

## LONGTIME BEHAVIOR OF A NON-AUTONOMOUS BEAM EQUATION

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*A perturbed non-autonomous equation of hyperbolic type with the structural damped terms, which arises from the vibration of a beam, is considered. By using a two-parameter operator family and decomposing the analytic semigroup, we obtain the compact kernel sections, which are the key ingredients to show the existence of the global attractor in the space  $H_0^2(\Omega) \times L^2(\Omega)$ .*

**Keywords:** Non-autonomous beam equation, Structural damped, Kernel sections, Global attractor

### 1. Introduction

This paper studies the longtime asymptotic behavior of a perturbed non-autonomous beam equation:

$$u_{tt} + \Delta^2 u + \mu \Delta^2 u_t + \eta u_t - [\alpha(t) + M(\int_{\Omega} |\nabla u|^2 dx) + N(\int_{\Omega} \nabla u \nabla u_t dx)] \Delta u = h(x, t), \quad x \in \Omega, \quad t > \tau, \quad \tau \in R^+, \quad (1)$$

associated with the following initial and boundary value conditions:

$$u(x, t)|_{x \in \partial\Omega} = \Delta u(x, t)|_{x \in \partial\Omega} = 0, \quad t \geq \tau, \quad (2)$$

$$u(x, \tau) = u_{0\tau}(x) \in H_0^2(\Omega), \quad u_t(x, \tau) = u_{1\tau}(x) \in L^2(\Omega), \quad x \in \Omega, \quad (3)$$

where  $u = u(x, t)$  describes the transversal motion of a beam.  $x$  is the space variable, and Eq.(1) is posed in an open bounded connected domain  $\Omega$  in  $R^n$  (with a sufficiently smooth boundary  $\partial\Omega$ ). The parameters  $\mu$  and  $\eta$  are nonnegative, and  $\Delta$  is the Laplacian in  $R^n$ . For the external force, we assume that  $h(x, t)$  is periodic with respect to  $t$  and satisfies:

$$h(x, t) \in L^\infty(R; L^2(\Omega)),$$

$$h'(x, t) \in C_b(R; L^2(\Omega)) = C(R; L^2(\Omega)) \cap L^\infty(R; L^2(\Omega)).$$

As it is well known (see [1]), attractor is an important problem studying the longtime asymptotic behavior of dynamical system. System (1)-(3) describes the nonlinear transversal vibrated motion of an elastic beam. In this paper we investigate the non-autonomous system (1)-(3) via the compact kernel

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sections of the corresponding family of processes  $\{S_\varepsilon(t, \tau), \varepsilon \in \Sigma\}$ , in which  $\Sigma$  is a two-parameter set.

Let us recall some investigation in this area. To the best of our knowledge, the classical equations in hyperbolic system were presented by Woinowsky as a new idea in the field of nonlinear analysis [2]. After that, hyperbolic problem with the nonlinearity analogous with the system (1)-(3) has drawn much attention. Recently, many classical results of the attractor have been obtained. For the autonomous beam equations, Chueshov and Lasiecka in [3] considered the existence and structure of the global attractor for dynamic von Karman equations with a nonlinear boundary dissipation. If the axial force was added, Yang in [4] and Kolbasin in [5] were concerned with the attractor of the nonlinear wave equation arising in elastic waveguide model. When the attractor is posed on unbounded domain, the case can be complex, which is mainly caused by the existence of a Lyapunov functional. To solve this problem, we refer to [6-8] for the detailed description of the growth exponent of  $f(x, u)$ . In addition, it was used to testify the existence of strong solutions and global attractors for the suspension bridge equations in the stronger space under the condition that the damped term is critical [9]. Finally, in the case of plate equations, the asymptotic behavior of solutions with a localized damping and a critical exponent was studied in [10-12].

Comparing with the autonomous case, the non-autonomous equations are more complex because the external force is time-dependent. Under appropriate assumptions, we need to prove that the external force belongs to the contractive function. In recent years, the non-autonomous string equations have attracted more attention than before. For instance, the pullback, uniform and global attractor of the string equations were explored in [13-17]. However, non-autonomous beam equations have been less discussed, which is our concern in this paper. Furthermore, Eq.(1) contains the structural damped terms of  $\Delta^2 u$ , and  $N(\int_\Omega \nabla u \nabla u, dx) \Delta u$ , and it makes our study more mechanically significant.

This paper is organized as follows. In section 2, we give some preparations for our consideration on forcing term  $h(x, t)$ , as well as on nonlinearities  $\alpha(\cdot)$  and  $M(\cdot)$ . Using the new sectorial operator approach, the existence of solutions is proved. In section 3, we show the boundedness of compact kernel sections. In the last section, by decomposing the analytic semigroup of Eq.(1), we obtain the existence of the global attractor generated by the system (1)-(3).

## 2. Preliminaries

In this section, we formulate the system (1)-(3) as abstract Cauchy initial-boundary value problem. With the usual notation, some other notations will be introduced and used throughout this paper. We denote the Hilbert spaces  $H = L^2(\Omega)$ ,  $V = H_0^2(\Omega)$ ,  $V \subset H$ ,  $V$  dense in  $H$ , the injection of  $V$  in  $H$  being continuous, and we endow these spaces with the usual scalar product and the norms in  $H$  and  $V$  are, respectively, denoted by  $(\cdot, \cdot)$ ,  $\|\cdot\|$  and  $((\cdot, \cdot))$ ,  $\|\cdot\|$ , where

$$(u, v) = \int_{\Omega} u(x)v(x)dx, \quad \|u\|^2 = (u, u), \quad \forall u, v \in L^2(\Omega),$$

$$((u, v)) = \int_{\Omega} \Delta u(x)\Delta v(x)dx, \quad \|u\|^2 = ((u, u)), \quad \forall u, v \in H_0^2(\Omega).$$

We identify  $H$  with its dual  $H^*$ , and  $H^*$  with a dense subspace of the dual  $V^*$  of  $V$  (norm  $\|\cdot\|_*$ ); thus,

$$V \subset H \subset V^*.$$

So, all embeddings are continuous and their domains are dense, where the injections are continuous and each space is dense. Let us denote by  $A: D(A) \subset H \rightarrow H$  the operator

$$Au = \Delta^2 u, \quad \text{for } u \in D(A).$$

It is well known that the linear unbounded operator  $A$  is an isomorphism from  $V$  onto  $V^*$  and it can also be considered as a self-adjoint positive operator strictly defined on a separable Hilbert space  $H$  with domain  $D(A) \subset V$ , where

$$D(A) = \left\{ \varphi \in H^4(\Omega), \varphi|_{x=\partial\Omega} = A^{\frac{1}{2}}\varphi|_{x=\partial\Omega} = 0 \right\}.$$

The space  $D(A)$  is dense in  $H$ , and it is a Banach space when endowed with the graph norm  $u \rightarrow |u|_H^2 + |A^{\frac{1}{2}}u|_H^2$ . We also assume that the resolvent of  $A$  is compact in  $H$ , and one denotes by  $\{e_k\}$  the orthonormal basis in  $H$ , consisting of eigenfunctions of the operator  $A^{\frac{1}{2}}$

$$A^{\frac{1}{2}}e_k = \lambda_k e_k,$$

and the eigenvalues  $\{\lambda_i\}_{i \in \mathbb{N}}$  of it satisfy:

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m \leq \dots, \quad \text{and} \quad \lim_{m \rightarrow +\infty} \lambda_m = +\infty.$$

Under these hypotheses, it is possible to define the powers  $A^s$  of  $A$  for  $s \in \mathbb{R}$ , which operate on the spaces  $D(A^s)$ . We have  $D(A^0) = H$ ,  $D(A^{\frac{1}{2}}) = V$ ,  $D(A^{-\frac{1}{2}}) = V^*$ , and  $D(A) \subset V \subset H = H^* \subset V^*$ . Meanwhile, we write

$$V_{2s} = D(A^s), \quad s \in \mathbb{R}.$$

This is a Hilbert space for the scalar product and the norm as follows:

$$(u, v)_{2s} = (A^s u, A^s v), \quad |u|_{2s}^2 = ((u, u)_{2s}), \quad \forall u, v \in D(A^s).$$

$A^r$  is an isomorphism from  $D(A^s)$  onto  $D(A^{s-r})$ ,  $\forall s, r \in R$ , and one shows that under several hypotheses its longtime behavior is also described by a global attractor. From the Poincaré inequality, there exists a proper constant  $\lambda_1 > 0$ , such that

$$\|v\| \geq \lambda_1 \|v\|, \quad \forall v \in V,$$

where  $\lambda_1$  denotes the first eigenvalue of  $A^{\frac{1}{2}}$ .

Now let us consider the system (1)-(3). In order to prove the existence of solutions of (1)-(3), we assume that

$$M(s), N(s) \in C^1(R), \quad M'(s), N'(s) \geq 0, \quad sN(s) \geq 0, \quad N(s) \geq s, \quad (4)$$

$$a(t) \text{ is time-periodic, } a(t) \geq 0. \quad (5)$$

It is usually more convenient to reduce the system (1)-(3) to an abstract ordinary differential equations of the first order in time in the following manner. This is easy to do by letting  $u_t = w$ , then the system (1)-(3) can be rewritten as the following initial-value problem in the Hilbert space  $E$  of the form:

$$\begin{cases} \dot{Y} = CY + F(Y, t), & x \in \Omega \quad t > \tau, \\ Y(\tau) = Y_{0\tau} = (u_{0\tau}, u_{1\tau})^T \in E, \end{cases} \quad (6)$$

where  $Y = (u, w)^T$ ,  $Y(\tau) = (u_{0\tau}, u_{1\tau})^T$ ,  $z(t) = |A^{\frac{1}{2}}u|^2$ ,  $E = H_0^2(\Omega) \times L^2(\Omega)$  with finite energy norm  $|Y|_E = |A^{\frac{1}{2}}u|^2 + |w|^2$ , and we can define  $C : E \rightarrow E$  by the operator matrix

$$C = \begin{pmatrix} 0 & I \\ -A & -(\mu A + \eta I) \end{pmatrix} \text{ with } D(C) = D(A) \times D(A^{\frac{1}{2}}),$$

$$F(Y, t) = (0, -[a(t) + M(z) + N(\dot{z})]A^{\frac{1}{2}}u + h(x, t))^T. \quad (7)$$

Next, it is going to be proved that  $C$  is a sectorial operator and generates an analytic semigroup on  $E$ .

**Lemma 1.** For  $\mu, \eta > 0$ , the operator  $C$  generates an analytic semigroup  $e^{Ct}$  on  $E$  for  $t > 0$  with  $D(C) = D(A) \times D(A^{\frac{1}{2}})$ , where  $A$  is sectorial operator, and  $C$  is defined in (6).

**Proof.** Let

$$B = \begin{pmatrix} 0 & -I \\ A & \mu A + \eta I \end{pmatrix},$$

then

$$(\mathcal{M} - B) = \begin{pmatrix} \lambda I & I \\ -A & (\lambda - \eta)I - \mu A \end{pmatrix}.$$

First, all that we need to show is that  $B$  is a sectorial operator. Obviously,  $B$  is closed and densely defined. Then, we can conclude the resolvent:

$$R(\lambda; B) = (\lambda I - B)^{-1}$$

$$= \begin{pmatrix} (\lambda - \eta)I - \mu A & -I \\ A & \lambda I \end{pmatrix} [(\lambda - \eta)\lambda + (1 - \mu\lambda)A]^{-1}.$$

Due to the fact that all of the operators can be commuted, it is easy to prove that this indeed is the resolvent. Since the linear operator  $A$  is sectorial operator, the sector

$$\begin{aligned} \rho(B) &\supset \Sigma_{a,\varphi} \\ &= \{\lambda \in C; \arg|\lambda - a| \leq \pi + \varphi, \lambda \neq a\} \cup \{a\}, \quad \varphi \in (0, \frac{\pi}{2}), \end{aligned}$$

is in the resolvent set and

$$|(\lambda I - A)^{-1}| \leq \frac{M}{|\lambda - a|}, \quad \text{for all } \lambda \in \Sigma_{a,\varphi}, \quad (8)$$

some  $M \geq 1$ , and some real  $a$ . After that, the boundedness should be proved to be able to be held (similar to [3]).

Hence, we can conclude that  $C = -B$  is sectorial. Meanwhile, it is known that if  $B$  is sectorial, then  $C$  generates an analytic semigroup  $e^{Ct}$ .

At last, by the assumptions above, it is easy to show that the function  $F(Y, t) : E \rightarrow E$  is locally Lipschitz continuous with respect to  $Y$ . Furthermore, by the classical semigroup theory concerning the existence and uniqueness of the solutions of abstract differential equations [1], we have the following theorem.

**Theorem 1.** Consider the initial value problem (6) in the Hilbert space  $E$ . Under the conditions (4), (5) and  $\mu, \eta > 0$ , for any  $Y_{0\tau} \in E$ , there exists a unique continuous function  $Y(\cdot) = Y(\cdot, Y_{0\tau}) \in C((\tau, +\infty); E)$  such that  $Y(\tau, Y_{0\tau}) = Y_{0\tau}$  and  $Y(t)$  satisfies the equation

$$Y(t, Y_{0\tau}) = e^{-B(t-\tau)} Y_{0\tau} + \int_{\tau}^t e^{-B(t-s)} F(Y(s), s) ds. \quad (9)$$

where  $Y(t)$  is called a mild solution of (6),  $Y(t, Y_{0\tau})$  is jointly continuous in  $t$  and  $Y_{0\tau}$ , meanwhile

$$\begin{aligned} (u, u_t) &\in C((\tau, +\infty); H_0^2(\Omega)) \\ &\times [C((\tau, +\infty); L^2(\Omega)) \cap L^2(\tau, \tau+T); H_0^2(\Omega))], \quad \forall T > 0. \end{aligned} \quad (10)$$

**Proof.** The existence and uniqueness of the solutions are obviously showed from [1], and the global existence of solution can be obtained from the existence of an absorbing set below(see Lemma 3).

For any  $t \geq \tau$ , let us introduce a map

$$S(t, \tau) : Y_{0\tau} = \{u_{0\tau}, u_{1\tau}\} \rightarrow \{u(t), u_t(t)\} = Y(t, Y_{0\tau}), \quad E \rightarrow E,$$

where  $Y(t, Y_{0\tau})$  is the (mild) solution of (4), then  $\{S(t, \tau), t \geq \tau\}$  is a continuous process on  $E$  which has the following properties:

- (1)  $S(t, \tau) : E \rightarrow E$ ,
- (2)  $S(\tau, \tau)$  is the identity on  $E$ ,

- (3)  $S(t,s)S(s,\tau) = S(t,\tau)$  for all  $t \geq s \geq T$ ,
- (4)  $S(t,\tau)Y \rightarrow Y$  as  $t \rightarrow \tau$  for all  $Y \in E$ ,
- (5)  $S(t,\tau) \in C(E,E)$ .

In this article, we will show the existence of non-empty compact kernel sections for the process  $\{S(t,\tau), t \geq \tau\}$ . Then, we also prove the existence of the global attractor.

### 3. Boundedness of Compact Kernel Sections

In this section, we will prove the uniform boundedness of solutions for the system (1)-(3) in the space  $E$ . For this purpose, we define a new weighted inner product and norm in  $E = H_0^2(\Omega) \times L^2(\Omega)$  by

$$(\varphi, \tilde{\varphi})_E = k((u_1, u_2)) + (v_1, v_2), \quad |\varphi|_E^2 = (\varphi, \varphi)_E, \quad (11)$$

for any  $\varphi = (u_1, v_1)^T$ ,  $\tilde{\varphi} = (u_2, v_2)^T \in E$ , where

$$k = \frac{4 + \mu^2 \lambda_1}{4 + 2\mu^2 \lambda_1} \in \left(\frac{1}{2}, 1\right). \quad (12)$$

Obviously, the norm  $|\cdot|_E$  in (11) is equivalent to the usual norm  $|\cdot|_E$  of  $E$ .

Let

$$\varphi = (u, v)^T, \quad v = u_t + \varepsilon u, \quad 0 < \varepsilon < \varepsilon_0 = \min\left(\frac{\mu}{4}, \frac{\lambda_1}{2\mu}\right),$$

where  $\varepsilon$  is chosen as

$$\varepsilon = \frac{\mu \lambda_1}{4 + 2\mu^2 \lambda_1}, \quad (13)$$

and then the system (6) can be written as

$$\begin{cases} \varphi_t + \Lambda_\varepsilon \varphi = \tilde{f}, \\ \varphi(\tau) = (u_{0\tau}, u_{1\tau} + \varepsilon u_{0\tau})^T, \end{cases} \quad (14)$$

where

$$\tilde{f} = \begin{pmatrix} 0 \\ -[a(t) + M(z) + N(\dot{z})]A^{\frac{1}{2}}u + h(x, t) \end{pmatrix}, \quad (15)$$

$$\Lambda_\varepsilon = \begin{pmatrix} \varepsilon I & -I \\ (1 - \varepsilon\mu)A + \varepsilon(\varepsilon - \eta)I & \mu A - (\varepsilon - \eta)I \end{pmatrix}. \quad (16)$$

Next, we present a positive property of the operator  $(\Lambda_\varepsilon \varphi, \varphi)$  defined in (16), which plays an important role in this article.

**Lemma 2.** For any  $\varphi = (u, v)^T \in E$ , we have

$$(\Lambda_\varepsilon \varphi, \varphi)_E \geq \sigma \|\varphi\|_E^2 + \frac{\mu}{2} \|v\|^2 \geq \sigma \|\varphi\|_E^2 + \frac{\mu \lambda_1^2}{2} |v|^2, \quad (17)$$

where

$$\begin{aligned}\sigma &= \frac{\mu\lambda_1}{\gamma_1 + \sqrt{\gamma_1\gamma_2}}, \\ \gamma_1 &= 4 + \mu^2\lambda_1, \quad \gamma_2 = \mu^2\lambda_1.\end{aligned}\tag{18}$$

**Proof.** For any  $\varphi = (u, v)^T \in D(C)$ , by (11), (16), the Poincaré inequality and  $k = 1 - \varepsilon\mu$ , we have

$$\begin{aligned} &(\Lambda_\varepsilon \varphi, \varphi)_E - \sigma \|\varphi\|_E^2 - \frac{\mu\lambda_1^2}{2} |v|^2 \\ &= \varepsilon k \|u\|^2 + \mu \|v\|^2 - (\varepsilon - \eta) |v|^2 + (1 - k - \varepsilon\mu)(Au, v) \\ &\quad + \varepsilon(\varepsilon - \eta)(u, v) - \sigma k \|u\|^2 - \sigma |v|^2 - \frac{\mu\lambda_1^2}{2} \\ &\geq (\varepsilon - \sigma)k \|u\|^2 + \left(\frac{\mu\lambda_1^2}{2} + \eta - \sigma - \varepsilon\right) |v|^2 - \frac{\varepsilon(\varepsilon - \eta)}{\lambda_1 \sqrt{k}}. \end{aligned}\tag{19}$$

By (13) and (19), an elementary computation shows:

$$4(\varepsilon - \sigma) \left(\frac{\mu\lambda_1^2}{2} + \eta - \sigma - \varepsilon\right) |v|^2 \geq \frac{\varepsilon^2(\varepsilon - \eta)^2}{\lambda_1^2 k}.\tag{20}$$

Thus,

$$(\Lambda_\varepsilon \varphi, \varphi)_E \geq \sigma \|\varphi\|_E^2 + \frac{\mu\lambda_1^2}{2} |v|^2, \quad \text{for any } \varphi = (u, v)^T \in D(C).$$

since  $E$  is dense in  $D(C)$ . By (20), the proof is completed.

Now, we consider the absorbing property of the semigroup  $\{S_\varepsilon(t, \tau), t \geq \tau\}$  on  $E$ . Obviously, the mapping

$$\begin{aligned} S_\varepsilon(t, \tau) : E &\rightarrow E, \quad t \geq \tau, \\ \varphi_0 = \{u_{0\tau}, v_{0\tau}\} &= \{u_{1\tau} + \varepsilon u_{0\tau}\}^T \\ &\rightarrow \varphi(t) = \{u(t), v(t) = u_{1\tau}(t) + \varepsilon u_{0\tau}(t)\}^T, \end{aligned}\tag{21}$$

has the following relation with  $S(t, \tau)$ :

$$S_\varepsilon(t, \tau) = R_\varepsilon S_\varepsilon(t, \tau) R_{-\varepsilon},\tag{22}$$

where  $\varphi$  is the solution of (14) which satisfies  $\varphi(0) = \varphi_0$ , and  $S(t, \tau)$  is the linear operator in  $E$ :

$$\{u_{0\tau}, u_{1\tau}\}^T \rightarrow \{u(t), u_{1\tau}(t)\}^T.$$

( $u$  is the solution of (1), (2)), and  $R_\varepsilon, \varepsilon \in R$ , is an isomorphism of  $E$ :

$$R_\varepsilon : \{u, v\} \rightarrow \{u, v + \varepsilon u\}.$$

So, we only need to consider the equivalent system (14). For the boundedness of solutions of (14), we have

**Lemma 3.** Suppose  $M_0 > 0$  (independent of  $\tau$ ), for any bounded set  $B$  of  $E$ , there exists  $T_0(B) \geq 0$  such that the solution  $\varphi(t) = \{u(t), v(t)\}^T$  of (14) with  $\varphi(\tau) \in B$  satisfies:

$$\|\varphi(t)\|_E^2 = k\|u(t)\|^2 + |v(t)|^2 \leq M_0^2, \quad \forall t \geq T_0(B) \geq \tau, \quad \tau \in R, \quad (23)$$

in which  $v = u_t + \varepsilon u$ .

**Proof.** Set  $\varphi(t) = \{u(t), v(t)\}^T$  be a solution of (14) with initial value  $\varphi(\tau) = \{u_{0\tau}, u_{1\tau} + \varepsilon u_{0\tau}\}^T \in E$ . As indicated above,

$$\begin{aligned} \varphi(t) &\in C(\tau, +\infty; H_0^2(\Omega)) \\ &\times [C(\tau, +\infty; L^2(\Omega)) \cap L^2(\tau, \tau+T; H_0^2(\Omega))], \quad \forall T > 0. \end{aligned}$$

Taking the scalar product on each side of (14) with  $\varphi = \{u, v\}^T$  in  $E$ , where  $v = u_t + \varepsilon u$ , we have

$$(\varphi_t, \varphi) + (\Lambda_\varepsilon \varphi, \varphi) + [\alpha(t) + M(z) + N(\dot{z})](A^{\frac{1}{2}}u, v) = (h(x, t), v).$$

Under the hypothesis above, we can conclude:

$$\begin{aligned} \alpha(t)(A^{\frac{1}{2}}u, v) &= \frac{1}{2} \frac{d}{dt} (\alpha(t)|A^{\frac{1}{2}}u|) - \dot{\alpha}(t)|A^{\frac{1}{2}}u|^2 + \varepsilon \alpha(t)|A^{\frac{1}{2}}u|^2, \\ M(z)(A^{\frac{1}{2}}u, v) &= \frac{d}{dt}(\tilde{M}(z)) + \varepsilon M(z)z, \\ N(\dot{z})(A^{\frac{1}{2}}u, v) &= \frac{1}{2} N(\dot{z})[\dot{z} + \varepsilon N(\dot{z})]z. \end{aligned}$$

Then,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} [\|\varphi\|_E^2 + \alpha(t)|A^{\frac{1}{2}}u|^2 + 2\tilde{M}(z)] + \varepsilon|A^{\frac{1}{2}}u|^4 &+ \sigma|\varphi|_E^2 \\ &+ (\varepsilon\alpha(t) - \dot{\alpha}(t))|A^{\frac{1}{2}}u|^2 + \varepsilon\tilde{M}(z) \leq \frac{1}{2\mu\lambda_1^2}|h(x, t)|^2, \quad \forall t \geq \tau. \end{aligned} \quad (24)$$

Let

$$\begin{aligned} Y(t) &= \sigma|\varphi|_E^2 + (\varepsilon\alpha(t) - \dot{\alpha}(t))|A^{\frac{1}{2}}u|^2 + \varepsilon\tilde{M}(z) \\ &= \varepsilon[\frac{\sigma}{\varepsilon}|\varphi|_E^2 + (\frac{\varepsilon\alpha(t) - \dot{\alpha}(t)}{\varepsilon})|A^{\frac{1}{2}}u|^2 + \tilde{M}(z)], \end{aligned} \quad (25)$$

and,

$$L(t) = \|\varphi\|_E^2 + \alpha(t)|A^{\frac{1}{2}}u|^2 + 2\tilde{M}(z) + \varepsilon|A^{\frac{1}{2}}u|^4. \quad (26)$$

So,

$$\begin{aligned} \frac{4}{\varepsilon}Y(t) - L(t) &\geq (\frac{4\sigma}{\varepsilon} - 1)|\varphi|_E^2 + (\sqrt{\frac{b}{2}}|A^{\frac{1}{2}}u|^2 + \sqrt{\frac{2}{b}}a)^2 \\ &+ (\frac{b}{2} - \varepsilon)|A^{\frac{1}{2}}u|^4 - \frac{2a^2}{b} - \frac{1}{4}(3a(t) - \frac{4}{\varepsilon}\dot{a}(t))^2 \\ &\geq -\frac{2a^2}{b} - \frac{1}{4}(3a(t) - \frac{4}{\varepsilon}\dot{a}(t))^2 \geq \frac{C_0}{2}. \end{aligned} \quad (27)$$

Set

$$k = \frac{\varepsilon}{2}, \quad \varepsilon < \min\left\{4\sigma, \frac{b}{2}\right\}.$$

Then, by (24) and (27),

$$\frac{d}{dt}L(t) + kL(t) \leq C_0 + \frac{1}{\mu\lambda_1^2}|h|_0^2, \quad (28)$$

where  $|h|_0 = \sup_{t \in R} |h(x, t)|$ . Applying the Gronwall's inequality, we obtain the following absorbing inequality in the space  $(E, \|\cdot\|_E)$ :

$$\begin{aligned} |\varphi(t)|_E^2 &\leq L(t) \leq L(\tau)e^{-k(t-\tau)} + \frac{2}{k} \left[ \frac{|h|_0^2}{\mu\lambda_1^2} + C_0 \right], \quad t \geq \tau, \\ \text{Or } \limsup_{t \rightarrow +\infty} |\varphi(t)|_E^2 &\leq M_0^2, \quad t \geq \tau. \end{aligned} \quad (29)$$

Taking

$$M_0^2 = L(\tau)e^{-k(t-\tau)} + \frac{2}{k} \left[ \frac{|h|_0^2}{\mu\lambda_1^2} + C_0 \right],$$

(independent of  $\tau$ ), the proof is completed.

Let  $B_0$  be a bounded closed ball of  $E$  centered at 0 of radius  $M_0$ :

$$B_0 = \{(u, v) \in E : \|u(t)\|^2 + \|v(t)\|^2 \leq \rho^2\}. \quad (30)$$

Then,  $B_0$  is the bounded absorbing set of the analytic semigroups  $\{S_\varepsilon(t, \tau)\}$ ,  $t \geq \tau$  of (1)-(3).

**Corollary 1.** For any initial value  $\forall \varphi(\tau) \in B_0$ , that is,  $|\varphi(\tau)|_E^2 = k\|u_{0\tau}\|^2 + |u_{1\tau} + \varepsilon u_{0\tau}|^2 \leq M_0^2$ , there exists  $M_1 > 0$  such that the solution of (14)  $\varphi(t) = \langle u(t), v(t) \rangle^T$  satisfies  $|\varphi(t)|_E \leq M_1$ ,  $t \geq \tau$ .

#### 4. Existence of the Global Attractor

To obtain the global attractor for the process  $\{S_\varepsilon(t, \tau), t \geq \tau\}$ , we need to show the uniform asymptotic compactness of the process  $\{S_\varepsilon(t, \tau), t \geq \tau\}$  in  $E$ , that is,  $\{S_\varepsilon(t, \tau), t \geq \tau\}$  possesses a uniformly attracting compact set in  $E$  with respect to  $\tau \in R$ . Next, let us recall some concepts in [1].

**Definition 1.** The kernel  $K$  of the process  $\{S_\varepsilon(t, \tau), t \geq \tau\}$  consists of all bounded complete trajectories of the process  $\{S_\varepsilon(t, \tau), t \geq \tau\}$ :

$$K = \{\varphi(\cdot) : \varphi(\cdot) \text{ is a solution of (14), } \|\varphi(t)\|_E \leq M_\varphi, \forall t \in R\}, \quad (31)$$

and the section  $K(s) \subset E$  of the kernel  $K$  at times  $s \in R$  is

$$K(s) = \{\varphi(s) : \varphi(\cdot) \in K\}.$$

**Definition 2.** A process  $\{S_\varepsilon(t, \tau), t \geq \tau\}$  possessing a compact uniformly attracting set is said to be uniformly asymptotically compact.

**Lemma 4.** Let  $\{S_\varepsilon(t, \tau), t \geq \tau\}$  be a uniformly compact process acting in the space  $E$ , with a compact uniformly attracting set  $\Lambda \subset E$ . Each mapping  $S_\varepsilon(t, \tau) : E \rightarrow E$  is assumed continuous. Then the kernel sections  $K(s)$  of the process  $\{S_\varepsilon(t, \tau), t \geq \tau\}$  are all compact, and  $K(s) \subseteq \Lambda$ .

**Lemma 5.** For the initial value  $\forall \varphi(\tau) \in B_0$ , the solution of (14)  $\varphi(t) = (u(t), v(t))^T$  can be decomposed, where  $\varphi_i(t) = (u_i(t), v_i(t))^T$ ,  $v_i(t) = u_{it(t)} + \varepsilon u_i(t)$ ,  $i = 1, 2$ , satisfy, respectively,

$$|\varphi_1(t)|_E^2 = k \|u_1(t)\|^2 + |v_1(t)|^2 \leq M_1^2, \quad t \geq \tau, \quad (33)$$

and

$$\left| A^{\frac{\beta}{2}} \varphi_2(t) \right|_E^2 = k \left| A^{\frac{\beta+1}{2}} u_2(t) \right|^2 + \left| A^{\frac{\beta}{2}} v_2(t) \right|^2 \leq M_2^2, \quad t \geq \tau, \quad (34)$$

where

$$\sigma_1 = \frac{\mu \lambda_1}{4 + \mu^2 \lambda_1 + \mu \sqrt{4 \lambda_1 + \mu^2 \lambda_1^2}}. \quad (35)$$

**Proof.** Let  $\varphi(t) = (u(t), v(t))^T$ ,  $t \geq \tau$  be a solution of (14). Thus,

$$|\varphi(\tau)|_E^2 = k \|u_{0\tau}\|^2 + |u_{1\tau} + \varepsilon u_{0\tau}|^2 \leq M_0^2. \quad (36)$$

Let  $\varphi(t) = (u_1(t), v_1(t))^T + (u_2(t), v_2(t))^T$ ,  $v_i(t) = u_{it}(t) + \varepsilon u_i(t)$ ,  $i = 1, 2$ , satisfy, respectively,

$$\begin{cases} u_{1tt} + \Delta^2 u_1 + \mu \Delta^2 u_{1t} + \eta u_{1t} = 0, & t \geq \tau, \\ u_1(\tau) = u_{0\tau}, \quad u_{1t}(\tau) = u_{1\tau}, \end{cases} \quad (37)$$

and

$$\begin{cases} u_{2tt} + \Delta^2 u_2 + \mu \Delta^2 u_{2t} + \eta u_{2t} = b_1(t) + b_2(t), & t \geq \tau, \\ u_2(\tau) = u_{2t}(\tau) = 0, \\ b_1(t) = h(x, t), \quad b_2(t) = [a(t) + M(z) + N(\dot{z})] \Delta u. \end{cases} \quad (38)$$

Let  $y = (u_1, u_{1t} + \varepsilon u_{1\tau})^T$ , then  $y(\tau) = (u_{0\tau}, u_{1\tau} + \varepsilon u_{0\tau})^T$ , where

$$\varepsilon_1 = \frac{\mu \lambda_1}{4 + 2\mu^2 \lambda_1}, \quad (39)$$

(37) can be written as

$$y_t + Hy = 0, \quad (40)$$

where

$$H = \begin{pmatrix} \varepsilon_1 I & -I \\ (1 - \varepsilon_1 \mu) A + \varepsilon_1 (\varepsilon_1 - \eta) I & \mu A - (\varepsilon_1 - \eta) I \end{pmatrix}. \quad (41)$$

Similar to Lemma 2, for any  $y = (u_1, \tilde{u}_1)^T \in E$ ,

$$(Hy, y)_E \geq \sigma_1 \|y\|_E^2 + \frac{\mu}{2} \|\tilde{u}_1\|^2 \geq \sigma_1 \|y\|_E^2 + \frac{\mu \lambda_1^2}{2} |\tilde{u}_1|^2, \quad (42)$$

where  $\sigma_1$  is as in (35). Taking the inner product of (40) with  $y$  in  $E$ , we obtain

$$\begin{aligned} |y(t)|_E^2 &\leq (k \|u_{0r}\|^2 + |u_{1r} + \varepsilon u_{0r} + (\varepsilon_1 - \varepsilon) u_{0r}|^2) \exp(-2\sigma_1(t-\tau)) \\ &\leq (2 + 4|\varepsilon_1 - \varepsilon|^2) \rho^2 \exp(-2\sigma_1(t-\tau)) \\ &\leq C \rho^2 \exp(-2\sigma_1(t-\tau)), \quad \forall t \geq \tau. \end{aligned} \quad (43)$$

Thus,

$$\begin{aligned} |\varphi(t)|_E^2 &= k \|u_1(t)\|^2 + |u_{1r}(t) + \varepsilon u_1(t)|^2 \\ &\leq (2 + 4|\varepsilon_1 - \varepsilon|^2) \rho^2 \exp(-2\sigma_1(t-\tau)), \quad \forall t \geq \tau. \end{aligned} \quad (44)$$

Setting  $M_1^2 = (2 + 4|\varepsilon_1 - \varepsilon|^2) \rho^2 \exp(-2\sigma_1(t-\tau))$ , one obtains (33).

In the following, we prove that  $u_2(t)$  satisfies (34). Setting  $\zeta = A^\beta u_2$ ,  $\zeta = \zeta_t + \varepsilon_1 \zeta$ , then, (38) can be written as

$$\begin{aligned} \tilde{\varphi}_2 + H\tilde{\varphi}_2 &= \tilde{B}(t), \quad \tilde{\varphi}_2 = (\zeta, \zeta)^T, \\ \tilde{B}(t) &= (0, A^\beta(b_1(t) + b_2(t)))^T, \quad \tilde{\varphi}(\tau) = 0. \end{aligned} \quad (45)$$

Taking the scalar product  $(\cdot, \cdot)_E$  of (45) with  $\tilde{\varphi}_2 = (\zeta, \zeta)^T$ , we have

$$\frac{1}{2} \frac{d}{dt} |\tilde{\varphi}_2|_E^2 + (H\tilde{\varphi}_2, \tilde{\varphi}_2)_E = (\zeta, A^\beta(b_1(t) + b_2(t))). \quad (46)$$

By the embedding theorem, we obtain

$$(\zeta, A^\beta b_1(t)) \leq |A^\beta \zeta| \|h(x, t)\| \leq 2k |A^{\frac{\beta+1}{2}} \zeta|^2 + \frac{\pi^{-4(1-\beta)}}{2k} |h(x, t)|^2.$$

Meanwhile, by (42) and (46), we have

$$\begin{aligned} (\zeta, A^\beta b_2(t)) &= -[a(t) + M(z) + N(\dot{z})](A^\frac{1}{2} u, A^\beta \zeta) \\ &\leq C(A^{\frac{\beta+1}{2}} u, A^{\frac{\beta}{2}} \zeta) \leq C |A^{\frac{\beta+1}{2}} u| |A^{\frac{\beta}{2}} \zeta| \leq C (|A^{\frac{\beta+1}{2}} u|^2 + |A^{\frac{\beta}{2}} \zeta|^2). \end{aligned}$$

By (45), (46) and the positivity of  $H$  in the new norm, we fine

$$\frac{d}{dt} |\tilde{\varphi}_2|_E^2 + 2\sigma_1 |\tilde{\varphi}_2|_E^2 \leq C, \quad \forall t \geq \tau. \quad (47)$$

By the Gronwall inequality and zero initial value at  $t = \tau$ , we obtain

$$|\tilde{\varphi}_2|_E^2 \leq M_2^2, \quad \forall t \geq \tau. \quad (48)$$

The proof is completed.

Let  $B_{\frac{\beta}{2}}$  be the ball of  $V_{\frac{\beta+1}{2}} \times V_{\frac{\beta}{2}} \subset E$  of radius  $M_2$ , i.e.  $\forall \varphi = (u, v)^T \in B_{\frac{\beta}{2}}$

satisfying  $|A^{\frac{\beta}{2}} \varphi|_E^2 \leq M_2^2$ .

**Theorem 2.** The process  $\{S_\varepsilon(t, \tau), t \geq \tau\}$  possesses a kernel  $K$  such that the kernel section  $K(s)$  at the time  $s$  is compact and  $K(s) \subseteq B_{\frac{\beta}{2}}$ ,  $\forall s \in R$ .

**Proof.** The proof is omitted.

**Theorem 3.** The analytic semigroup  $\{S_\varepsilon(t, \tau), t \geq \tau\}$  of (14) possesses a global attractor  $B$  in  $E$ .

**Proof.** By Theorem 2, (9) and  $e^{Ct}$  is compact, we complete the proof of Theorem 3.

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