

## DETERMINATION OF DEFINING HYPERPLANES OF DEA PRODUCTION POSSIBILITY SET

Majid Zohrehbandian<sup>1</sup>

*The ability of determining all defining hyperplanes of DEA production possibility set (efficient frontier) prior to the DEA computations is of extreme importance. Specially, access to efficient frontier permits a complete analysis (e.g. calculation of efficiency scores, returns to scale, sensitivity analysis and so on) in second phase for the corresponding model. This paper presents a linear system of constraints which its extreme points correspond to defining hyperplanes (both weak and strong ones). Numerical examples are provided to explore the advantage of using the proposed method.*

**Keywords:** Data Envelopment Analysis; Production Possibility Set; Defining Hyperplane; Efficient Frontier

### 1. Introduction

Data envelopment analysis (DEA), Charnes et al. [4], is a non-parametric method for evaluation of the relative efficiency of a number of decision making units (DMUs), each of which consume varying amount of  $m$  inputs to produce  $s$  outputs. A DEA domain is completely specified by a finite list of data points in  $R^{m+s}$ , one for each DMU. The combined data about the DMUs and the assumptions about the technology, generate an empirical production possibility set (PPS). The boundary of the PPS (efficient frontier) includes all the efficient observations as well as linear combinations obtained from efficient units, while the rest (those considered as inefficient) remain below it.

DEA is computationally intensive and, as the scale of application grows, this intensity rapidly becomes one of the limiting factors in its utility. Therefore, many published works address the problem of reducing computational time in DEA. In this field, it is argued that an identification of defining hyperplanes provides a highly appropriate framework for an analysis of important frontier characteristics; see Ali [1], Barr and Durchholz [3], Dulá [5], Dulá et al [6], Dulá and Thrall [7], Jahanshahloo et al [9,10,11], Olesen and Petersen [14,15], Wei et al [16], Yu et al [17,18].

Even though a good amount of research work carried out on identification of efficient frontier, there is still a need for simple and efficient mathematical

---

<sup>1</sup> Assistant Prof., Department. of Mathematics, Karaj Branch, Islamic Azad University, Iran.,  
e-mail: zohrehbandian@yahoo.com

methods to solve this problem. The aim of this study is to develop a way to obtain efficient frontier by enumerating the extreme points of a convex polytope specified by some linear constraints. It is very important to note that, the number of extreme points of the proposed set is about equal to the number of defining hyperplanes. Moreover, the number of DMUs (n) is usually much larger than the number of defining hyperplanes. So, for large n, enumerating small amount of extreme points may be more preferable than solving so many linear problems (LP), because it imparts greater flexibility to the analysis in second phase, especially when DEA studies are over several models and multiple orientations.

The plan for the rest of this paper is as follows. Section 2 formalizes a linear system and proposes an approach for identifying the equations of DEA efficient frontier. Section 3 describes the useful application of efficient frontier equations. Section 4 provides numerical examples and finally, section 5 draws the conclusive remarks.

## 2. Specifying the Efficient Frontier

Consider n DMUs, each of which consume varying amount of m inputs to produce s outputs. Suppose  $x_{ij} \geq 0$  denotes the amount consumed of the i-th input measure and  $y_{rj} \geq 0$  denotes the amount produced of the r-th output measure by the j-th DMU. The PPS of obviously most widely used DEA model, BCC with variable returns to scale (VRS) characteristic, is defined as semi-positive vectors (x,y) as follows:

$$T_v = \{(x, y) \mid x \geq \sum_{j=1}^n \lambda_j x_j, \quad y \leq \sum_{j=1}^n \lambda_j y_j, \quad \sum_{j=1}^n \lambda_j = 1, \quad \lambda_j \geq 0 \quad j = 1, \dots, n\}$$

The other polyhedral sets, are explicitly defined by different constraint on sum of  $\lambda_j$ 's.

A standard formulation of DEA creates a separate LP for each DMU. It is instructive to apply the input oriented version of the multiplier BCC model, where  $DMU_p$  is under consideration and each optimal solution of the problem is associated to coefficients  $(-v^*, u^*, w^*)$  of a supporting hyperplane  $-v^*x + u^*y + w^* = 0$  (a hyperplane which contains  $T_v$  in only one of the halfspaces and pass among at least one of the points of  $T_v$ ).

$$\begin{aligned} \max \quad & \sum_{r=1}^s u_r y_{rp} + w \\ \text{s.t.} \quad & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} + w \leq 0 \quad j = 1, \dots, n \\ & \sum_{i=1}^m v_i x_{ip} = 1 \\ & u, v \geq 0 \end{aligned} \quad (BCC)$$

**Definition 1.** By separable hyperplane, we refer to coefficients  $(-v, u, w)$  of a hyperplane that contains  $T_v$  in only one of the halfspaces and does not support it.

**Definition 2.** The set of points in PPS that correspond to the points on a supporting hyperplane is called proper face of PPS. Any  $(m+s-1)$ -dimensional proper face is called a facet of PPS, where the PPS itself is a subset of  $R^{m+s}$ .

**Definition 3.** By defining hyperplane, we refer to coefficients  $(-v, u, w)$  of a hyperplane that its intersection with PPS is a facet of PPS.

**Definition 4** Let  $S$  be a nonempty, closed convex set in  $R^n$ . A nonzero vector  $d$  in  $R^n$  is called a recession direction of  $S$  if for each  $x$  in  $S$ ,  $x + \lambda d \in S \quad \forall \lambda \geq 0$ .

As discussed in Ali [1] computational constructs (e.g. identification of efficient frontier) that allow streamlining of DEA computations, have been necessary to circumvent intensive time consuming calculations. However, up to now, all of the approaches which have been proposed for identification of efficient frontier are inefficient. Jahanshahloo et al [10,11] proposed a method which solve an integer problem for production of each defining hyperplane. Note that obtaining the exact solution of each of these integer problems is computationally intractable and is based on an enumerative method like branch and bound. The approach proposed by Jahanshahloo et al [9] only produced the strong defining hyperplanes where it was still computationally expensive. Because, their approach was based on evaluation of so many DMUs (observed and virtual) and solving so many equality equations. Jahanshahloo et al [12] proposed an approach for production of strong defining hyperplanes, but it has been shown that their method is failed to compute all of the strong defining hyperplanes. Finally, Jahanshahloo et al [8] proposed another method for production of weak defining hyperplanes which was still computationally expensive due to solving so many linear problems. Moreover, Amirteimoori and Kordrostami [2] proposed a method for production of strong defining hyperplanes which was based on evaluation of so many perturbed DMUs. In other words, by solving  $n(m+s)$  linear problems they produced linearly independent defining hyperplanes passing through a specific DMU.

Therefore, there is still a need for simple and efficient mathematical methods to produce weak and strong defining hyperplanes. Here, we propose a method to produce efficient frontier of  $T_v$  in terms of the defining hyperplanes of PPS. Nevertheless, the idea extends easily to other production possibility sets. By definition of  $T_v$ , the PPS is a polyhedral set and the number of efficient facets of it is finite. We propose a system of linear constraints to produce the equation of some supporting hyperplanes of  $T_v$  which identify an efficient facet of it. We refer to these hyperplanes as defining hyperplanes, to signal that these hyperplanes are constructors of PPS. To this end, consider the following linear system.

$$-\sum_{i=1}^m v_i x_{ij} + \sum_{r=1}^s u_r y_{rj} + w_1 - w_2 \leq 0 \quad j=1, \dots, n \quad (1)$$

$$\sum_{i=1}^m v_i + \sum_{r=1}^s u_r = 1 \quad (2)$$

$$u, v, w_1, w_2 \geq 0 \quad (3)$$

Let  $S$  be the feasible region of the above system. We only need to ascertain whether the coefficients  $(-v, u, w)$  of the defining hyperplanes are produced by the extreme points of the region  $S$  and, whether almost all of the extreme points of  $S$  are associated to defining hyperplanes. If so, we can utilize one of the extreme points enumerating algorithms, for construction of all DEA efficient facets.

**Theorem 1.** Number of extreme points of  $S$ , which identify a separable hyperplane, are at most  $m+s+2$ .

**Proof.** The coefficients  $(-v', u', w'_1, w'_2)$  are associated to a separable hyperplane (support non of the DMUs) if non of the constraints (1) are binding at  $(v', u', w'_1, w'_2) \in S$ . The system has  $m+s+2$  variables and  $n+1+m+s+2$  constraints. By the definition of extreme points, there are some  $m+s+2$  linearly independent constraints binding at any extreme point  $(v', u', w'_1, w'_2) \in S$ . Since constraint (2) is binding at all feasible solutions of  $S$ , for a separable hyperplane it must be choose  $m+s+1$  constraints among  $m+s+2$  sign constrains. Therefore, number of separable hyperplanes associated to the extreme points of  $S$  are at most  $C(m+s+2, m+s+1) = m+s+2 \square$

Recall that PPS is intersection of finite number of halfspaces and production of separable hyperplanes do not influence on the shape of it. Fortunately, due to the above theorem, the number of separable hyperplanes produced by extreme points of  $S$  has a low density.

**Theorem 2.** A vector  $(-v, u, w_1, w_2)$  associated to the normalized coefficient of a defining hyperplane is an extreme point of  $S$ .

**Proof.** For each efficient facet  $H$  of PPS there exist exactly one normalized coefficient  $(-v^0, u^0, w^0_1, w^0_2)$  of the defining hyperplane which itself is associated to a feasible solution of  $S$ . Suppose that this feasible solution is not an extreme point of  $S$ . Two possibilities arise here.

i) The case where the point  $(v^0, u^0, w^0_1, w^0_2) \in S$  is a convex combination of extreme points  $(v^k, u^k, w^k_1, w^k_2) \in S \quad \forall k \in K \neq \{0\}$ .

Hence,  $(v^0, u^0, w^0_1, w^0_2) = \sum_{k \in K} \lambda_k (v^k, u^k, w^k_1, w^k_2)$ , where  $\sum_{k \in K} \lambda_k = 1$ . Suppose that hyperplane  $-v^0 x + u^0 y + w^0_1 - w^0_2 = 0$ , pass among DMU<sub>j</sub>  $\forall j \in J \neq \{\}$ . We can deduce that  $-v^k x_j + u^k y_j + w^k_1 - w^k_2 = 0 \quad \forall k \in K, j \in J$ . This is because if there exist a

DMU<sub>p</sub>  $p \in J$  and index  $q$ ,  $q \in K$  where  $-v^q x_p + u^q y_p + w_1^q - w_2^q < 0$ , then, we conclude that:

$$-v^o x_p + u^o y_p + w_1^o - w_2^o = -\left(\sum_{k \in K} \lambda_k v^k\right) x_p + \left(\sum_{k \in K} \lambda_k u^k\right) y_p + \left(\sum_{k \in K} \lambda_k (w_1^k - w_2^k)\right) < 0,$$

which is a contradiction. Hence, we have:

$$-v^k x_j + u^k y_j + w_1^k - w_2^k = 0 \quad \forall k \in K, j \in J,$$

and the extreme points  $(v^k, u^k, w_1^k, w_2^k) \forall k \in K \neq \{o\}$  are associated to the same normalized coefficient of a supporting hyperplane at efficient facet H of PPS. This will result in one obvious corollary  $K=\{o\}$ .

**ii)** The other case is where the point  $(v^o, u^o, w_1^o, w_2^o) \in S$  is a convex combination of extreme points  $(v^k, u^k, w_1^k, w_2^k) \in S \forall k \in K \neq \{o\}$  accompanied by linear combinations of extreme recession directions of region S. It is straightforward that the extreme recession directions of S are  $d_1=(0,0,0,1)$ ,  $d_2=(0,0,1,1) \in \mathbb{R}^{m+s} \times \mathbb{R}^1 \times \mathbb{R}^1$ . Therefore we must have,

$$(v^o, u^o, w_1^o, w_2^o) = \sum_{k \in K} \lambda_k (v^k, u^k, w_1^k, w_2^k) + \sum_{i=1}^2 \mu_i d_i, \text{ where } \sum_{k \in K} \lambda_k = 1, \quad \mu_1, \mu_2 \geq 0.$$

By virtue of the directions  $d_1, d_2$ , we have,

$$\sum_{k \in K} \lambda_k (v^k, u^k, w_1^k, w_2^k) + \sum_{i=1}^2 \mu_i d_i = \sum_{k \in K} \lambda_k (v^k, u^k, w_1^k + \mu_1, w_2^k + \mu_1 + \mu_2).$$

This shows that, if  $w_1^k + \mu_1$  replaced by  $W_1^k$  and  $w_2^k + \mu_1 + \mu_2$  replaced by  $W_2^k$ , in a similar manner outlined in case (i), we can conclude that  $K=\{o\}$ .

The foregoing arguments lead to the conclusion that the point  $(v^o, u^o, w_1^o, w_2^o)$  is an extreme point of region S  $\square$

**Theorem 3.** Every extreme point of S, which is not associated to a separable hyperplane, is associated to a normalized coefficient of a defining hyperplane.

**Proof.** Given an extreme point  $(v^o, u^o, w_1^o, w_2^o) \in S$  which is associated to a normalized coefficient  $(-v^o, u^o, w_1^o, w_2^o)$  of a supporting hyperplane passing among DMU<sub>j</sub>,  $\forall j \in J \neq \{o\}$ . Suppose that intersection of this hyperplane with the PPS is a face of PPS, and not a facet. This face is produced by intersection of some facets of PPS and we saw earlier that associated to these facets, there are normalized coefficients of defining hyperplanes  $(-v^k, u^k, w_1^k, w_2^k) \forall k \in K$ , passing among DMU<sub>j</sub>  $\forall j \in J$ , which themselves are associated to the extreme points of S. Since, the normalized vector  $(-v^o, u^o)$  lies in the cone constructed by the normalized vectors  $(-v^k, u^k)$ , there exist multipliers  $\lambda_k$  where

$$(-v^o, u^o) = \sum_{k \in K} \lambda_k (-v^k, u^k), \quad \sum_{k \in K} \lambda_k = 1. \quad (4)$$

Let  $p \in J$ . Then  $-v^k x_p + u^k y_p + w_1^k - w_2^k = 0 \quad \forall k \in K$ , and a convex combination of these constraints is as follows:

$$-(\sum_{k \in K} \lambda_k v^k) x_p + (\sum_{k \in K} \lambda_k u^k) y_p + (\sum_{k \in K} \lambda_k (w_1^k - w_2^k)) = 0 \quad \forall k \in K.$$

Following the relation (4) we can deduce that

$$-v^o x_p + u^o y_p + (\sum_{k \in K} \lambda_k (w_1^k - w_2^k)) = 0.$$

On the other hand, since  $-v^o x_p + u^o y_p + w_1^o - w_2^o = 0$ , we can conclude that

$$w_i^o = \sum_{k \in K} \lambda_k w_i^k \quad i = 1, 2.$$

In conclusion, we have  $(-v^o, u^o, w_1^o, w_2^o) = \sum_{k \in K} \lambda_k (-v^k, u^k, w_1^k, w_2^k)$ , where

$\sum_{k \in K} \lambda_k = 1$ , which is a contradiction. Since, the normalized coefficient

$(-v^o, u^o, w_1^o, w_2^o)$  of the supporting hyperplane is associated to an extreme point of  $S$  and can't be represented by a strict convex combination of points in  $S$   $\square$

The foregoing theorems lead to the conclusion that the procedure of identification of all DEA efficient facets, consists of enumerating the extreme points of the system of constraints 1-3, where almost all of them are associated to defining hyperplanes. Based on the formulation of these hyperplanes, the set of weights for DEA multiplier model s at hand and we can easily compute the efficiency scores of al of DMUs. To this end, we used the software pdd v0.2 (by Komei Fukuda, EPFL Lausanne, Switzerland and University of Tsukuba, Japan) for finding all of the extreme points of  $S$ . The program pdd.p, which we used here, is a Pascal implementation of the double description method, Motzkin et al. [13], for generating all extreme points and extreme directions of a general  $d$ -dimensional convex polyhedron given by a system of linear inequalities.

### 3. The usefulness of generating efficient frontier

For large data set with many inputs and outputs, maybe, it is still a time (and memory) consuming task to generate all defining hyperplanes but the information made available by an efficient frontier representation is of sufficient value to warrant the effort needed for its identification [15]. In other words, in using DEA in practice we typically go far beyond the computation of a simple measure of the relative efficiency of a unit. Indeed, we wish to know what operating practices, mix of resources, scale sizes and so on. The construction of DEA efficient frontier will bring the analysis of production efficiencies to depth. For example, access to the equations of efficient frontier permits expeditious scoring of the rest of the DMUs especially if the efficient DMUs are a small subset of the data. This is especially true for large problems. Reports indicate that this relation can be less than 1% [3].

Suppose that the equations of efficient frontier are produced as  $-V_k X + U_k Y + U_{ko} = 0 \quad k = 1, \dots, L$ . Where,  $L$ , is the number of defining hyperplanes produced by using the proposed approach, and  $(-V_k, U_k)$  is associated to the gradient vector of  $k$ -th supporting hyperplane. Then, for example, efficiency score of  $DMU_j$  in both input/output oriented, will be obtained by a simple comparison as follows:

$$(Eff. score of DMU_j)_{Input} = \max \left\{ \frac{U_k Y_j + U_{ko}}{V_k X_j} : k=1,2,\dots,L \right\}$$

$$(Eff. score of DMU_j)_{output} = \min \left\{ \frac{V_k X_j - U_{ko}}{U_k Y_j} : k=1,2,\dots,L \right\}$$

Note that in classical DEA, these scores are obtained by solution of  $2n$  Linear Programming problems.

There are other advantages to construction of efficient frontier besides faster computer times. This information can be exploited to gain knowledge about returns to scales (RTS) by using the free variable in the BCC model and without paying the full computational price that individual analysis would require. Moreover, since DEA is data based, it is very useful to assess possible input/output changes (data perturbation and sensitivity analysis) of a DMU such that its obtained efficiency classification does not change. Another advantage to construction of efficient frontier is in this context and to identifying the region of efficiency for an efficient  $DMU_o$ .

#### 4. Numerical Example

**Example 1.** In order to motivate our approach we apply a simple example involving just eleven DMUs, each using one input to produce one output; see Table 1. Enumeration result of the extreme points of the system of constraints 1-3, by using pdd software and based on the raw data from Table 1, are depicted in Figure 1.

Table 1

The Raw Data set for example 1

DMUs	1	2	3	4	5	6	7	8	9	10	11
Input	3.8	6	2	6.5	5	3	4	6	4	5	4
Output	2.6	4	1	3.2	2.9	3	3.5	2	1	2	1.5

The output shows that the polyhedron has seven vertices:

$$(-v, u, w_1, w_2) = (-1, 0, 0, 0), (-5.0E-01, 5.0E-01, 0, 0), (-1, 0, 2, 0), (0, 1, 0, 4),$$

$$(-6.667E-01, 3.333E-01, 1, 0), (-3.333E-01, 6.667E-01, 0, 1),$$

$$(-2.0E-01, 8.0E-01, 0, 2)$$

corresponding to the seven hyperplanes as follows:

$$\begin{aligned} -x = 0 & \quad -0.5x + 0.5y = 0 \quad -x + 2 = 0 \quad y - 4 = 0 \\ -\frac{2}{3}x + \frac{1}{3}y + 1 = 0 & \quad -\frac{1}{3}x + \frac{2}{3}y - 1 = 0 \quad -0.2x + 0.8y - 20 = 0 \end{aligned}$$

Among them, the last five vertices are corresponding to the five defining hyperplanes, and the first two of which are surplus hyperplanes.

```
* pdd: Double Description Method Code: Version 0.22 (November 20, 1993)
Copyright 1993, Komei Fukuda, EPFL, Switzerland fukuda@dma.epfl.ch
Input File:frontier.ine ( 17 x 5 )
HyperplaneOrder: LexMin
AdjacencyTest: Combinatorial
FINAL RESULT:
Number of Vertices = 7, Rays = 2
begin
  v           u           w1           w2
  1           0           0           0
  0           1           0           4
  5.000E-01   5.000E-01   0           0
  1           0           2           0
  6.667E-01   3.333E-01   1           0
  3.333E-01   6.667E-01   0           1
  2.000E-01   8.000E-01   0           2
End
```

Fig. 1. The execution of Pdd software

**Example 2.** Let us consider an example with ten DMUs, each using two inputs to produce two outputs, see Table 2.

Table 2

The Raw Data set for example 2

DMUs	1	2	3	4	5	6	7	8	9	10
Input 1	30	93	50	80	35	105	97	100	90	98
Input 2	60	40	70	30	45	75	67	50	60	65
Output 1	180	170	190	180	140	120	100	140	140	140
Output 2	70	60	130	120	82	90	82	40	105	50

The extreme points of the system of constraints 1-3 which have been produced for this example are corresponding to the following hyperplanes:

$$\begin{aligned}
& -0.31579x_1 - 0.52632x_2 + 0.1579y_1 + 12.63158 = 0, \\
& -0.36226x_1 - 0.3449x_2 + 0.29284y_2 + 4.18655 = 0, \\
& -0.1579x_1 - 0.26316x_2 + 0.57895y_1 - 83.68421 = 0, \\
& -0.71329x_1 - 0.04196x_2 + 0.24476y_2 + 6.78322 = 0, \\
& -0.75x_1 + 0.25y_2 + 5 = 0, \\
& -0.25x_1 - 0.75x_2 + 42.5 = 0, \\
& -0.33334x_1 + 0.66667y_1 - 110 = 0, \\
& -0.36511x_1 - 0.34604x_2 + 0.13137y_1 + 0.15748y_2 - 2.94583 = 0, \\
& -x_1 + 30 = 0, \\
& -x_2 + 30 = 0, \\
& y_1 - 190 = 0, \\
& y_2 - 130 = 0, \\
& -0.75x_1 - 0.25x_2 + 37.5 = 0, \\
& -0.2x_2 + 0.8y_1 - 138 = 0, \\
& -0.2x_2 + 0.8y_2 - 90 = 0.
\end{aligned}$$

## 6. Conclusions

If contributions to reduce time and increase the information yield of a DEA study are up to the task, DEA will emerge as one of the available tools for mining massive data sets. It has been demonstrated that a dual representation of a polyhedral empirical production possibility set in terms of its defining facets provides a highly appropriate framework for an analysis of important frontier characteristics and possibility of reducing times while increasing flexibility of DEA studies, especially when these are over several models and multiple orientations.

For dealing with this difficulty, this paper examines the application of a system of linear inequalities and vertex enumeration algorithms. The procedure proposed here attempts to find out all extreme points of a linear model, where almost all of its extreme points are associated to an efficient facet of PPS. The proposed approach has been characterized under conditions of variable returns to scale, but it easily generalized to the other cases.

## Acknowledgments

The author is grateful to anonymous referees for their valuable suggestions and to the editors for ensuring a timely review process.

## R E F E R E N C E S

- [1] *A.I. Ali*, "Streamlined computation for data envelopment analysis", in European Journal of Operational Research, **vol. 64**, 1993, pp. 61-67.
- [2] *A. Amirteimoori and S. Kordrostami*, "Generating strong defining hyperplanes of the production possibility set in data envelopment analysis", in Appl. Math. Lett. **vol. 25**, no. 3, 2012, pp. 605-609.
- [3] *R.S. Barr and M.L. Dorchholz*, "Parallel and hierarchical decomposition approaches for solving large-scale data envelopment analysis models", in Annals of Operations Research **vol. 73**, 1997, pp. 339-372.
- [4] *A. Charnes, W.W. Cooper and E. Rhodes*, "Measuring the efficiency of decision making units", in European Journal of Operational Research, **vol. 2**, 1978, pp. 429-444.
- [5] *J.H. Dulá*, "Computations in DEA", in Pesquisa Operacional, **vol. 22**, no. 2, 2002, pp. 165-182.
- [6] *J.H. Dulá, R.V. Helgason and N. Venugopal*, "An Algorithm for Identifying the Frame of a Pointed Finite Conical Hull", in INFORMS Journal on Computing, **vol. 10**, 1998, pp. 323-330.
- [7] *J.H. Dulá and R.M. Thrall*, "A computational framework for accelerating DEA", in Journal of Productivity Analysis, **vol. 16**, 2001, pp. 63-78.
- [8] *G.R. Jahanshahloo, F.H. Lotfi and D. Akbarian*, "Finding weak defining hyperplanes of PPS of the BCC model", in Applied Mathematical Modelling, **vol. 34**, 2010, pp. 3321-3332.
- [9] *G.R. Jahanshahloo, F.H. Lotfi, H. Zhiani Rezai and F. RezaiBafal*, "Finding strong defining hyperplanes of production possibility set", Eur. J. Oper. Res., **vol. 177**, 2007, pp. 42-54.
- [10] *G.R. Jahanshahloo, F.H. Lotfi and M. Zohrehbandian M*, "Finding the efficiency score and RTS characteristic of DMUs by means of identifying the efficient frontier in DEA", in Appl. Math. Comput., **vol. 170**, 2005, pp. 985-993.
- [11] *G.R. Jahanshahloo, F.H. Lotfi and M. Zohrehbandian M*, "Finding the piecewise linear frontier production functionin in data envelopment analysis", in Appl. Math. Comput., **vol. 163**, 2005, pp. 483-488.
- [12] *G.R. Jahanshahloo, A. Shirzadi and S.M. Mirdehghan*, "Finding strong defining hyperplanes of PPS using multiplier form", in European Journal of Operational Research, **vol. 194**, 2009, pp. 933-938.
- [13] *T.S. Motzkin, H. Raiffa, G.L. Thompson, and R.M. Thrall*, "The double description method", in Contribution to the Theory of Games, **Vol. II** (Kuhn H.V. and Tucker A.W., eds.), Annals of Mathematical Studies: Princeton University Press, 28;81-103.
- [14] *O.B. Olesen, and N.C. Petersen*, "Indicators of ill-conditioned data sets and model misspecification in data envelopment analysis: An extended facet approach", in Management Science, **Vol. 42**, 1996, pp. 205-219.
- [15] *O.B. Olesen, and N.C. Petersen*, "Identification and use of efficient faces and facets in DEA", in Journal of Productivity Analysis, **Vol 20**, 2003, pp. 323-360.
- [16] *Q.L. Wei, H. Yan and G. Hao*, "Characteristics and structures of weak efficient surfaces of production possibility sets", in J. Math. Anal. Appl., **Vol. 327**, 2007, pp. 1055-1074.
- [17] *G. Yu, Q. Wei and P. Brockett*, "A generalized data envelopment model: A unification and extension of existing methods for efficiency analysis of decision making units", in Annals of Operations Research, **Vol. 66**, 1996, pp. 47-89.
- [18] *G. Yu, Q. Wei, P. Brockett and L. Zhou*, "Construction of all DEA efficient surfaces of the production possibility set under generalized data envelopment analysis model", in European Journal of Operational Research, **Vol. 95**, 1996, pp. 491-510.