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DECAY OF A POTENTIAL VORTEX IN A TIME FRACTIONAL SECOND GRADE FLUID

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Exact solutions for the dimensionless velocity field and the adequate non-trivial shear stress corresponding to a potential vortex through a time fractional second grade fluid have been established by means of the integral transform technique. The known similar solutions for ordinary or fractional Newtonian fluids, as well as those for ordinary second grade fluids, are easily obtained as limiting cases of these solutions. The influence of fractional parameter on the fluid motion, as well as a comparison with Newtonian fluids is graphically depicted and discussed. In all cases the vortex decreases in time and space and the diagrams corresponding to fractional fluids tend to superpose over those of ordinary fluids when the fractional parameter tends to one.

Keywords: Second grade fluids, potential vortex, fractional model, exact solutions.

1. Introduction

Usually, a vortex is associated with the rotating motion of a fluid around a common centerline. The fluid vorticity, defined as the “curl” of the fluid velocity, is a measure of the rate of local fluid rotation. Vortices arise in nature and technology in a large range of sizes [1] like tornadoes, hurricanes or vortices in superfluids. Atmospheric vortices are generated by temperature gradients, the Coriolis force due to the Earth's rotation and spatial land scape variations and instabilities. Temperature differences between poles and Equator and the Earth's rotation can also lead to vortices such as polar vortex, polar jet stream or subtropical jet stream. As it results from the work of Gieser [2], C.W. Oseen (1879-1944) studied the vortices and formulated a mathematical model of vortex with exponential azimuthally velocity. The frequent occurrence of vortices in nature and technology determined the researchers in fluid mechanics to study both their generation and evolution.

The velocity field corresponding to the decay of a potential vortex in a Newtonian fluid has been determined by Zierep [3] using similarity by transformation of variables. Since similarity solutions for motions of non-Newtonian fluids seem to not exist (see Taipel [4] for the motion of a second grade fluid over an infinite plate), the decay of a potential vortex in second grade

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or Oldroyd-B fluids has been studied using the Hankel transform by Fetecau C. and Fetecau Corina [5]. Recently, Zierep's results have been extended to Newtonian fluids with time fractional derivative [6]. The corresponding non-trivial shear stress, as well as the circulation on a circle of arbitrary radius, has been also determined.

It is well known the fact that the fractional models are more flexible in describing the complex behavior of viscoelastic materials. In the last time, they gained much importance and popularity due to their vast potential of applications in various fields including rheology. The first authors who applied fractional calculus in viscoelasticity have been Bagley and Torvik [7] and a very good agreement with experimental results using fractional derivatives has been obtained by Caputo and Mainardy [8, 9]. Moreover, Makris et al. [10] used experimental data to calibrate a fractional Maxwell model. More precisely, they determined the value of fractional parameter in order to have an excellent agreement between experimental and theoretical results. On the other hand, the behavior of viscoelastic fluids depends on the flow history and the memory formalism can be represented by means of fractional derivatives. For an interesting review regarding applications and the importance of fractional calculus see Sheoran et al. [11].

Bearing in mind the above-mentioned remarks, as well as the increasing interest of fractional models in different domains of science, the purpose of this paper is to extend the results from [6] to fractional second grade fluids. More exactly, it uses the computational advantages of the new fractional derivative with non-singular kernel defined by Caputo and Fabrizio [12] in order to see how the fractional parameter affects the fluid motion due to a potential vortex. To do that, exact solutions are established for dimensionless velocity and shear stress fields. These solutions, which are presented in integral form in terms of Bessel functions $J_1(\cdot)$ and $J_2(\cdot)$, can be easily reduced to the similar solutions corresponding to ordinary second grade fluids and ordinary or fractional Newtonian fluids. The influence of fractional parameter on the fluid motion, as well as a comparison with the Newtonian fluid is graphically underlined and discussed.

2. Statement of the Problem

In the following it is considered the circular motion of a fractional second grade fluid whose velocity field, in a cylindrical coordinate system r, θ and z, is

$$v = v(r,t) = \left(0, \omega(r,t), 0\right) \tag{1}$$

The initial distribution of velocity is assumed to be that of a potential vortex of circulation Γ_0, namely [3]

$$\omega(r,0) = \frac{\Gamma_0}{2\pi r}. \tag{2}$$

For such a motion the continuity equation is identically satisfied while the constitutive relationship of second grade fluids and the motion equations reduce to the relevant equations

$$\tau(r,t) = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r,t); \quad \rho \frac{\partial \omega(r,t)}{\partial t} = \frac{\partial \tau(r,t)}{\partial r} + \frac{2}{r}\tau(r,t), \tag{3}$$

where $\tau(r,t)$ is the non-trivial shear stress, μ and ρ are the fluid viscosity and density and α_1 is a material constant. The value of the circulation $\Gamma(r,t)$ on a circle of radius r is given by [3]

$$\Gamma(r,t) = 2\pi r \omega(r,t). \tag{4}$$

Eliminating $\tau(r,t)$ between Eqs. (3), it results that

$$\frac{\partial \omega(r,t)}{\partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t); \quad r,t > 0, \tag{5}$$

where $\nu = \mu / \rho$ is the kinematic viscosity and $\alpha = \alpha_1 / \rho$.

In order to provide solutions that are free of the flow geometry, let us introduce the next non-dimensional variables, functions or constants

$$t^* = \frac{t}{t_0}, \quad r^* = \frac{r}{\sqrt{\nu t_0}}, \quad \omega^* = \omega\sqrt{\frac{t_0}{\nu}}, \quad \tau^* = \frac{t_0}{\mu}\tau, \quad \alpha^* = \frac{\alpha}{\nu t_0}, \quad \Gamma^* = \frac{\Gamma}{\nu}, \tag{6}$$

where t_0 is a characteristic time. Substituting Eqs. (6) in (2), $(3)_1$, (4) and (5) and dropping out the star notation, it is found that

$$\frac{\partial \omega(r,t)}{\partial t} = \left(1 + \alpha \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t); \quad \omega(r,0) = \frac{\Gamma_0}{2\pi r}, \tag{7}$$

$$\tau(r,t) = \left(1 + \alpha \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r,t); \quad \Gamma(r,t) = 2\pi r \omega(r,t). \tag{8}$$

Since the flow domain is unbounded, the natural conditions

$$\omega(r,t), \quad \frac{\partial \omega(r,t)}{\partial r} \to 0; \quad \text{as} \quad r \to \infty \quad \text{and} \quad t > 0, \tag{9}$$

have to be also satisfied. They assure the fact that the fluid is quiescent at infinity and there is no shear in the free stream [13, 14].

The dimensionless fractional model corresponding to this problem is based on the fractional partial differential equation

$$D_t^\beta \omega(r,t) = (1 + \alpha D_t^\beta)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t); \quad r,t > 0, \tag{10}$$

with the initial condition $(7)_2$. Temporal Caputo-Fabrizio fractional derivative [12]

$$D_t^{\beta}\omega(r,t)=\frac{1}{1-\beta}\int_0^t\frac{\partial\omega(r,s)}{\partial s}\exp\left[-\frac{\beta(t-s)}{1-\beta}\right]ds;\quad 0<\beta<1, \tag{11}$$

satisfies the useful properties

$$\lim_{\alpha\to 1}D_t^{\beta}\omega(r,t)=\frac{\partial\omega(r,t)}{\partial t},\quad L\{D_t^{\beta}\omega(r,t)\}=\frac{q\overline{\omega}(r,q)-\omega(r,0)}{(1-\beta)q+\beta}, \tag{12}$$

where $\overline{\omega}(r,q)=L\{\omega(r,t)\}$ is the Laplace transform of $\omega(r,t)$ and q is the transform parameter.

3. Solution of the problem

In order to solve this problem, the integral transform technique is used. Denoting by $\omega_H(\rho,t)$ the Hankel transform of $\omega(r,t)$, then [15]

$$\omega_H(\rho,t)=\int_0^{\infty}r\omega(r,t)J_1(\rho r)dr\quad\text{and}\quad\omega(r,t)=\int_0^{\infty}\rho\omega_H(\rho,t)J_1(\rho r)d\rho. \tag{13}$$

Applying the Laplace transform to Eq. (10) and bearing in mind Eqs. $(7)_2$ and $(12)_2$, it results that

$$q\overline{\omega}(r,q)-\frac{\Gamma_0}{2\pi r}=[(\alpha+1-\beta)q+\beta]\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\overline{\omega}(r,q). \tag{14}$$

Lengthy but straightforward computations show that [15]

$$\int_0^{\infty}rJ_1(\rho r)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\omega(r,t)dr=-\rho^2\omega_H(\rho,t), \tag{15}$$

if the following conditions

$$\lim_{r\to 0}r\omega(r,t)=0,\ \lim_{r\to\infty}r\omega(r,t)<\infty,\ \lim_{r\to 0}r\frac{\partial\omega(r,t)}{\partial r}<\infty,\ \lim_{r\to\infty}r\frac{\partial\omega(r,t)}{\partial r}<\infty, \tag{16}$$

are satisfied. Consequently, it is assumed that all conditions (16) are fulfilled.

Now, multiplying Eq. (14) by $rJ_1(\rho r)$, integrating the result with respect to r from zero to infinity and using Eq. (15), it results that

$$\overline{\omega}_H(\rho,q)=\frac{\Gamma_0}{2\pi\rho}\frac{1}{\rho^2(\alpha+1-\beta)q+1}\frac{1}{q+\dfrac{\rho^2\beta}{\rho^2(\alpha+1-\beta)+1}}. \tag{17}$$

Finally, successively applying the inverse Laplace and Hankel transforms to the equality (17), one obtains the dimensionless velocity field in the form

$$\omega(r,t)=\frac{\Gamma_0}{2\pi}\int_0^{\infty}\frac{J_1(\rho r)}{\rho^2(\alpha+1-\beta)+1}\exp\left(-\frac{\rho^2\beta t}{\rho^2(\alpha+1-\beta)+1}\right)d\rho. \tag{18}$$

Introducing Eq. (18) in (8), the corresponding expressions

$$\tau(r,t) = -\frac{\Gamma_0}{2\pi}\int_0^\infty \frac{\rho J_2(\rho r)}{\rho^2(\alpha+1-\beta)+1}\exp\left(-\frac{\rho^2\beta t}{\rho^2(\alpha+1-\beta)+1}\right)d\rho$$
$$+\frac{\Gamma_0}{2\pi}\alpha\beta\int_0^\infty \frac{\rho^3 J_2(\rho r)}{[\rho^2(\alpha+1-\beta)+1]^2}\exp\left(-\frac{\rho^2\beta t}{\rho^2(\alpha+1-\beta)+1}\right)d\rho, \tag{19}$$

$$\Gamma(r,t) = r\Gamma_0\int_0^\infty \frac{J_1(\rho r)}{\rho^2(\alpha+1-\beta)+1}\exp\left(-\frac{\rho^2\beta t}{\rho^2(\alpha+1-\beta)+1}\right)d\rho, \tag{20}$$

of the adequate non-dimensional shear stress and circulation are obtained.

However, a simple analysis shows that the initial condition $(7)_2$ is not satisfied. Indeed, making $t=0$ in Eq. (18) and bearing in mind Eq. (A1) from Appendix, it results that

$$\omega(r,0) = \frac{\Gamma_0}{2\pi r} - \frac{\Gamma_0}{2\pi\sqrt{\alpha+1-\beta}}K_1\left(\frac{r}{\sqrt{\alpha+1-\beta}}\right), \tag{21}$$

where $K_1(\cdot)$ is the modified Bessel function of second kind and order one. Consequently, the initial condition is not satisfied and the obtained solution seems to be wrong. In order to remove this doubt, another way will be followed to show that the last equality is correct.

For this, let us write Eq. (14) in the equivalent form

$$r^2\overline{\omega}''(r,q) + r\overline{\omega}'(r,q) - [1+a(q)r^2]\overline{\omega}(r,q) = rb(q), \tag{22}$$

where $a(q) = \dfrac{q}{(\alpha+1-\beta)q+\beta}$ and $b(q) = -\dfrac{\Gamma_0}{2\pi}\dfrac{1}{(\alpha+1-\beta)q+\beta}$ and $\overline{\omega}'(r,q)$ is the partial derivative of $\overline{\omega}(r,q)$ with respect to *r*.

The general solution of Eq. (22) can be written in the form

$$\overline{\omega}(r,q) = \frac{\Gamma_0}{2\pi rq} + C_1 I_1\left(r\sqrt{a(q)}\right) + C_2 K_1\left(r\sqrt{a(q)}\right); \quad r>0, \tag{23}$$

where C_1 and C_2 are constants and $I_1(\cdot)$ is the modified Bessel function of the first kind and order one. Bearing in mind the conditions $(9)_1$, $(16)_1$ and the approximate evaluation (A_2) from Appendix, it results that $C_1 = 0$ and $C_2 = -\dfrac{\Gamma_0\sqrt{a(q)}}{2\pi q}$. Consequently, Eq. (23) can be written in the form

$$q\overline{\omega}(r,q) = \frac{\Gamma_0}{2\pi}\left\{\frac{1}{r} - a(q)K_1\left(r\sqrt{a(q)}\right)\right\}; \quad r>0. \tag{24}$$

Now, taking the limit of Eq. (24) when $q\to\infty$ and using the property

(A3), Eq. (21) is recovered. Consequently, this result is correct and the solution (18) does not satisfy the initial condition. Similar results have been also obtained by Bandelli et al. [14] and Bandelli and Rajagopal [16] for two different motions of the same fluids. Their solutions do not satisfy the initial conditions due to the incompatibility between the prescribed data. However, as it will be later seen, the limit of the solution (18) when $\alpha \to 0$ and $\beta \to 1$ satisfies the initial condition.

Nevertheless, in order to give a more suitable evaluation of the magnitude of the deviation from the initial value, namely

$$\omega(r,0) = \frac{\Gamma_0}{2\pi r} - \frac{\Gamma_0}{2\sqrt{2\pi r}(\alpha+1-\beta)^{1/4}} \exp\left(-\frac{r}{\sqrt{\alpha+1-\beta}}\right), \tag{25}$$

the approximate evaluation (A7) for the function $K_1(\cdot)$ has been used. For large values of r, as expected, this deviation becomes negligible. It tends to zero for $r \to \infty$. Of course, taking the limit of Eq. (25) when $\alpha \to 0$ and $\beta \to 1$, as it was already mentioned before the initial condition $(7)_2$ is recovered.

4. Limiting Cases. Numerical Results

Case 1. $\alpha = 0$ (time fractional Newtonian fluid)

Making $\alpha = 0$ in Eq. (18)-(20), the solutions

$$\omega(r,t) = \frac{\Gamma_0}{2\pi} \int_0^\infty \frac{J_1(\rho r)}{\rho^2(1-\beta)+1} \exp\left(-\frac{\rho^2 \beta t}{\rho^2(1-\beta)+1}\right) d\rho, \tag{26}$$

$$\tau(r,t) = -\frac{\Gamma_0}{2\pi} \int_0^\infty \frac{\rho J_2(\rho r)}{\rho^2(1-\beta)+1} \exp\left(-\frac{\rho^2 \beta t}{\rho^2(1-\beta)+1}\right) d\rho, \tag{27}$$

$$\Gamma(r,t) = r\Gamma_0 \int_0^\infty \frac{J_1(\rho r)}{\rho^2(1-\beta)+1} \exp\left(-\frac{\rho^2 \beta t}{\rho^2(1-\beta)+1}\right) d\rho, \tag{28}$$

corresponding to the decay of potential vortex in a time fractional Newtonian fluid are obtained. These solutions are in accordance with those obtained in [6] where the dimensionless entities r, ω and Γ have been taken in a little different form.

Case 2. $\beta = 1$ (ordinary second grade fluid)

By now letting $\beta \to 1$ in Eq. (18)-(20), the solutions corresponding to the decay of potential vortex in a second grade fluid, namely

$$\omega(r,t) = \frac{\Gamma_0}{2\pi} \int_0^\infty \frac{J_1(\rho r)}{\alpha\rho^2+1} \exp\left(-\frac{\rho^2 t}{\alpha\rho^2+1}\right) d\rho, \tag{29}$$

$$\tau(r,t) = -\frac{\Gamma_0}{2\pi}\int_0^\infty \frac{\rho J_2(\rho r)}{(\alpha\rho^2+1)^2}\exp\left(-\frac{\rho^2 t}{\alpha\rho^2+1}\right)d\rho, \tag{30}$$

$$\Gamma(r,t) = r\Gamma_0\int_0^\infty \frac{J_1(\rho r)}{\alpha\rho^2+1}\exp\left(-\frac{\rho^2 t}{\alpha\rho^2+1}\right)d\rho, \tag{31}$$

are obtained. The solutions (29) and (30) have to be the dimensionless forms of Eq. (25) and (26) from [5]. Unfortunately, at the denominators of these relations from [5] the term "$1+\alpha\xi^2$" has been omitted.

Case 3. $\alpha = 0$, $\beta = 1$ (**ordinary Newtonian fluid**)

The dimensionless forms of the velocity and shear stress fields and the circulation corresponding to the decay of a potential vortex in a Newtonian fluid

$$\omega(r,t) = \frac{\Gamma_0}{2\pi r}\left\{1-\exp\left(-\frac{r^2}{4t}\right)\right\},\ \tau(r,t) = \frac{\Gamma_0}{\pi r^2}\left\{\left\{1+\frac{r^2}{4t}\right\}\exp\left(-\frac{r^2}{4t}\right)-1\right\}, \tag{32}$$

$$\Gamma(r,t) = \Gamma_0\left\{1-\exp\left(-\frac{r^2}{4t}\right)\right\}, \tag{33}$$

are immediately obtained making $\beta \to 1$ in Eqs. (26)-(28) or $\alpha = 0$ in Eqs. (29)-(31) and using Eqs. (A4)-(A6) from Appendix (see also [17, Table VII and Appendix A]). Direct computations clearly show that Eq. $(32)_1$ is the dimensionless form of Eq. (2.80) from [3], while Eq. $(32)_2$ represents the non-dimensional form of Eq. (34) from [5]. Furthermore, the products $r\omega(r,t)$, $r^2\tau(r,t)$ as well as the circulation $\Gamma(r,t)$ depend of *r* and *t* only by means of the similarity variable $r/(2\sqrt{t})$.

Finally, it is worth pointing out the fact that the velocity fields (26) and (29) corresponding to the decay of a potential vortex in a fractional Newtonian fluid or in an ordinary second grade fluid also do not satisfy the initial condition $(7)_2$. However, as well as the general solution (18), they satisfy the natural conditions (9) and the corresponding governing equations and reduce to the classical solution $(32)_1$ as limiting cases. Consequently, according to Bandelli and Rajagopal [16], they do not represent smooth solutions (cf. [14]) but are physically interesting.

Now, in order to obtain some physical insight of results that have been here obtained the variations of the velocity and shear stress fields given by Eqs. (18), (19), (29), (30) and (32) against *r* are presented in Figs. 1-6 for different values of second grade parameter α, fractional parameter β and time *t*. The influence of the fractional parameter β on the velocity $\omega(r,t)$ and the shear stress $\tau(r,t)$ corresponding to the decay of a potential vortex through a time fractional second grade fluid is brought to light in Figs. 1 and 2 at times $t=3$ and $t=5$. Fluid velocity, as well as the shear stress in absolute value, increases from the zero value up to a maximum value and then smoothly decreases to the asymptotic value for increasing values of *r*. It is a decreasing function with respect to β on the entire flow domain. Consequently, intensity of the vortex is stronger through fractional fluids in comparison to ordinary fluids. Moreover, for each *t*, there exists a critical value of *r* up to which the influence of the fractional parameter is significant. After this value, its influence is negligible.

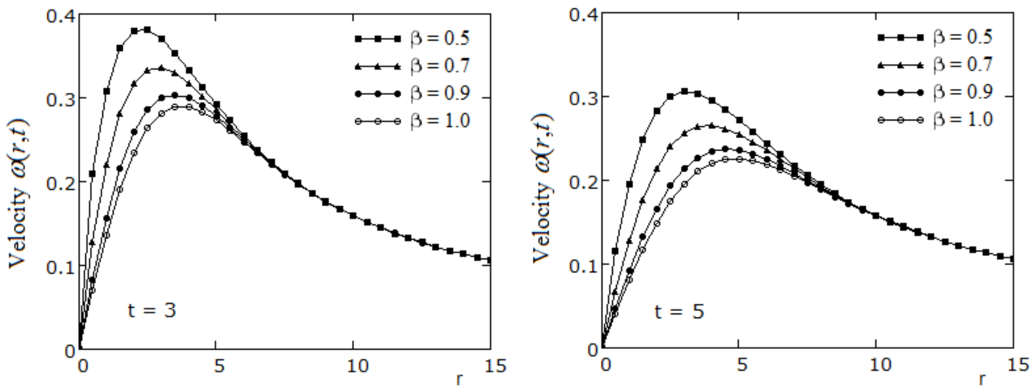

Fig. 1. Profiles of the dimensionless velocity $\omega(r,t)$ given by Eq. (18) for $\Gamma_0 = 10$, $\alpha = 0.2$ and different values of β

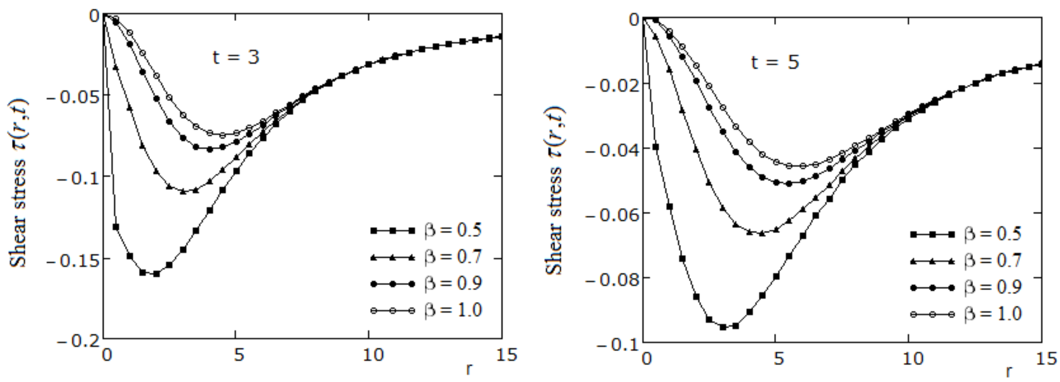

Fig. 2. Profiles of the dimensionless shear stress $\tau(r,t)$ given by Eq. (19) for $\Gamma_0 = 10$, $\alpha = 0.2$ and different values of β

Figs. 3 and 4 present the variations of $\omega(r,t)$ and $\tau(r,t)$, given by the same relations as before, for different sets of values of fractional and second grade parameters. On the same graphs, for comparison, the profiles of dimensionless classical solutions have been also included. Actually, the main interest is to show that the diagrams of present solutions tend to superpose over those of classical solutions when $\alpha \to 0$ and $\beta \to 1$. Furthermore, as it results from these figures, for $\alpha = 0$ and $\beta = 1$ the corresponding profiles are identical to those of classical solutions (32). Velocity and shear stress profiles have the same form as before and the vortex intensity is the lowest for Newtonian fluids.

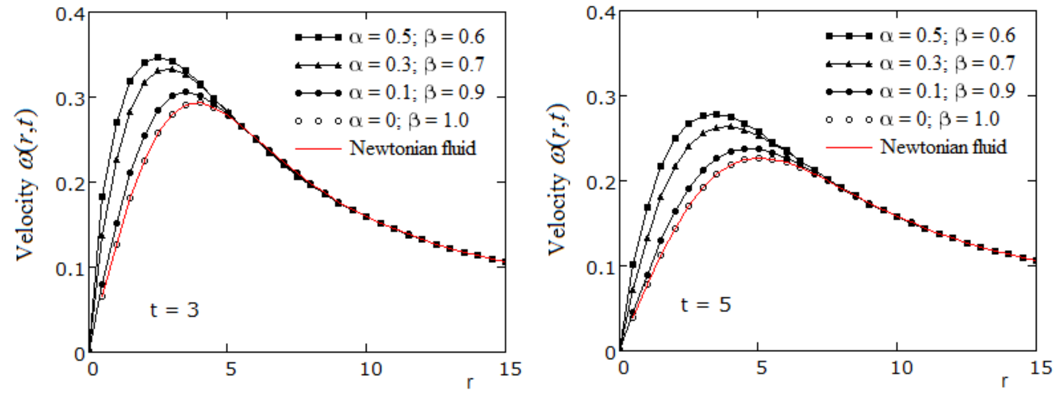

Fig. 3. Profiles of the dimensionless velocity $\omega(r,t)$ given by Eqs. (18) and $(32)_1$ (Newtonian fluid) for $\Gamma_0 = 10$ and different sets of values for α and β

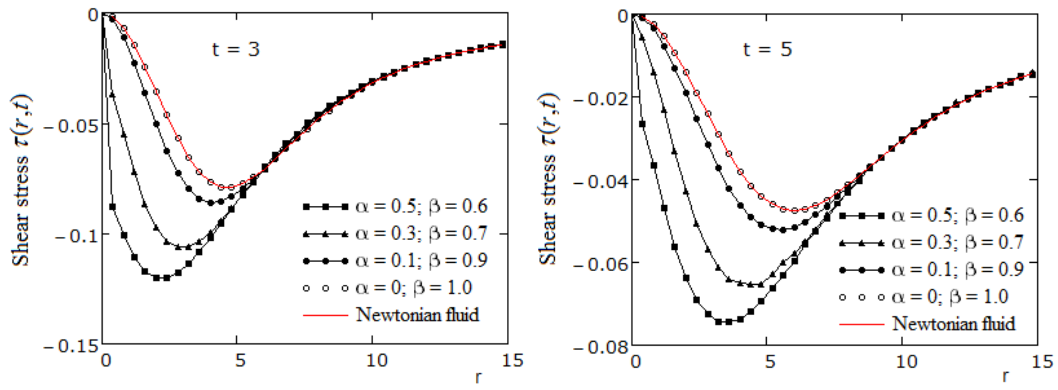

Fig. 4. Profiles of the dimensionless shear stress $\tau(r,t)$ given by Eqs. (19) and $(32)_2$ (Newtonian fluid) for $\Gamma_0 = 10$ and different sets of values for α and β

A comparison between the behavior of Newtonian and second grade fluids in such a potential vortex is presented in Figs. 5 and 6.

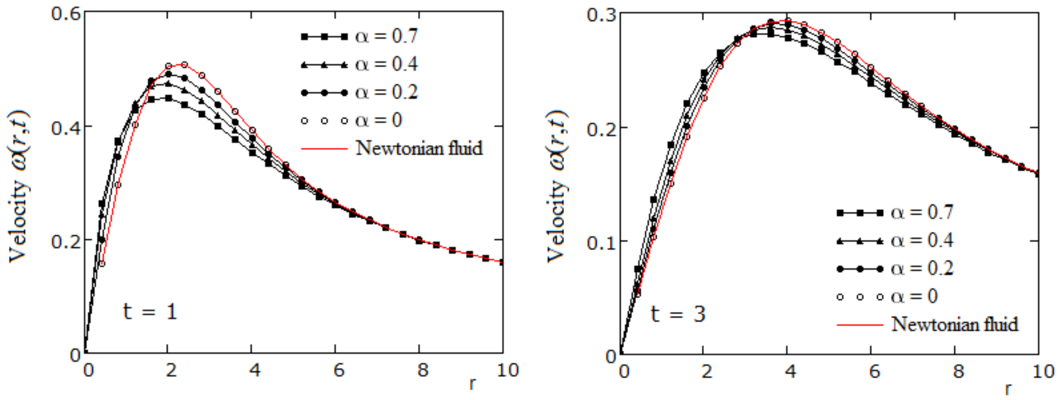

Fig. 5. Profiles of the dimensionless velocity $\omega(r,t)$ given by Eqs. (29) and $(32)_1$ (Newtonian fluid) for $\Gamma_0 = 10$ and different values of α

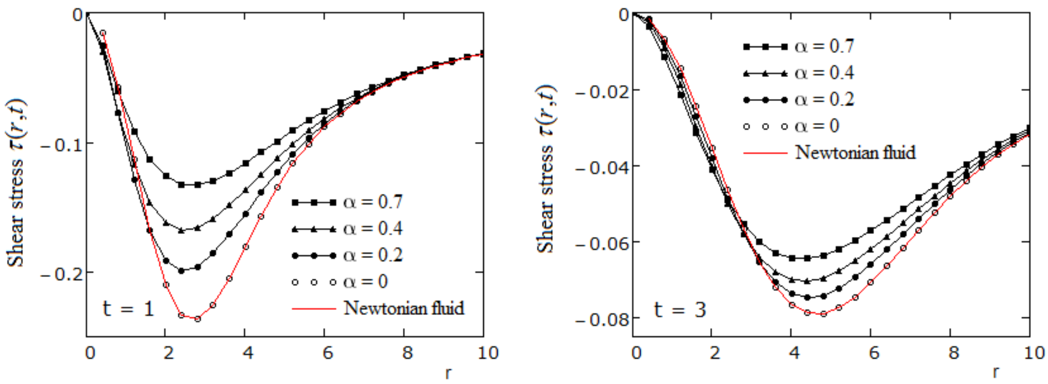

Fig. 6. Profiles of the dimensionless shear stress $\tau(r,t)$ given by Eqs. (30) and $(32)_2$ (Newtonian fluid) for $\Gamma_0 = 10$ and different values of α

The fluids velocity, as well as the shear stress in absolute value, also increases up to a maximum value and smoothly decreases to the zero value for large values of r but it is an increasing function with respect to α only up to a critical value of r that increases in time. Consequently, unlike the fractional case, the vortices in Newtonian fluids are stronger than those in second grade fluids for values of r greater than some time dependent critical values. Furthermore, the values of t have been diminished in order to show that for small values of t (less than one) both the velocity $\omega(r,t)$ and the shear stress $\tau(r,t)$ cannot be determined in $r=0$. This is due to the fact that $\omega(r,0)$ is not defined for $r=0$. In all cases, the vortex decreases in time and space at once it reached the maximum intensity.

5. Conclusions

Decay of a potential vortex in a time fractional incompressible second grade fluid is analytically studied by means of integral transforms. Exact solutions, under integral form in terms of Bessel functions $J_1(\cdot)$ and $J_2(\cdot)$, are established for dimensionless velocity and shear stress fields and the circulation $\Gamma(r,t)$ on a circle of radius *r*. These solutions, which have been easily reduced as limiting cases to the classical solutions of Newtonian fluids, satisfy the natural conditions at infinity and the governing equations. Unfortunately, the velocity field $\omega(r,t)$ does not satisfy the initial condition although it was enforced in the present calculi. However, this is not a singular case in the literature [14, 16]. Other similar case appears in the problem of a block mass *m* subjected to a blow *P* [18]. In our case, this inconvenience is due to the incompatibility between the initial condition $(7)_2$ and the natural condition $(9)_1$.

The main results of this paper are:

- Exact solutions are established for the decay of a potential vortex in fractional second grade fluids. They reduce to well-known classical solutions.
- Due to the incompatibility between the imposed conditions $(7)_2$ and $(9)_1$, the solution (18) for the fluid velocity does not satisfy the initial condition.
- The vortex intensity is lower in ordinary fluids as compared to fractional fluids. It decreases in time and, after a maximum value, diminishes to zero.

Appendix

$$\int_0^\infty \frac{\rho^{\nu+1} J_\nu(\rho r)}{(\rho^2+a^2)^{\mu+1}} d\rho = \frac{a^{\nu-\mu} r^\mu}{2^\mu \Gamma(\mu+1)} K_{\nu-\mu}(ar). \tag{A1}$$

$$K_n(z) \approx \frac{(n-1)!}{2}\left(\frac{z}{2}\right)^{-n} \quad \text{if} \quad z \le n \quad \text{and} \quad n>0. \tag{A2}$$

$$\omega(r,0) = \lim_{q\to\infty} q\overline{\omega}(r,q) \quad \text{if} \quad \overline{\omega}(r,q) = L\{\omega(r,t)\}. \tag{A3}$$

$$\int_0^\infty J_1(\rho r)\exp(-\rho^2 t)\, d\rho = \frac{1}{r}\left\{1-\exp\left(-\frac{r^2}{4t}\right)\right\}; \quad r \ne 0. \tag{A4}$$

$$\int_0^\infty \rho J_2(\rho r)\exp(-\rho^2 t)\,d\rho = \frac{r^2}{16t^2}F\left(2,3;-\frac{r^2}{4t}\right); \quad r \neq 0. \tag{A5}$$

$$x^2 F(2,3;-x) = 2\{1-(1+x)\exp(-x)\}; \quad x \neq 0. \tag{A6}$$

$$K_n(z) \approx \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z} \quad \text{for} \quad z >> n \quad \text{and} \quad n > 0. \tag{A7}$$

R E F E R E N C E S

[1]. *A. J. Majda*, *A. L. Bertozzi*, Vorticity and Incompressible Flow, Cambridge: Cambridge University Press, 2002.

[2]. *S. Gieser*, Philosophy and modern physics in Sweden: C.W. Oseen, Oskar Klein, and the intellectual traditions of Uppsala and Lund, 1920-1940. In Svante Lindquist. Center on the Periphery: Historical Aspects of 20th-century Swedish Physics. Science History Publications, pp. 24-41, 1993.

[3]. *J. Zierep*, Similarity Laws and Modeling, Marcel Deker, Inc., New York, 1971.

[4]. *I. Taipel*, “The impulsive motion of a flat plate in a viscoelastic fluid”, Acta Mech. **Vol. 39**, 1981, pp. 277-279.

[5]. *C. Fetecau*, *Corina Fetecau*, “Decay of a potential vortex in an Oldroyd-B fluid”, Int. J. Eng. Sci. **Vol. 43**, Issues 3-4, February 2005, pp. 340-351.

[6]. *C. Fetecau*, *D. Vieru*, *A. T. Ahmed*, “Time fractional Oseen problem for viscous fluids”, Ann. Acad. Rom. Sci. Ser. Math. Appl. **Vol. 10**, No. 1, 2018, pp. 165-178

[7]. *R. L. Bagley*, *P. J. Torvik*, “A theoretical basis for the application of fractional calculus to viscoelasticity”, J. Rheol. **Vol. 27**, Issue 3, 1983, pp. 201-210.

[8]. *M. Caputo*, *F. Mainardi*, “A new dissipation model based on memory mechanism”, Pure Appl. Geophys. **Vol. 91**, Issue 1, December 1971, pp. 137-147.

[9]. *M. Caputo*, *F. Mainardi*, “Linear models of dissipation in inelastic solids”, Riv. Nuovo Cimento **Vol. 1**, No. 2, 1971, pp. 161-198.

[10]. *N. Markis*, *G. J. Dargush*, *M. C. Constantinou*, “Dynamic analysis of generalized viscoelastic fluids”, J. Eng. Mech. **Vol. 119**, Issue 8, August 1993, pp. 1663-1679.

[11]. *S. S. Sheoran*, *P. Kundu*, “Fractional order generalized thermoelasticity theories: A review”, Int. J. Adv. Appl. Math. and Mech. **Vol. 3**, Issue 4, 2016, pp. 76-81.

[12]. *M. Caputo*, *M. Fabrizio*, “A new definition of fractional derivative without singular kernel”, Progr. Fract. Differ. Appl. **Vol. 1**, Issue 2, 2015, pp. 73-85.

[13]. *K. R. Rajagopal*, *A. S. Gupta*, “An exact solution for the flow of a non-Newtonian fluid past an infinite porous plate”, Meccanica **Vol. 19**, Issue 2, June 1984, 158-160.

[14]. *R. Bandelli*, *K. R. Rajagopal*, *G. P. Galdi*, “On some unsteady motions of fluids of second grade”, Arch. Mech. **Vol. 47**, Issue 4, January 1995, pp. 661-676.

[15]. *L. Debnath*, *D. Bhatta*, Integral Transforms and Their Applications, Second Ed., Chapman and Hall/CRC Press, Boca-Raton, 2007.

[16]. *R. Bandelli*, *K. R. Rajagopal*, “Start-up flows of second grade fluids in domains with one finite dimension”, Int. J. Non-Linear Mech. **Vol. 30**, Issue 6, November 1995, pp. 817-839.

[17]. *I. N. Sneddon*, Fourier Transforms, McGRAW-HILL BOOK COMPANY, INC., New York, Toronto, London, 1951.

[18]. *H. S. Carslaw*, *J. C. Jaeger*, Operational Methods in Applied Mathematics, 2nd ed., Oxford University Press, London, 1947.