

ORDERED JOIN HYPERLATTICES

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In this paper, we consider join hyperlattices and we define ordered join hyperlattices. Then, we consider product of two ordered join hyperlattices and we study prime ideals on them. Moreover, we define semiprime ideals on ordered join hyperlattices and prove some results about them. Also, we define a regular relation on ordered join hyperlattice L such that the quotient of L is an ordered join hyperlattice. Then, we investigate isomorphism on product of two ordered join hyperlattices with such regular relations.

Keywords: Join hyperlattice, ordered hyperlattice, prime ideal, regular relation, quasi-ordered relation.

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1. Introduction and basic definitions

The first step in the history of the development of hyperstructures theory was the 8th congress of Scandinavian mathematicians from 1934, when Marty [13] introduced the notion of hypergroup, analyzed its properties and applied them to non-commutative groups, algebraic functions, rational fractions. One of the structures that are most extensively used and discussed in mathematics and its applications is lattice theory (see [1]). The notion of partial and lattice order goes back to 19th century investigations in logic. Konstantinidou and Mittas introduced the concept of hyperlattices in [12] and the concept of ordering hypergroup introduced by Chvalina [3] as a special class of hypergroups and studied by many authors, see [2, 6, 9]. Now, by considering ordered hypergroups, we define ordered hyperlattice. Product of two hyperlattice is defined in [5]. In first section, we investigate conditions on product of two ordered hyperlattices such that prime ideals on them is defined. Rav introduced the concept of semiprime ideals and filters in lattices [16]. Also, semiprime ideals in ordered structures such as posets and other structures are studied in [17, 11, 10]. In second section, we generalize semiprime ideals and filters to ordered hyperlattices. Then, we get results which connect this concepts to distributivity and other concepts in ordered hyperlattices. Also, 0-distributive lattice are studied in [7, 8]. In the end of this section, we introduce 0-distributive hyperlattice and we investigate the connection between semiprime ideals and this category of hyperlattices. The main tools in the theory of hyperstructures are the fundamental relations and we study the quotient of hyperstructures with them. The quotient hyperlattices studied by Xiao guang Li and Xiao long Xin [18]. In third section, we define a regular relation on ordered join hyperlattice such that its quotient is an ordered hyperlattice and we study some properties of such relations.

Let H be a non-empty set. A hyperoperation on H is a map \circ from $H \times H$ to $\wp^*(H)$, the family of non-empty subsets of H . The couple (H, \circ) is called a hypergroupoid. For any two non-empty subsets A and B of H and $x \in H$, we define $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$; $A \circ x =$

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$A \circ \{x\}$ and $x \circ B = \{x\} \circ B$. A hypergroupoid (H, \circ) is called a semihypergroup if for all a, b, c of H we have $(a \circ b) \circ c = a \circ (b \circ c)$. Moreover, if for any element $a \in H$ equalities $a \circ H = H \circ a = H$ hold, then the pair (H, \circ) is called a hypergroup.

Join hyperlattice. Let L be a non-empty set, $\bigvee : L \times L \longrightarrow \wp^*(L)$ be a hyperoperation, and $\wedge : L \times L \longrightarrow L$ be an operation. Then, (L, \bigvee, \wedge) is a join hyperlattice if for all $x, y, z \in L$ the following conditions hold:

- (1) $x \in x \bigvee x$ and $x = x \wedge x$;
- (2) $x \bigvee (y \bigvee z) = (x \bigvee y) \bigvee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;
- (3) $x \bigvee y = y \bigvee x$ and $x \wedge y = y \wedge x$;
- (4) $x \in x \wedge (x \bigvee y) \cap x \bigvee (x \wedge y)$.

L is called a strong join hyperlattice, if $y \in x \bigvee y$ implies that $x = x \wedge y$ and if for all $x \in L$, there exists $1 \in L$ such that $x \leq 1$, we say L is bounded. Also, L is said to be distributive if for all $x, y, z \in L$, we have $x \wedge (y \bigvee z) = (x \wedge y) \bigvee (x \wedge z)$ and L is s -distributive if $x \bigvee (y \wedge z) = (x \bigvee y) \wedge (x \bigvee z)$. Moreover, we call $x \in L$ is complemented if there exist $y \in L$, such that $x \wedge y = 0$ and $1 \in x \bigvee y$. If each $x \in L$ is complemented, we say hyperlattice L is complemented.

Example 1.1. Let (L, \leq) be a partial order set. we define hyperoperations as follows: $a \bigvee b = \{x \in L : x \leq a, x \leq b\}$ and $a \wedge b = \{x \in L : a \leq x, b \leq x\}$. Then, (L, \bigvee, \wedge) is a join hyperlattice.

Definition 1.1. Let I be a non-empty subset of L . Then, I is called an ideal of L if: (1) for every $x, y \in I$, $x \bigvee y \subseteq I$; (2) $x \leq I$ implies $x \in I$. The intersection of all ideals of L containing A is denoted by $[A]$ and by [6], in ordered hyperlattice, we have $[A] = \{x \in L : \exists a \in A, x \leq a\}$.

Now, let (L, \bigvee, \wedge) be a join hyperlattice. We call (L, \leq) is an ordered hyperlattice, if \leq is an equivalence relation and $x \leq y$ implies that $x \bigvee z \leq y \bigvee z$ and $x \wedge z \leq y \wedge z$. Note that for any $A, B \subseteq L$, $A \leq B$ means that there exist $x \in A, y \in B$ such that $x \leq y$. Notice that if L is a s -distributive hyperlattice, then L is an ordered hyperlattice but the converse is not true in general.

Example 1.2. Let $H = \{0, x_1, x_2, 1\}$. Consider the following tables:

\bigvee	0	x_1	x_2	1
0	0	x_1	x_2	1
x_1	x_1	$\{0, x_1\}$	1	$\{x_2, 1\}$
x_2	x_2	1	$\{0, x_2\}$	$\{x_1, 1\}$
1	x_2	1	1	H

\wedge	0	x_1	x_2	1
0	0	0	0	0
x_1	0	x_1	0	x_1
x_2	0	0	x_2	x_2
1	0	x_1	x_2	1

We define \leq as $\{(x_1, x_1), (x_2, x_2), (x_1, x_2)\}$. Then, $(L, \bigvee, \wedge, \leq)$ is not s -distributive but it is an ordered hyperlattice.

Example 1.3. Let (L, \bigvee, \wedge) be a strong join hyperlattice such that $x \bigvee y = x \bigvee x \cup y \bigvee y$ and if $x \bigvee x = y \bigvee y$, then $x = y$. We define the relation \leq as $x \leq y$ implies $x \in y \bigvee y$. Thus, $(L, \bigvee, \wedge, \leq)$ is an ordered hyperlattice.

Definition 1.2. [5] Let $(L_1, \bigvee_1, \wedge_1, \leq_1)$ and $(L_2, \bigvee_2, \wedge_2, \leq_2)$ be two ordered hyperlattice. Give $(L_1 \times L_2, \bigvee', \wedge', \leq)$, \bigvee' and \wedge' are two hyperoperations on $L_1 \times L_2$ such that for any $(x_1, y_1), (x_2, y_2) \in L_1 \times L_2$, we have $(x_1, y_1) \bigvee' (x_2, y_2) = \{(u, v) : u \in x_1 \bigvee_1 x_2, v \in y_1 \bigvee_2 y_2\}$,

$(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq_1 x_2, y_1 \leq_2 y_2$. The hyperoperation \wedge' is defined similar to \vee' .

Definition 1.3. [4] Let \mathcal{R} be an equivalence relation on a nonempty set L and $A, B \subseteq L$, $A \bar{\mathcal{R}} B$ means that $\forall a \in A, \exists b \in B$ such that $a \mathcal{R} b$ and $\forall b' \in B, \exists a' \in A$ such that $a' \mathcal{R} b'$.

Also, \mathcal{R} is called a regular relation respect to \vee (respect to \wedge) if $x \mathcal{R} y$ implies that $x \vee z \bar{\mathcal{R}} y \vee z (x \wedge z \bar{\mathcal{R}} y \wedge z)$, for all $x, y, z \in L$. \mathcal{R} is called a regular relation if it is regular respect to \vee and \wedge , at the same time.

Theorem 1.1. [14] Let (L, \vee, \wedge) be a hyperlattice and v be an equivalence relation on L . Then, $(L/v, \vee, \wedge)$ is a hyperlattice if and only if v is a regular relation.

2. Properties of prime ideals in product of two ordered join hyperlattices

In this section, we consider strong join hyperlattices. First we define prime ideals in strong join hyperlattices. Then, we investigate sufficient conditions of a subset of product of two ordered strong join hyperlattices is a prime ideal. Also, we define special elements in ordered strong join hyperlattices and we investigate the connection between these elements and ideals in ordered strong join hyperlattices.

By [14] an ideal P of a join hyperlattice L is prime if for all $x, y \in L$ and $x \wedge y \in P$, we have $x \in P$ or $y \in P$.

Proposition 2.1. Let L be a join hyperlattice. A subset P of a hyperlattice L is prime if and only if $L \setminus P$ is a subhyperlattice of L .

Proof. Let $x, y \in L \setminus P$. Then, by definition of prime ideal we have $x \wedge y \notin P$. So, $x \wedge y \in L \setminus P$. Now, we show that $x \vee y \subseteq L \setminus P$. Let $x \vee y \subseteq P$ and $x \notin P, y \notin P$. Since P is an ideal of hyperlattice L , we have $(x \vee y) \wedge x \in P, (x \vee y) \wedge y \in P$. So, by definition of join hyperlattice we have $x \in (x \vee y) \wedge x$. Therefore, $x \in P, y \in P$ and this is contradiction. Thus, $x \vee y \not\subseteq P$ and $x \vee y \subseteq L \setminus P$. Similarly, we show that $x \wedge y \in L \setminus P$ and $L \setminus P$ is a subhyperlattice of L . Now, Let $L \setminus P$ is a subhyperlattice of L and $x \wedge y \in P, x, y \notin P$. Thus, $x, y \in L \setminus P$ and we have $x \wedge y \in L \setminus P$. Therefore, $x \wedge y \notin P$ and this is contradiction and we conclude that $x \in P$ or $y \in P$. \square

Theorem 2.1. Let $(L_1, \vee_1, \wedge_1, \leq_1)$ and $(L_2, \vee_2, \wedge_2, \leq_2)$ be two ordered strong join hyperlattices and $L \subseteq L_1 \times L_2$. L is a prime ideal if and only if there exist a prime ideal $I \subseteq L_1$ and $J \subseteq L_2$ with the properties that for any $x \in L_1, x' \in L_2$ and $y \in I, y' \in J$, we have $x \vee_1 y \subseteq I, x' \vee_2 y' \subseteq J$ and $L = (I \times L_2) \cup (L_1 \times J)$.

Proof. First we prove the converse. We show that L is a prime ideal of $L_1 \times L_2$. Let $(x, z), (y, w) \in L$. If $x, y \in I$, we have $x \vee_1 y \subseteq I$. Since L_2 is a join hyperlattice, we have $z \vee_2 w \subseteq L_2$. Thus, $(x, z) \vee'(y, w) = x \vee_1 y \times z \vee_2 w \subseteq I \times L_2 \subseteq L$. If $z \in J, w \in L_2$, by condition of L_1 we have $(x, z) \vee'(y, w) = x \vee_1 y \times z \vee_2 w \subseteq I \times J \subseteq I \times L_2 \subseteq L$. Now, let $(x, y) \in L_1 \times L_2, (z, w) \in L$. If $z \in I$ and $w \in L_2$, we have $(x, y) \wedge'(z, w) \in I \times L_2 \subseteq L$ and if $z \in L_1$ and $w \in J$, we have $(x, y) \wedge'(z, w) \in L_1 \times J \subseteq L$. Now, we show that L is prime. Let $(x, y) \wedge'(z, w) \in L$. Thus, we have $x \wedge_1 z \times y \wedge_2 w \in L$. Therefore, (1) $x \wedge_1 z \in I, y \wedge_2 w \in L_2$ or (2) $x \wedge_1 z \in L_1, y \wedge_2 w \in J$. In the first case, since I is prime we have $x \in I$ or $z \in I$. Thus, $(x, y) \in I \times L_2$ or $(z, w) \in I \times L_2$. The second case is similar to first. Therefore, L is a prime ideal of L . Now, let L is a prime ideal and $(x, z) \in L$. We show that $\{x\} \times L_2 \subseteq L$ or $L_1 \times \{z\} \subseteq L$. If these relations are not true, there exist $y \in L_2$ and $w \in L_1$ such that $(x, y) \notin L$ and $(w, z) \notin L$. Since L is prime, we have $(x \wedge_1 w) \times (y \wedge_2 z) \notin L$. Also, since L is an ideal and $(x, z) \in L, (w, y) \in L_1 \times L_2$, we have $(x \wedge_1 w) \times (y \wedge_2 z) \in L$ and this is contradiction. Now, we define $A = \{x \in L_1; \{x\} \times L_2 \subseteq L\}$ and $B = \{z \in L_2; L_1 \times \{z\} \subseteq L\}$ and $I = \{y \in L_1; y \leq_1 a \text{ for some } a \in A\}$ and $J = \{y \in L_2; y \leq_1 b \text{ for some } b \in B\}$.

By these definitions, we can easily prove $L = (I \times L_1) \cup (L_2 \times J)$. Now, we show that I is a prime ideal and it has the property which is stated in the assumption of theorem. Let $x, y \in L_1$ and $x \wedge_1 y \in I$. Thus, there exists $a \in A$ such that $x \wedge_1 y \leq_1 a$. Also, $(x, a) \wedge'(y, a) = (x \wedge_1 y, a \wedge_2 a) \in I \times L_1 \subseteq L$. Since L is a prime ideal, we have $(x, a) \in L$ or $(y, a) \in L$. Therefore, since $a \in L_1$, we have $x \in I$ or $y \in I$. Now, let $x \in L_1, y \in I$. So, there exists $a \in A$ such that $y \leq_1 a$. Thus, $x \vee_1 y \leq_1 a \vee_1 y$. Since $a \vee_1 y \subseteq I$ and I is an ideal, we have $x \vee_1 y \in I$ and proof is completed. \square

Definition 2.1. Let (L, \vee, \wedge, \leq) be an ordered strong join hyperlattice which is not bounded and $x \in L$. If $(x \wedge L] = \bigcup_{y \in L} x \wedge y = \{y \in L; y \leq a \text{ for some } a \in x \wedge L\} = L$, then, we call x is right simple element. Now, let L is bounded with greatest element 1, $x \in L$ is right simple element, if $(x \wedge L] = L \setminus \{1\}$.

Example 2.1. [15] Let $L = \{0, x_1, x_2, 1\}$. Consider the following tables:

\vee	0	x_1	x_2	1
0	0	$\{x_1, x_2, 1\}$	$\{x_2, 1\}$	1
x_1	$\{x_1, x_2, 1\}$	x_1	$\{x_2, 1\}$	1
x_2	$\{x_2, 1\}$	$\{x_2, 1\}$	x_2	1
1	1	1	1	1

\wedge	0	x_1	x_2	1
0	0	0	0	0
x_1	0	x_1	x_1	x_1
x_2	0	x_1	x_2	x_2
1	0	x_1	x_2	1

Then, $x_2 \in L$ is a right simple element of L .

Theorem 2.2. Let L be a distributive ordered strong join hyperlattice and R be the set of all right simple elements of L . Then, we have R is a subhyperlattice of L . Also, if for any arbitrary subset $A, B \subseteq L, y \in L$ we have $y \in A \vee B$ implies that $y \in A$ or $y \in B$ and $L \setminus R$ is nonempty, then $L \setminus R$ is a maximal ideal of L .

Proof. Let $a, b \in R$. Thus, we have $(a \wedge L] = L, (b \wedge L] = L$ and $L = (a \wedge L] = (a \wedge (b \wedge L)) \subseteq (a \wedge b \wedge L)$. Therefore, $a \wedge b \in R$. Since $(A] \vee (B] \subseteq (A \vee B]$ and L is distributive, we have $L = (a \wedge L] \vee (b \wedge L] \subseteq ((a \vee b) \wedge L]$. So, $a \vee b \in R$. Now, let $x, y \in L \setminus R$. If $x \vee y \subseteq R$, we have $L = ((x \vee y) \wedge L]$. Let $z \in L$. Thus, there exists $z' \in (x \vee y) \wedge L$ such that $z \leq z'$. Therefore, there exists $w \in L$ such that $z' = (x \vee y) \wedge w$. Thus, $z' \in (x \wedge w) \vee (y \wedge w)$. By assumption, we conclude that $z' \in x \wedge w$ or $z' \in y \wedge w$. Therefore, $z \leq z' \in x \wedge w$ or $z \leq z' \in y \wedge w$. Thus, $z \in (x \wedge L]$ or $z \in (y \wedge L]$ and $L = (x \wedge L]$ or $L = (y \wedge L]$. So, $x \vee y \not\subseteq R$ and $x \vee y \subseteq L \setminus R$. Let $x \in L \setminus R$ and $y \in L$. We show that $x \wedge y \in L \setminus R$. Let $x \wedge y \in R$. We have $L = (x \wedge y \wedge L] \subseteq (x \wedge L] = L$ and this is contradiction. So, $x \wedge y \in L \setminus R$. We can easily show that $L \setminus R$ is a maximal ideal of L and proof is completed. \square

Theorem 2.3. Let (L, \vee, \wedge, \leq) be an ordered strong join hyperlattice and I be an ideal of L such that for any $y \in I$ and $x \in L \setminus I$ there exists $z \in x \wedge L$ such that $y \leq z$. Then, $L \setminus I$ is the set of all right simple elements of L .

Proof. Let $x \in L \setminus I$ and $y \in L$. By assumption there exists $z \in x \wedge L$ such that $y \leq z$. Thus, $y \in (x \wedge L]$ and $L = (x \wedge L]$. Now, let $x \in R$ and $x \in I$. Thus, for any $b \in L$ we have $a \wedge b \leq a \in I$ and $L = (a \wedge L] \subseteq I$. Thus, $L \setminus I$ is right simple elements of L . \square

3. Semiprime ideals in ordered join hyperlattices

In this section, we consider order relation \leq as $x \leq y$ if and only if $x = x \wedge y$ and we introduce semiprime ideals in ordered join hyperlattices. Then, we prove some results about them.

Definition 3.1. Let (L, \vee, \wedge, \leq) be an ordered join hyperlattice and $I \subseteq L$ be an ideal and F be a filter of L . We call I is a semiprime ideal if for every $x, y, z \in L$, $x \wedge y \in I$ and $x \wedge z \in I$ implies that $x \wedge (y \vee z) \subseteq I$. Also, we call F is a semiprime filter if $x \vee y \subseteq F$ and $x \vee z \subseteq F$ implies that $x \vee (y \wedge z) \subseteq F$.

Notice that every prime ideal I is semiprime. Since if $x \wedge y \in I$ and $x \wedge z \in I$, we have $x \in I$ or $y \in I$ and $x \in I$ or $z \in I$. If $x \in I$, by $x \wedge (y \vee z) \leq x$ we have $x \wedge (y \vee z) \subseteq I$. Otherwise, we have $y, z \in I$. So, $y \vee z \subseteq I$ and $x \wedge (y \vee z) \subseteq I$.

Proposition 3.1. Let (L, \vee, \wedge, \leq) be an ordered join hyperlattice and I be a semiprime ideal of L . Also, for any $A, B \subseteq L$, $A \leq B \subseteq I$ implies that $A \subseteq I$. Then, $I_1 = \{J \in Id(L); J \subseteq I\}$ is a semiprime ideal of L . If L is a finite hyperlattice, $I_2 = \cup\{J; J \subseteq I\}$ is a semiprime ideal of L .

Proof. Let $J_1, J_2 \subseteq I$. Then, $J_1 \vee J_2 \subseteq I \vee I$. Since I is an ideal of L , we have $I \vee I \subseteq I$. Therefore, $J_1 \vee J_2 \subseteq I$. Now, Let $J_1 \wedge J_2 \subseteq I_1$, $J_1 \wedge J_3 \subseteq I_1$ for any $J_1, J_2, J_3 \in Id(L)$. Then, let $x' \in J_1 \wedge (J_2 \vee J_3)$. Thus, $x' = x \wedge y$ for $x \in J_1, y \in J_2 \vee J_3$. Therefore, $y = y' \vee y''$ for some $y' \in J_2$ and $y'' \in J_3$. We have $x \wedge y' \in J_1 \wedge J_2 \subseteq I$ and $x \wedge y'' \in J_1 \wedge J_3 \subseteq I$. Therefore, since I is semiprime, we have $x \wedge (y' \vee y'') \subseteq I$ and $J_1 \wedge (J_2 \vee J_3) \subseteq I$. Now, if L is finite, we show that I_2 is a semiprime ideal. Let $x, y \in I_2$. Thus, $x \in J_1 \subseteq I$ and $y \in J_2 \subseteq I$. Therefore, $x \vee y \subseteq J_1 \vee J_2 \subseteq I$. Let $x \leq y \in J_1 \subseteq I$. Since I is an ideal, we have $x \in I$ and $x \in I_2$. Since L is finite, the semiprimeness of I_2 is ensumbered. \square

Theorem 3.1. Let L be a s -good($x \vee 0 = x$) bounded ordered join hyperlattice and I be an ideal and F be a filter of L such that $I \cap F = \emptyset$ and for any $A \subseteq L$, $A \vee 1 \subseteq F$ implies that $A \subseteq F$. If F is a semiprime filter, there exists a semiprime ideal J such that $I \subseteq J$ and $J \cap F = \emptyset$.

Proof. Let F be a semiprime filter and θ be a congruence on L which is defined as $a\theta b$ if and only if $F : a = F : b$ where $F : a = \{x \in L; a \vee x \subseteq F\}$. Clearly, θ be an equivalence relation. Now, we show that θ is compatible with \vee and \wedge . Let $a\theta b$, since F is semiprime filter, we have $F : a \wedge c = (F : a) \cap (F : c) = (F : b) \cap (F : c) = F : b \wedge c$. Thus, $a \wedge c \theta b \wedge c$. Let $y \in F : a \vee c$. Thus, $y \vee a \vee c \subseteq F$ and therefore $y \vee c \subseteq F : a = F : b$. So, $y \vee c \vee b \subseteq F$ and $y \in F : c \vee b$. Therefore, θ is compatible with \vee . Clearly, θ is a strongly regular relation and therefore L/θ is a lattice. Now, we show that L/θ is a distributive lattice. Let $s\theta x \wedge (y \vee z)$ and $u \in F : s = F : x \wedge (y \vee z)$. Therefore, $A = u \vee (x \wedge (y \vee z)) \subseteq F$. Since L is bounded, we have $A \leq u \vee (1 \wedge (y \vee 1)) \leq u \vee (y \vee 1)$. So, we have $u \vee y \subseteq F$ and $u \vee x \subseteq F$. By semiprime property of F , we have $u \vee (x \wedge y) \subseteq F$ and since $u \vee (x \wedge y) \leq u \vee (x \wedge y) \vee (x \wedge z)$. Therefore, $u \in F : (x \wedge y) \vee (x \wedge z)$ and L/θ is a distributive lattice. Also, in L/θ we have $I_\theta \cap F_\theta = \emptyset$. Becaue if there exists $y \in I_\theta \cap F_\theta$, we have $I\theta F$. Thus, $F : I = F : F$ and since $0 \vee F = F \subseteq F$, we have $0 \in F : I$. Therefore, $0 \vee I = I \subseteq F$ and this is contradiction with $I \cap F = \emptyset$. So, $I_\theta \cap F_\theta = \emptyset$. Now, By theorem of [15] there exists $P_\theta \in L/\theta$ such that $I_\theta \subseteq P_\theta$ and P_θ is a prime ideal. We consider canonical map $h : L \longrightarrow L/\theta$ by $h(a) = \theta(a)$. So, we have $I \subseteq h^{-1}(P_\theta) = P$, $P \cap F = \emptyset$ and P is a prime(semiprime) ideal of L . \square

Theorem 3.2. Let (L, \vee, \wedge, \leq) be an ordered join hyperlattice. L is a distributive hyperlattice if and only if for every ideal I and filter F of L such that $I \cap F = \emptyset$, there exist ideal J and filter G of L such that $I \subseteq J$, $F \subseteq G$, $J \cap G = \emptyset$, J or G is semiprime and for every $x \in L$, we have $x \in J \cup G$.

Proof. Let L be a distributive hyperlattice. By [15] one side proof of theorem is completed. Now, we consider such ideal and filter with their properties exist. We show that L is distributive. Let $x, y, z \in L$ and I be the ideal which is generated by $(x \wedge y) \vee (x \wedge z)$ and F be a filter which is generated by $x \wedge (y \vee z)$. Now, let $x \wedge (y \vee z) \not\leq (x \wedge y) \vee (x \wedge z)$. Therefore, $I \cap F = \emptyset$. Thus, there exist ideal J and filter G such that $I \subseteq J$ and $F \subseteq G$, $J \cap G = \emptyset$. If J is semiprime ideal, since $x \wedge y \in J$ and $x \wedge z \in J$, we have $x \wedge (y \vee z) \subseteq J$. Since $x \wedge (y \vee z) \subseteq G$, we have $J \cap G \neq \emptyset$ and this is contradiction. If G is semiprime, we have $x \in G$ and $y \vee z \subseteq G$. If $y \in G$, since $x \in G$, we have $x \wedge y \in G$, and if $z \in G$, we have $x \wedge z \in G$, this is contradict with $J \cap G = \emptyset$. So, neither y nor z are not in G . If both $y, z \in J$, $y \vee z \subseteq J$. This is contradiction with $J \cap G = \emptyset$. So, both $y, z \in J$ is impossible. Let $y \notin J$ and $z \in J$. We have $x \wedge z \in J$. Since $x \wedge y \leq (x \wedge y) \vee (x \wedge z) \in J$, we have $x \wedge y \in J$. But $x \wedge y \in G$, and this is contradiction. So, we have $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$. Now, let $(x \wedge y) \vee (x \wedge z) \not\leq x \wedge (y \vee z)$ and I is an ideal which is generated by $x \wedge (y \vee z)$, F be a filter which is generated by $(x \wedge y) \vee (x \wedge z)$. Similar to above arguments, we get to contradiction and the proof is completed. \square

Theorem 3.3. *Let (L, \vee, \wedge, \leq) be an ordered join hyperlattice and for each $x \in L$, we have $x \vee x = x$. Then, L is distributive if and only if for every $x, y \in L$, $(x \vee y]$ is a semiprime ideal.*

Proof. Let L be a distributive hyperlattice and $a \wedge b \in (x \vee y]$, $a \wedge c \in (x \vee y]$ for any $a, b, c \in L$. Then, $(a \wedge b) \vee (a \wedge c) \leq (x \vee y) \vee (x \vee y) = x \vee y$. So, $a \wedge (b \vee c) \in (x \vee y)$ and $(x \vee y]$ is semiprime. Now, we show that $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$, for any $x, y, z \in L$. Since $x \wedge y \leq (x \wedge y) \vee (x \wedge z)$ and $x \wedge z \leq (x \wedge y) \vee (x \wedge z)$, there exist $v, w \in (x \wedge y) \vee (x \wedge z)$ such that $x \wedge y \leq v$, $x \wedge z \leq w$. Notice that $v \leq v \vee w$ and $w \leq v \vee w$. So, since $(v \vee w]$ is semiprime ideal, we have $x \wedge (y \vee z) \leq v \vee w$. We have $v \vee w \subseteq ((x \wedge y) \vee (x \wedge z)) \vee (x \wedge y) \vee (x \wedge z) = (x \wedge y) \vee (x \wedge z)$. Therefore, $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$. Now, Let $I = (x \wedge (y \vee z)]$. Since I is an ideal of L and $x \wedge y \in I$, $x \wedge z \in I$, we have $(x \wedge y) \vee (x \wedge z) \subseteq I$. So, $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$. \square

Theorem 3.4. *Let (L, \vee, \wedge, \leq) be an ordered join hyperlattice which has an element c such that $(c], [c)$ is semiprime and for each $x \in L$, $x \vee x = x$ and has the properties of 3.1. Also, for any ideal $I, J \subseteq L$ with $I \subseteq J$, if I is semiprime, we have J is semiprime and for any filter $F, J \subseteq L$ with $J \subseteq F$, if J or F is semiprime, we have other subset is semiprime. Then, L is a distributive hyperlattice.*

Proof. It suffices to show that for any $x, y, z \in L$, $K = ((x \wedge y) \vee (x \wedge z)]$ is semiprime. Since by $x \wedge y \in K$, $x \wedge z \in K$ and semiprime property of K , we have $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$. First, we show that for any $x \in L$, $(x]$ is semiprime. If we show this, we have $(x \wedge y) \subseteq (x \wedge y) \vee (x \wedge z) \subseteq ((x \wedge y) \vee (x \wedge z)]$, and by supposition proof is completed. To show that $(x]$ is semiprime, let $x \in L$. If $c \leq x$, then $(c] \subseteq (x]$. Since $(c]$ is semiprime, we have $(x]$ is semiprime and proof is completed. Now, let $c \not\leq x$. If we show that $(c \wedge x]$ is semiprime, by $(c \wedge x] \subseteq (x]$ proof is completed. Let A be the intersection of semiprime ideals of L which contains $(c \wedge x]$. So, A is semiprime ideal. If we show that $A = (c \wedge x]$, proof is completed. Since A contains $(c \wedge x]$, we have $(c \wedge x] \subseteq A$. Now, let $a \in A$. Therefore $a \in I_i$ such that I_i are semiprime ideals which contains $(c \wedge x]$. If $a \leq c \wedge x$, $A \subseteq (c \wedge x]$. Otherwise, let $a \not\leq c \wedge x$. Then, let $I = (x \wedge c]$ and $F = [a \vee (x \wedge c)]$. Clearly, $I \cap F = \emptyset$ and by hypothesis $[a \vee c] \subseteq [c]$ is semiprime. Also, $[a \vee c] \subseteq [a \vee (x \wedge c)]$. So, F is semiprime filter and by theorem 3.1, there exists a semiprime ideal J such that $I \subseteq J$ and $J \cap F = \emptyset$. Since $c \wedge x \in J$ and J is semiprime ideal which contains $(c \wedge x]$, we conclude that $a \in J$. Therefore, $a \vee (c \wedge x) \subseteq J$. But $a \vee (c \wedge x) \subseteq F$ and this is contradiction with $J \cap F = \emptyset$. So, $a \leq c \wedge x$ and $A = (c \wedge x]$ is a semiprime ideal. Now, Let $I = (x \wedge (y \vee z)]$. Since $x \wedge y \in I$, $x \wedge z \in I$ and I is an ideal, we have $(x \wedge y) \vee (x \wedge z) \subseteq I$. So, $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$. \square

Definition 3.2. Let (L, \vee, \wedge, \leq) be an ordered join hyperlattice. We call L be 0-distributive if $a \wedge b = a \wedge c = 0$, then $a \wedge (b \vee c) = 0$. For each $x \in L$, we define $x^\perp = \{y \in L; x \wedge y = 0\}$.

Proposition 3.2. Let (L, \vee, \wedge, \leq) be an ordered join hyperlattice which is 0-distributive. Then, x^\perp is a semiprime ideal of L .

Proof. Let $u, v \in x^\perp$. Thus, $x \wedge u = 0$ and $x \wedge v = 0$. Since L is 0-distributive, $x \wedge (u \vee v) = 0$. So, $u \vee v \subseteq x^\perp$. If $a \leq b$ and $b \in x^\perp$, we have $a \wedge x \leq b \wedge x = 0$. Therefore, $a \in x^\perp$. Now, let $a \wedge b \in x^\perp$ and $a \wedge c \in x^\perp$. So, $a \wedge b \wedge x = 0$ and $a \wedge c \wedge x = 0$. Since L is a 0-distributive hyperlattice, $a \wedge x \wedge (b \vee c) = 0$. Therefore, $x \wedge (a \wedge (b \vee c)) = 0$ and $a \wedge (b \vee c) \subseteq x^\perp$. So, x^\perp is a semiprime ideal of L . \square

Theorem 3.5. Let (L, \vee, \wedge, \leq) be a good($0 \vee 0 = 0$) ordered join hyperlattice. Then, the following statements are equivalent:

- : (1) L is 0-distributive;
- : (2) If for any $a, b \in L$ which $a \not\leq b$, we have $a \wedge b = 0$, then $[p]$ is a prime filter (for any $p \in L$);
- : (3) for every $x \in L$, x^\perp is an ideal of L .

Proof. (1) \implies (2). Let $x, y \in [p]$. Thus, $p \leq x$ and $p \leq y$. Since \leq is an order relation, we have $p \wedge p \leq x \wedge y$. Thus, $x \wedge y \in [p]$. Now, let $x \leq y$ and $x \in [p]$. We have $p \leq x \leq y$. Therefore $y \in [p]$. Let $x \vee y \subseteq [p]$. If $x \notin [p]$ and $y \notin [p]$, we have $p \not\leq x$ and $p \not\leq y$. By hypothesise $p \wedge x = 0$ and $p \wedge y = 0$. Since L is 0-distributive, $p \wedge (x \vee y) = 0$ and this result is contradiction with $p \leq x \vee y$. So, $x \in [p]$ or $y \in [p]$.

(2) \implies (3). Let $a, b \in x^\perp$. If $x \not\leq a \vee b$ or $a \vee b \not\leq x$, we have $x \wedge (a \vee b) = 0$. Now, let $x \neq 0 \leq a \vee b$. There exists $t \in a \vee b$ such that $x \leq t$. Therefore, $a \vee b \in [x]$ and since by hypothesise $[x]$ is a semiprime filter, $a \in [x]$ or $b \in [x]$. Thus, $x \wedge a = x$ or $x \wedge b = x$. By $a, b \in x^\perp$, $x = 0$ and this is contradiction. So, $x \leq a \vee b$ is not hold. (3) \implies (1). Let $a \wedge b = 0$ and $a \wedge c = 0$. Therefore, $b \in a^\perp$ and $c \in a^\perp$. Since a^\perp is an ideal of L , $b \vee c \subseteq a^\perp$. So, $a \wedge (b \vee c) = 0$ and L is a 0-distributive hyperlattice. \square

Theorem 3.6. Let (L, \vee, \wedge, \leq) be a s -good bounded 0-distributive join hyperlattice which is s -distributive and not complemented. Also, L has the property that for every $A, B \subseteq L$, $1 \in A \leq B$ implies that $1 \in B$ and $A \vee 1 \subseteq F$ implies $A \subseteq F$. Then, there exist semiprime ideals I, J such that $I \subseteq J$.

Proof. Let $c \in L$ has no complement in L . Since L is not complemented, such element exists. Now, consider $c^\perp = \{x \in L; x \wedge c = 0\}$. By proposition 3.2, c^\perp is a semiprime ideal of L . Consider $F = \{x \in L; 1 \in x \vee c\}$. We show that F is semiprime filter of L . Let $x, y \in F$. Thus, $1 \in x \vee c$ and $1 \in y \vee c$ and $(x \vee c) \wedge (y \vee c) = c \vee (x \wedge y)$. Therefore, $1 \in c \vee (x \wedge y)$ and we conclude that $x \wedge y \subseteq F$. Also, if $x \leq y$ and $1 \in x \vee c$, we have $x \vee c \leq y \vee c$. So, $1 \in y \vee c$ and $y \in F$. Now, let $a \wedge b \in F$, $a \wedge d \in F$. Thus, $1 \in (a \wedge b) \vee c$ and $1 \in (a \wedge d) \vee c \leq (a \wedge (b \vee d)) \vee c$. Therefore, $1 \in (a \wedge (b \vee d)) \vee c$ and $a \wedge (b \vee d) \in F$. Notice that $I \cap F = \emptyset$. Since if there exists $x \in I \cap F$, we have $x \wedge c = 0$ and $1 \in x \vee c$. This is contradiction with c has not complement. Now, by theorem 3.1 there exists a semiprime ideal J such that $I \subseteq J$ and $J \cap F = \emptyset$. \square

4. Quotient of ordered join hyperlattices with a regular relation

In this section, we study special relation which is regular on ordered join hyperlattices which has connection with order on L and we derive ordered join hyperlattice from an ordered join hyperlattice with such regular relation.

Let (L, \vee, \wedge, \leq) be an ordered strong join hyperlattice and v be a relation which is transitive and contains the relation \leq . Moreover, for any $x, y \in L$, if $x v y$, we have $x \vee z \bar{v} y \vee z$ and $x \wedge z \bar{v} y \wedge z$, for all $z \in L$ and $x \in y \vee z$ implies that $x v y$, $y v x$, $x v z$, $z v x$,

we call such relations as quasi-ordered relations. We know that if v is a regular relation, the quotient L/v is a hyperlattice. But this relation is not equivalence relation. So, we define $v^* = \{(a, b) \in v \times v; avb, bva\}$ and in the following theorem we show that L/v^* is an ordered hyperlattice.

Theorem 4.1. *Let (L, \vee, \wedge, \leq) be an ordered strong join hyperlattice and v^* be a relation which is defined above. Thus, L/v^* is an ordered hyperlattice.*

Proof. We can easily show that v^* is an equivalence relation. Now, we show that v^* is a regular relation. Let $xv^*y, z \in L$ and $x' \in x \vee z$. Thus, xvy and yvx . Therefore, $x \vee z \bar{v} y \vee z$, $y \vee z \bar{v} x \vee z$ and we conclude there exists $y' \in y \vee z$ such that $x'vy'$. By property of v , we have $y'vz$ and zvx' . So, $y'vx'$ and $x \vee z \bar{v}^* y \vee z$. Since \wedge is binary operation, we can easily show that $x \wedge z \bar{v}^* y \wedge z$. So, v^* is a regular relation and L/v^* is a hyperlattice. Now, we show L/v^* is ordered. Let $v^*(x) \preceq v^*(y)$. Thus, since L/v^* is a hyperlattice, we have $v^*(x) \vee v^*(z) = v^*(z')$ where $z' \in x \vee z$ and $v^*(y) \vee v^*(z) = v^*(w)$, $w \in y \vee z$. Thus, there exist $x' \in v^*(x)$ and $y' \in v^*(y)$ such that $x' \leq y'$. Therefore, we have $x' \vee z \leq y' \vee z$ and therefore $x' \vee z \bar{v} y' \vee z$. So, $v^*(x') \vee v^*(z) \preceq v^*(y') \vee v^*(z)$. Since $v^*(x') = v^*(x)$ and $v^*(y') = v^*(y)$, we have $v^*(x) \vee v^*(z) \preceq v^*(y) \vee v^*(z)$. Therefore, L/v^* is an ordered hyperlattice. \square

Theorem 4.2. *Let (L, \vee, \wedge, \leq) be an ordered strong join hyperlattice and v be a quasi-ordered relation. There is one to one correspondence between quasi-ordered relations on L which contain v and quasi ordered relations on L/v^* .*

Proof. Let η be a quasi-ordered relation on L/v^* . We show

$$\tau = \{(x, y); (v^*(x), v^*(y)) \in \eta\}$$

is a quasi-ordered relation on L which contains v . Let $x \leq y$. So, xvy and $(v^*(x), v^*(y)) \in L/v^*$. Since η is a quasi-ordered relation, it follows that $(v^*(x), v^*(y)) \in \eta$ and so $(x, y) \in \tau$, $\leq \subseteq \tau$. We can easily show that τ has transitive property. Now, let $x \in y \vee z$. So, $v^*(x) \in v^*(y) \vee v^*(z)$ where \vee is hyperoperation on L/v^* . Therefore, $(v^*(x), v^*(y)) \in \eta$ and $(v^*(x), v^*(z)) \in \eta$. So, $(x, y) \in \tau$, $(x, z) \in \tau$, $(y, x) \in \tau$, $(z, x) \in \tau$ and let $(x, y) \in \tau$, $a \in x \vee z$. So, $v^*(a) \in v^*(x) \vee v^*(z)$ and since η is a quasi-ordered, there exists $v^*(b) \in v^*(y) \vee v^*(z)$ such that $(v^*(a), v^*(b)) \in \eta$. So, $(a, b) \in \tau$. Also, we can prove for \wedge and τ is a quasi-ordered relation on L . Similarly, we can show that if we have a quasi-ordered relation on L which contains v , then there exists a quasi-ordered relation on L/v^* . \square

Proposition 4.1. *Let (L, \vee, \wedge) be an ordered strong join hyperlattice and v, τ be two quasi-ordered relations on L such that $v \subseteq \tau$ and $v^*(x) \tau/v v^*(y)$ if and only if $\exists a \in v^*(x), \exists b \in v^*(y), avb$. Then, τ/v is a quasi-ordered relation on L/v^* .*

Proof. Let $(v^*(x), v^*(y)) \in \tau/v$. Thus, there exist $x \in v^*(a)$, $y \in v^*(b)$ such that xvy . Therefore, avx and yvb . So, avb and since $v \subseteq \tau$, we have $a\tau b$. We can easily show that τ/v contains \preceq on L/v^* and has transitive property. Now, we let $(v^*(x), v^*(y)) \in \tau/v$ and $v^*(z) \in L/v^*$, $v^*(c) \in v^*(x) \vee v^*(z)$. Thus, $(x, y) \in v$ and $c \in x \vee z$. Since τ is a quasi-ordered relation, there exists $u \in y \vee z$ such that $c\tau u$. Therefore, there exists $v^*(u) \in v^*(y) \vee v^*(z)$ such that $(v^*(c), v^*(u)) \in \tau/v$. Also, let $v^*(z) \in v^*(x) \vee v^*(y)$. Thus, $z \in x \vee y$ and $z\tau x, z\tau y, x\tau z, y\tau z$. Therefore, $(v^*(z), v^*(x)) \in \tau/v$, $(v^*(z), v^*(y)) \in \tau/v$, $(v^*(x), v^*(z)) \in \tau/v$, $(v^*(y), v^*(z)) \in \tau/v$. Similarly, we prove for \wedge and proof is completed. \square

In the previous section, we study prime hyperideals in $L_1 \times L_2$. Now, we investigate quasi-ordered relation on $L_1 \times L_2$ in the following theorem.

Theorem 4.3. Let $(L_1, \vee_1, \wedge_1, \leq_1)$ and $(L_2, \vee_2, \wedge_2, \leq_2)$ be two ordered strong join hyperlattices and v_1, v_2 be quasi-ordered relations on L_1 and L_2 . Then, $(L_1 \times L_2)/v^*$ isomorphic to $L_1/v_1^* \times L_2/v_2^*$.

Proof. We define $f : (L_1 \times L_2)/v^* \longrightarrow L_1/v_1^* \times L_2/v_2^*$ by $f(v^*(a), v^*(b)) = (v_1^*(a), v_2^*(b))$. We can easily show that f is well defined and one to one. Now, we show that f is a homomorphism between two ordered join hyperlattices. $f(v^*(a_1, b_1) \vee v^*(a_2, b_2)) = f(v^*(u, v))$ where $u \in a_1 \vee a_2, v \in b_1 \vee b_2$. So, we have $f(v^*(a_1, b_1) \vee v^*(a_2, b_2)) = (v^*(a_1) \vee_1 v^*(a_2), v^*(b_1) \vee_2 v^*(b_2)) = f(v^*(a_1, b_1)) \times f(v^*(a_2, b_2))$. Similarly, these relations holds for binary operation \wedge and by definition of order on $L_1 \times L_2$, if $v^*(a_1, b_1) \preceq v^*(a_2, b_2)$, we have $(a_1, b_1)v(a_2, b_2)$. Therefore, $(a_1, a_2) \in v_1$ and $(b_1, b_2) \in v_2$. Thus, $v_1^*(a_1) \leq_1 v_1^*(a_2)$ and $v_2^*(b_1) \leq_2 v_2^*(b_2)$. Therefore, f is an order preserving map and it is clear that f is onto. So, f is isomorphism and proof is completed. \square

REFERENCES

- [1] G. Birkhoff, Lattice Theory, Amer. Math. Soc, New York, 1940.
- [2] I. Chajda and S. Hoskova, factorization of quasiorder hypergroups, Comment. Math. Univ. Carolin., **45(4)** (2004), 573-581.
- [3] J. Chvalina, Commutative hypergroup in the sense of Marty and ordered Sets. Gen. Alg. and Ordered Sets, Proc. Inter. Conf. Olomouc, (1994), 19-30.
- [4] B. Davvaz, Polygroup Theory and Related Systems, World Scientific, 2013.
- [5] X. Z. Guo and X. L. Xin, Hyperlattice, Pure Appl. Math., **20** (2004), 40-43.
- [6] D. Heidari and B. Davvaz, On ordered hyperstructures, University Politehnica of Bucharest Scientific Bulletin, Series A: Applied Mathematics and Physics, **73(2)** (2011), 85-96.
- [7] V. V. Joshi and N. Mundlik, Prime ideals in 0-distributive posets, Central European Journal of Mathematics, **11(5)** (2013), 940-955.
- [8] V. V. Joshi and B. N. Waphare, Characterizations of 0-distributive posets, Math. Bohem., **130(1)** (2005), 73-80.
- [9] Y.B. Jun, K.J. Lee and A. Khan, Soft ordered semigroups, Mathematical Logic Quarterly, **56** (2010), 42-50.
- [10] V. S. Kharat and K. A. Mokbel, Semiprime ideals and separation theorems for posets, Order, **25(3)** (2008), 195-210.
- [11] V. S. Kharat and K. A. Mokbel, Primeness and semiprimeness in posets, Math. Bohem., **134(1)** (2009), 19-30.
- [12] M. Konstantinidou and J. Mittas, An introduction to the theory of hyperlattices, Mathematica Balkanica, **7** (1977), 187-193.
- [13] F. Marty, Sur une generalisation de la notion de groupe, in 8th congrès des Mathématiciens Scandinaves, (1934) 45-49.
- [14] S. Rasouli and B. Davvaz, Lattices derived from hyperlattices, Comm. Algebra, **38(8)** (2010), 2720-2737.
- [15] S. Rasouli and B. Davvaz, Construction and spectral topology on hyperlattices, Mediterranean Journal of Mathematics, **7(2)** (2010), 249-262.
- [16] Y. Rav, Semiprime ideals in general lattices, J. Pure Appl. Algebra, **56(2)** (1989), 105-118.

- [17] *J. Varlet*, Relative annihilators in semilattices, *Bulletin Australian Mathematical Society*, **9** (1973), 169-185.
- [18] *X. L. Xin and X. G. Li*, On hyperlattices and quotient hyperlattices, *Southeast Asian Bulletin of Mathematics*, **33** (2009), 299-311.