.
\end{aligned} \tag{11}$$

Finally, according to (8) and (11), we get

$$\begin{aligned}
& \|u_i^{n+1}\|_0^2 - \|u_i^n\|_0^2 + \|u_i^{n+\frac{1}{2}} - u_i^n\|_0^2 + \|u_i^{n+1} - u_i^{n+\frac{1}{2}}\|_0^2 + \Delta t(\theta_i + \nu_i)\|\nabla u_i^{n+1}\|_0^2 \\
& + \Delta t(\theta_i + \nu_i)\|\nabla(u_i^{n+1} - u_i^{n+\frac{1}{2}})\|_0^2 + \theta_i\Delta t\|\nabla(u_i^{n+\frac{1}{2}} - u_i^n)\|_0^2 - \theta_i\Delta t\|\nabla u_i^n\|_0^2 \\
& + \Delta t \int_I \kappa |u_i^n - u_j^n|(u_i^{n+\frac{1}{2}})^2 ds - \Delta t \int_I \kappa |u_i^{n-1} - u_j^{n-1}|(u_j^n)^2 ds \\
& + \Delta t \int_I \kappa (|u_i^n - u_j^n|^{\frac{1}{2}} u_i^{n+\frac{1}{2}} - |u_i^{n-1} - u_j^{n-1}|^{\frac{1}{2}} u_j^n)^2 ds + \Delta t \int_I \kappa |u_i^n - u_j^n|(u_i^{n+1})^2 ds \\
& - \Delta t \int_I \kappa |u_i^n - u_j^n|(u_j^{n+\frac{1}{2}})^2 ds + \Delta t \int_I \kappa (|u_i^n - u_j^n|^{\frac{1}{2}} u_i^{n+1} - |u_i^n - u_j^n|^{\frac{1}{2}} u_j^{n+\frac{1}{2}})^2 ds \\
& \leq \frac{\Delta t}{\nu_i} \|f_i^{n+1}\|_{-1}^2.
\end{aligned} \tag{12}$$

Due to

$$\begin{aligned}
& \sum_{i=1}^2 \sum_{n=1}^m \left( \Delta t \int_I \kappa |u_i^n - u_j^n|(u_i^{n+\frac{1}{2}})^2 ds - \Delta t \int_I \kappa |u_i^n - u_j^n|(u_j^{n+\frac{1}{2}})^2 ds \right) = 0, \\
& \sum_{i=1}^2 \sum_{n=1}^m \left( \Delta t \int_I \kappa |u_i^n - u_j^n|(u_i^{n+1})^2 ds - \Delta t \int_I \kappa |u_i^{n-1} - u_j^{n-1}|(u_j^n)^2 ds \right) \\
& = \sum_{i=1}^2 \Delta t \int_I \kappa |u_i^m - u_j^m|(u_i^{m+1})^2 ds - \sum_{i=1}^2 \Delta t \int_I \kappa |u_i^0 - u_j^0|(u_j^1)^2 ds,
\end{aligned}$$

summing (12) over  $i = 1, 2$  and  $n = 1, 2, \dots, m$ , yields inequality (5).

#### 4. Numerical experiment

In this section, some numerical tests are presented to show the effectiveness of Algorithm 3.1. We assume  $\Omega_1 = [0, 1] \times [0, 1]$ ,  $\Omega_2 = [0, 1] \times [-1, 0]$  and  $I = (0, 1) \times \{0\}$ , and consider the following analytic solutions of the problem (1)

$$\begin{aligned}
u_{1,1}(t, x, y) &= -ax^2 \exp(-t)(x-1)^2(y-1), \\
u_{1,2}(t, x, y) &= axy \exp(-t)(6x+y-3xy+2x^2y-4x^2-2), \\
u_{2,1}(t, x, y) &= -ax \exp(-t)(x-1) \left( y^2x(x-1) \left( \frac{\nu_1}{\nu_2} + 1 \right) - \frac{\nu_1^{1/2} y^2 \exp(t/2)}{(a\kappa)^{1/2}} \right. \\
&\quad \left. - x(x-1) + \frac{\nu_1^{1/2} \exp(t/2)}{(a\kappa)^{1/2}} + \frac{\nu_1 xy(x-1)}{\nu_2} \right),
\end{aligned}$$

and

$$\begin{aligned}
u_{2,2}(t, x, y) &= -\frac{ay \exp(-t)(2x-1)}{3\nu_2(a\kappa)^{1/2}} \left( 6\nu_2 x^2(a\kappa)^{1/2} - 6\nu_2 x(a\kappa)^{1/2} - 3\nu_1^{1/2} \nu_2 \exp(t/2) \right. \\
&\quad \left. - 2\nu_1 x^2 y^2(a\kappa)^{1/2} - 2\nu_2 x^2 y^2(a\kappa)^{1/2} + 3\nu_1 xy(a\kappa)^{1/2} + 2\nu_1 xy^2(a\kappa)^{1/2} \right. \\
&\quad \left. - 3\nu_1 x^2 y(a\kappa)^{1/2} + 2\nu_2 xy^2(a\kappa)^{1/2} + \nu_1^{1/2} \nu_2 y^2 \exp(t/2) \right), \\
p_1(t, x, y) &= p_2(t, x, y) = \exp(-t) \cos(\pi x) \sin(\pi y),
\end{aligned}$$

where  $a > 0$  is a parameter.

The right-hand sides  $f_1 = (f_{1,1}(t, x, y), f_{1,2}(t, x, y))$  and  $f_2 = (f_{2,1}(t, x, y), f_{2,2}(t, x, y))$  are taken so as to tender to  $(u_1, p_1)$  and  $(u_2, p_2)$  as solutions of the problem (1).

In the following numerical tests, we use the parameter values  $\nu_1 = 0.5$ ,  $\nu_2 = 0.05$ ,  $a = 1$  and  $\kappa = 100$  as in [34]. Let  $(u_i(t_n), p_i(t_n))$  denote the continuous solution at  $t = t_n$  in  $\Omega_i, i = 1, 2$ , which is approximated by  $(u_i^n, p_i^n)$ . Then the errors are denoted by

$$Err(u_i) = \left( \Delta t \sum_{n=1}^N \|\nabla(u_i(t_n) - u_i^n)\|_0^2 \right)^{\frac{1}{2}}, \quad Err(p_i) = \left( \Delta t \sum_{n=1}^N \|p_i(t_n) - p_i^n\|_0^2 \right)^{\frac{1}{2}}.$$

#### 4.1. Convergence rate verification

On one hand, we test the convergence rates of Algorithm 3.1 with respect to  $h$ . In order to avoid the influence of the time step,  $\Delta t$  should be chosen small enough. Hence, we set  $\Delta t = 0.0001$  and  $0.001$  with  $T = 0.1$ , and take  $h = 1/4, 1/8, 1/16$  and  $1/32$  successively.

We display the convergence rates of Algorithm 3.1 ( $\theta_i \neq 0$ ) and the viscosity-splitting method ( $\theta_i = 0$ ) [18] in Table 1 and 2 with  $\Delta t = 0.0001$  and  $0.001$ , respectively. From Table 1 and 2, when the time step is small, it can be easily to see that both Algorithm 3.1 and the viscosity-splitting method work well and keep the optimal convergence rates. In fact, the expected first-order convergence rates for the velocity are achieved. However, the pressure convergence rates are above the predicted value, but decreasing rates with slightly large time step, because of the application of the MINI-element which shows a superconvergence behavior for the pressure approximation.

**Table 1** Convergence orders with respect to  $h$  with  $\Delta t = 0.0001$ .

$1/h$	$(\theta_1, \theta_2)$	$Err(u_1)$	Rate	$Err(u_2)$	Rate	$Err(p_1)$	Rate	$Err(p_2)$	Rate
4	(0,0)	0.05729	—	0.25725	—	0.20388	—	0.89323	—
8	(0,0)	0.02877	0.99	0.12363	1.06	0.05134	1.99	0.22398	2.00
16	(0,0)	0.01428	1.01	0.05955	1.05	0.01180	2.12	0.05200	2.11
32	(0,0)	0.00713	1.00	0.02962	1.01	0.00292	2.02	0.01283	2.02
4	(1,2)	0.05721	—	0.25725	—	0.19493	—	0.89323	—
8	(1,2)	0.02871	0.99	0.12363	1.04	0.04907	1.99	0.22398	2.00
16	(1,2)	0.01430	1.01	0.06045	1.03	0.01209	2.02	0.05476	2.03
32	(1,2)	0.00713	1.00	0.03009	1.01	0.00298	2.02	0.01346	2.02

**Table 2** Convergence orders with respect to  $h$  with  $\Delta t = 0.001$ .

$1/h$	$(\theta_1, \theta_2)$	$Err(u_1)$	Rate	$Err(u_2)$	Rate	$Err(p_1)$	Rate	$Err(p_2)$	Rate
4	(0,0)	0.05808	—	0.25972	—	0.06354	—	0.27132	—
8	(0,0)	0.02921	0.99	0.12422	1.06	0.01627	1.97	0.06732	2.01
16	(0,0)	0.01456	1.00	0.06064	1.03	0.00414	1.98	0.01647	2.03
32	(0,0)	0.00727	1.00	0.03022	1.00	0.00126	1.72	0.00404	2.03
4	(1,2)	0.05837	—	0.27253	—	0.07271	—	0.36398	—
8	(1,2)	0.02939	0.99	0.13816	0.98	0.01843	1.98	0.08976	2.02
16	(1,2)	0.01465	1.00	0.06907	1.00	0.00464	1.99	0.02189	2.04
32	(1,2)	0.00732	1.00	0.03461	1.00	0.00143	1.70	0.00562	1.96



On the other hand, we test the convergence rates of Algorithm 3.1 with respect to  $\Delta t$ . Here, we set  $\Delta t = h$ . Further, we take  $\Delta t = 1/4, 1/8, 1/16$  and  $1/32$  successively.

Table 3 lists the numerical results solving by Algorithm 3.1 and the viscosity-splitting method for  $T = 1$ . According to Table 3, when we choose  $\theta_i = 0$  (viscosity-splitting method), we find that the pressure convergence rates are decreasing; but based on Algorithm 3.1, by choosing suitable parameters  $\theta_i$ , we can obtain the good numerical results of pressure, i.e., the convergence order of velocity and pressure with respect to  $\Delta t$  is approximately 1, which is the optimal order for first-order time discretization.

**Table 3** Convergence orders with respect to  $\Delta t$ .

$1/\Delta t$	$(\theta_1, \theta_2)$	$Err(u_1)$	Rate	$Err(u_2)$	Rate	$Err(p_1)$	Rate	$Err(p_2)$	Rate
4	(0,0)	0.20323	—	0.84080	—	0.06115	—	0.02680	—
8	(0,0)	0.09050	1.17	0.37864	1.15	0.02771	1.14	0.01635	0.71
16	(0,0)	0.04030	1.17	0.17469	1.12	0.01157	1.26	0.00962	0.77
32	(0,0)	0.01789	1.17	0.08131	1.10	0.00414	1.48	0.00600	0.68
4	(0,0.2)	0.20456	—	0.90688	—	0.06512	—	0.14794	—
8	(0,0.2)	0.09345	1.13	0.42802	1.08	0.03402	0.94	0.04843	1.61
16	(0,0.2)	0.04029	1.21	0.19775	1.11	0.01132	1.59	0.01922	1.33
32	(0,0.2)	0.01773	1.18	0.08854	1.16	0.00382	1.57	0.01033	0.90
4	(0.1,0.2)	0.20394	—	0.90646	—	0.06667	—	0.14775	—
8	(0.1,0.2)	0.09301	1.13	0.42727	1.09	0.03366	0.99	0.04831	1.61
16	(0.1,0.2)	0.04058	1.20	0.19779	1.11	0.01176	1.52	0.01923	1.33
32	(0.1,0.2)	0.01767	1.20	0.08809	1.17	0.00395	1.57	0.00983	0.97

At the end of this subsection, we will test the performance of Algorithm 3.1 and viscosity-splitting method under larger time step. In Table 4, the numerical results solving by Algorithm 3.1 with different value of  $\theta_i$  and the viscosity-splitting method are listed for  $\Delta t = 0.01$  and  $T = 0.1$ . From this table, we can see that better convergence results are obtained with the larger value of  $\theta_i$ . Hence, for the nonlinear fluid-fluid interaction problem (1) solving with larger time step, the desired numerical results of both velocity and pressure can be gained by Algorithm 3.1 with the larger value of  $\theta_i$ , while the viscosity-splitting method can not get the desired convergence results.

From Table 4, we can see that the better convergence results can be obtained by Algorithm 3.1 with the larger value of  $\theta_i$ . However, it is questionable whether the best numerical results can be obtained when  $\theta_i \rightarrow \infty$ . In practice there are limitations on how large  $\theta_i$  may be chosen, because as  $\theta_i$  increases the resulting linear system becomes ill-conditioned. Hence, in the next subsection, we will consider the effect of parameters  $\theta_i$  for the fixed time step and mesh size.

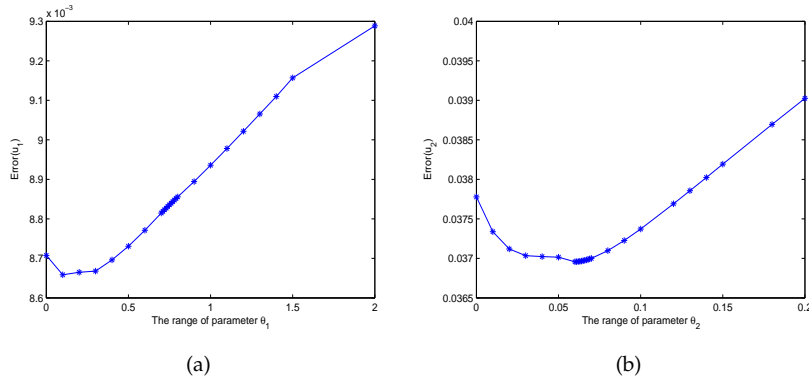
#### 4.2. The effectiveness of parameter $\theta_i$

This subsection is mainly devoted to the influence of  $\theta_i$  ( $i = 1, 2$ ) on the error. Figure 1 shows the effectiveness of various  $\theta_1$  and  $\theta_2$  to the accuracy of velocity in  $H^1$ -norm with the time step  $\Delta t = 0.01$  and  $1/h = 32$  at  $T = 0.1$ , respectively. One can see that the optimal  $\theta_i$  for the accuracy of velocity is not obtained from the largest value of  $\theta_i$ .

Next, we test the performance of Algorithm 3.1 and viscosity-splitting method for the nonlinear fluid-fluid interaction problem with small viscosities. The numerical errors with

**Table 4** Convergence rates with larger time step  $\Delta t = 0.01$  with  $T = 0.1$ .

$1/h$	$(\theta_1, \theta_2)$	$Err(u_1)$	Rate	$Err(u_2)$	Rate	$Err(p_1)$	Rate	$Err(p_2)$	Rate
4	(0,0)	0.06669	—	0.28958	—	0.02561	—	0.08837	—
8	(0,0)	0.03398	0.97	0.14309	1.02	0.00746	1.78	0.02168	2.03
16	(0,0)	0.01710	0.99	0.07166	1.00	0.00287	1.38	0.00610	1.83
32	(0,0)	0.00871	0.97	0.03778	0.92	0.00224	0.36	0.00345	0.82
4	(1,2)	0.06905	—	0.34704	—	0.05035	—	0.34747	—
8	(1,2)	0.03541	0.96	0.18544	0.90	0.01467	1.78	0.09853	1.82
16	(1,2)	0.01783	0.99	0.09467	0.97	0.00524	1.49	0.03024	1.70
32	(1,2)	0.00903	0.98	0.04795	0.98	0.00348	0.59	0.01352	1.16
4	(3,2)	0.07393	—	0.34703	—	0.10617	—	0.34746	—
8	(3,2)	0.03839	0.95	0.18544	0.90	0.03184	1.74	0.09852	1.82
16	(3,2)	0.01942	0.98	0.09467	0.97	0.01046	1.61	0.03022	1.71
32	(3,2)	0.00986	0.98	0.04795	0.98	0.00541	0.95	0.01348	1.16
4	(4,2)	0.07581	—	0.34703	—	0.13471	—	0.34746	—
8	(4,2)	0.03951	0.94	0.18544	0.90	0.04077	1.72	0.09852	1.82
16	(4,2)	0.02003	0.98	0.09467	0.97	0.01331	1.61	0.03021	1.71
32	(4,2)	0.01019	0.98	0.04795	0.98	0.00660	1.01	0.01347	1.17

**FIGURE 1.** Effectiveness of various  $\theta_1$  (a) and  $\theta_2$  (b) on the accuracy of velocity in  $H^1$ -norm.

various small viscosities at  $T = 0.1$  are listed in Table 5. One can see that compared with viscosity-splitting method the better errors are obtained by Algorithm 3.1. Therefore, Algorithm 3.1 can solve the nonlinear fluid-fluid interaction problem (1) with small viscosities well.

## 5. Conclusion

In this work we have designed a two-step decoupled finite element algorithm for a nonlinear fluid-fluid interaction model. In the first step we apply the GA method while in the second step we use the IMEX method. The advantage of the two-step decoupled scheme rather than a coupled method is the decoupling of the convective effects from incompressibility. Besides, the decoupled scheme can solve the considered model with smaller

**Table 5** The error of velocity for the small viscosities and with  $\Delta t = 0.01$ .

$(\theta_1, \theta_2)$	$(\nu_1, \nu_2)$	$Err(u_1)$	$Err(u_2)$
(0,0)	(5E-5,5E-5)	0.232333	0.236648
(0,0)	(5E-5,5E-6)	0.232974	0.284403
(0,0)	(5E-6,5E-6)	0.258397	0.265243
(1,0.1)	(5E-5,5E-5)	0.012174	0.019347
(1,0.1)	(5E-5,5E-6)	0.012172	0.050380
(1,0.1)	(5E-6,5E-6)	0.012175	0.019410

viscosity and larger time step. We prove stability results of the method. Numerically, we investigate the performances of the method by the some numerical tests. All computational results demonstrate the effectiveness of the method, showing that the method is suitable to solve the nonlinear fluid-fluid interaction model with smaller viscosity.

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