

**THE KSGNS CONSTRUCTION ASSOCIATED WITH A
PROJECTIVE u -COVARIANT COMPLETELY POSITIVE LINEAR
MAP**

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In this paper we prove a covariant projective version of the Stinespring theorem in terms of Hilbert C^ -modules. We also present an extension of a projective u -covariant completely positive linear map on the twisted crossed product $A \times_{\alpha}^{\omega} G$ to a unique completely positive linear map in the case of locally compact groups and discrete groups. The main result of the paper is the KSGNS construction associated with a projective u -covariant completely positive linear map.*

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1. Introduction

The term of completely positive map was introduced by Stinespring in [21] and showed that every positive map of a commutative C^* -algebra into the algebra $\mathcal{L}(H)$ of operators on the Hilbert space H is completely positive, as is every scalar-valued positive linear map of a general C^* -algebra. He also proved a well known theorem about characterization of completely positive maps.

The physical significance of completely positive linear maps of C^* -algebras has been observed over the years in many papers, in which it was shown that the completely positive linear maps describe the change of states of quantum dynamical systems, produced by quantum measurement or describe the time development of open quantum dynamical systems. The completely positive linear maps, which appear in the theory of quantum measurements, in the operational approach to quantum mechanics and in the theory of the open quantum dynamical systems are called respectively imperfect measurements, covariant instruments, generalized observables and dynamical maps. The notion of completely positive maps generalizes the notion of state, representation, conditional expectation and the notion of semi-spectral measure.

The GNS (Gel'fand-Naimark-Segal) representation theorem is one of the most useful theorems frequently applied to mathematical physics. The GNS construction applied to an invariant state, gives a cyclic covariant representation with an

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invariant cyclic vector. To every positive linear functional on a C^* -algebra A can be associated a cyclic representation on a Hilbert space H by GNS construction. In [21], Stinespring extended this theorem for a completely positive linear map (as a generalization of a positive linear functional) from a C^* -algebra A into $\mathcal{B}(H)$, the C^* -algebra of linear bounded operators on a Hilbert space H , in order to obtain a representation of A on another Hilbert space K . On the other hand, Paschke [18] (respectively, Kasparov [14]) showed that a completely positive linear map from A to another C^* -algebra B (respectively, from A to the C^* -algebra of all adjointable operators on the Hilbert C^* -module H_B) induces a $*$ -representation of A on a Hilbert B -module, generalizing the Stinespring representation theorem. Since then, the generalization of a C^* -algebra of the GNS representation by Stinespring and Kasparov is called the KSGNS representation of a C^* -algebra with a completely positive map. By KSGNS (Kasparov-Stinespring-Gel'fand-Naimark-Segal) construction [15], to a strictly completely positive map ρ from a C^* -algebra A on a Hilbert C^* -module F over a C^* -algebra B can be associated a triple $(F_\rho, \pi_\rho, v_\rho)$ consisting of a Hilbert B -module F_ρ , a $*$ -homomorphism $\pi_\rho: A \rightarrow \mathcal{L}_B(F_\rho)$ and an adjointable operator $v_\rho: F \rightarrow F_\rho$ which is unique up to a unitary equivalence. If $F = B = \mathbb{C}$, then the KSGNS construction reduces to the classical GNS construction. If $B = \mathbb{C}$ (so F is a Hilbert space), then we get the Stinespring construction. In the context of Hilbert C^* -modules the construction was given by Kasparov [14]. In [9], Joița extended KS-GNS construction for strict continuous completely multi-positive linear maps from a locally C^* -algebra A to $\mathcal{L}_B(E)$, the C^* -algebra of all adjointable B -module morphisms from E into E , and showed in Theorem 4.3, [9] a covariant version of this construction.

The relation of covariant completely positive maps to twisted crossed products, that we approach in this paper, was explored in recent works [22], [4], [5], where an abstract covariant version of Stinespring theorem has been proved for a unital C^* -dynamical system (A, M, α) , with M a left-cancellative semigroup with unit [22], covariant projective homomorphisms have been extended to the twisted crossed product $C_\theta^*(A, M, \alpha)$ of a C^* -algebra A by the semigroup M under the action α relative to the cocycle θ [22], completely multi-positive projective u -covariant non-degenerate linear maps from a C^* -algebra A on a Hilbert C^* -module E have induced completely multi-positive linear maps on the twisted crossed product $A \times_\alpha^\omega G$ [4] and also projective u -covariant completely bounded multilinear maps have been extended on the twisted crossed product $(A \times_\alpha^\omega G)^k$, in the case of amenable groups, to completely bounded multilinear maps [5].

In Section 2 we prove a covariant projective version of Stinespring theorem, with a different approach than the one presented in Theorem 1, [3]. The third section is dedicated to the extension on the twisted crossed product $A \times_\alpha^\omega G$ of projective u -covariant completely positive linear maps in the case of discrete groups and locally compact groups. The main result of the paper is presented in Section 4. Given a C^* -dynamical system (G, A, α) , we show that a projective u -covariant non-degenerate completely positive linear maps from A to $\mathcal{L}_B(E)$ induces a projective covariant non-degenerate representation of (G, A, α) on a Hilbert B -module, uniquely up to unitary equivalence, called the KSGNS construction.

Now we remind some definitions and notations that will be used throughout the paper.

Hilbert C^* -modules were first introduced by Kaplansky in 1953 in [13]. His idea was to generalize Hilbert space by allowing the inner product to take values in a commutative unital C^* -algebra rather than in the field of complex numbers.

Definition 1.1. ([15]) A **pre-Hilbert A -module** is a complex vector space E which is also a right A -module, compatible with the complex algebra structure, equipped with an A -valued inner product $\langle \cdot, \cdot \rangle : E \times E \rightarrow A$ which is \mathbb{C} - and A -linear in its second variable and satisfies the following relations:

- (1) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$;
- (2) $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$;
- (3) $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$.

We say that E is a **Hilbert A -module** if E is complete with respect to the topology determined by the norm $\|\cdot\|$ given by $\|\xi\| = \sqrt{\|\langle \xi, \xi \rangle\|}$.

The theory of projective representations of finite groups was founded by I. Schur [20]. Projective representations help us understand numerous physical systems. For example, they are used to describe the symmetry operations of a crystal lattice and to label the energy levels of quantum systems. Mackey remarked in [16] that a problem arising in quantum field theory can be formulated as a problem of finding certain projective representations. Brown has used projective representations of translation groups to discuss the energy-level degeneracy occurring when a crystal is subjected to a uniform magnetic field. Projective representations of abelian groups arise naturally in the study of energy bands in the presence of a magnetic field. A projective representation is also relevant for describing the symmetries of quantum mechanical systems.

Let G be a locally compact group and let A be a C^* -algebra.

Definition 1.2. A map $\omega: G \times G \rightarrow \mathcal{U}(\mathcal{Z}(A))$, where

$$\mathcal{U}(\mathcal{Z}(A)) = \{u \in A \mid u \text{ unitary}, \quad ua = au, \quad \forall a \in A\}$$

is called a **multiplier** on G if

- i) $\omega(x, e) = \omega(e, x) = 1_A$ for all $x \in G$, where e is the identity of G ;
- ii) $\omega(x, y)\omega(xy, z) = \omega(x, yz)\omega(y, z)$ for all $x, y, z \in G$.

Let E be a Hilbert C^* -module over A and let $\mathcal{L}_A(E)$ be the Banach space of all adjointable module homomorphisms from E to E (i.e. T is a bounded module homomorphism such that there is T^* a bounded module homomorphism from E to E satisfying $\langle \eta, T\xi \rangle = \langle T^*\eta, \xi \rangle$ for all $\xi, \eta \in E$).

Definition 1.3. A **projective representation** of G on E with multiplier ω is a map $\pi: G \rightarrow \mathcal{L}_A(E)$ such that

- i) $\pi(xy) = \omega(x, y)\pi(x)\pi(y)$ for all $x, y \in G$;
- ii) $\pi(e) = I_E$, where I_E is the identity operator on E .

Definition 1.4. A **projective unitary representation** of G on E with multiplier ω is a map u from G to $\mathcal{L}_A(E)$ such that:

- i) u_g is a unitary element in $\mathcal{L}_A(E)$ for all $g \in G$;
- ii) $u_{gt} = \omega(g, t)u_g u_t$ for all $g, t \in G$.

We remind some definitions that will be used throughout the paper.

Definition 1.5. ([23]) A **C^* -dynamical system** is a triple (G, A, α) , where G is a locally compact group, A is a C^* -algebra and α is a continuous action of G on A , i.e. a continuous homomorphism $\alpha: G \rightarrow \text{Aut}(A)$, where $\text{Aut}(A)$ is the group of automorphism of A .

Definition 1.6. A **projective covariant representation** of a C^* -dynamical system (G, A, α) on a Hilbert B -module E is a triple (Φ, v, E) , where Φ is a $*$ -representation of A on E , v is a projective unitary representation of G on E with multiplier ω and $\Phi(\alpha_g(a)) = v_g \Phi(a) v_g^*$, for all $g \in G$ and $a \in A$.

Definition 1.7. ([21], [1]) Let A and B be two C^* -algebras and let $M_n(A)$, respectively $M_n(B)$ denote the $*$ -algebra of all $n \times n$ matrices over A , respectively B with the algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of A , respectively B . A linear map $\rho: A \rightarrow B$ is **completely positive** if the linear map $\rho^{(n)}: M_n(A) \rightarrow M_n(B)$, defined by $\rho^{(n)}([a_{ij}]_{i,j=1}^n) = [\rho(a_{ij})]_{i,j=1}^n$ is positive for all positive integers n .

Definition 1.8. Let (G, A, α) be a C^* -dynamical system and let u be a projective unitary representation of G on a Hilbert B -module E with multiplier ω . We say that a completely positive linear map ρ from A into $\mathcal{L}_B(E)$ is **projective u -covariant** with respect to the C^* -dynamical system (G, A, α) if $\rho(\alpha_g(a)) = u_g \rho(a) u_g^*$ for all $a \in A$ and $g \in G$.

2. A covariant projective version of the Stinespring theorem in terms of Hilbert C^* -modules

The next theorem is a covariant projective version of the Stinespring theorem in terms of Hilbert C^* -modules ([3]) proved by following the steps of one of Heo's results [7].

Theorem 2.1. Let E be a Hilbert B -module and let (G, A, α) be a unital C^* -dynamical system. If u is a projective unitary representation of G on E with the multiplier ω and if a linear map ρ from A into $\mathcal{L}_B(E)$ is projective u -covariant completely positive, then there are :

- (i) a Hilbert B -module E_ρ ;
- (ii) a representation Φ_ρ of A into $\mathcal{L}_B(E_\rho)$;
- (iii) a projective unitary representation v^ρ from G into $\mathcal{L}_B(E_\rho)$ with the multiplier ω ;
- (iv) an isometry $V_\rho \in \mathcal{L}_B(E, E_\rho)$,

such that for all $a \in A$ and $g \in G$,

- (a) $\rho(a) = V_\rho^* \Phi_\rho(a) V_\rho$ and $u_g = V_\rho^* v_g^\rho V_\rho$,
- (b) $(v_g^\rho)^* \Phi_\rho(\alpha_g(a)) v_g^\rho = P_{V_\rho} \Phi_\rho(a) |_{X_V}$, where P_{V_ρ} is the projection on the image X_{V_ρ} of V_ρ .

Proof. By Theorem 2.4, [17] there are : a B -module E_ρ , a $*$ -representation Φ from A into $\mathcal{L}_B(E_\rho)$ and an isometry $V_\rho \in \mathcal{L}_B(E, E_\rho)$ such that $\rho(a) = V_\rho^* \Phi_\rho(a) V_\rho$ for all $a \in A$. Moreover, $E_{\Phi_\rho} = \overline{\text{sp}} \Phi_\rho(A) V_\rho E$.

We define a map v^ρ on G by $v_g^\rho = V_\rho u_g V_\rho^*$ for all $g \in G$. Then each element v_g^ρ is in $\mathcal{L}_B(E_\rho)$. Moreover, $(v_g^\rho)^* = V_\rho u_g^* V_\rho^* = V_\rho u_{g^{-1}} V_\rho^* = v_{g^{-1}}^\rho$.

We show that v^ρ is a projective representation with the multiplier ω . We have

$$v_{g_1 g_2}^\rho = V_\rho u_{g_1 g_2} V_\rho^* = V_\rho \omega(g_1, g_2) u_{g_1} u_{g_2} V_\rho^* = \omega(g_1, g_2) V_\rho u_{g_1} u_{g_2} V_\rho^* =$$

$$\omega(g_1, g_2) V_\rho u_{g_1} V_\rho^* V_\rho u_{g_2} V_\rho^* = \omega(g_1, g_2) v_{g_1}^\rho v_{g_2}^\rho$$

for all $g_1, g_2 \in G$.

It remains to prove that $(v_g^\rho)^* \Phi_\rho(\alpha_g(a)) v_g^\rho = P_{V_\rho} \Phi_\rho(a)|_{X_{V_\rho}}$ for all $a \in A$. Therefore, for all $a \in A$ and $g \in G$, we have

$$(v_g^\rho)^* \Phi_\rho(\alpha_g(a)) v_g^\rho = V_\rho u_g^* V_\rho^* \Phi_\rho(\alpha_g(a)) V_\rho u_g V_\rho^* =$$

$$V_\rho u_g^* \rho(\alpha_g(a)) u_g V_\rho^* = V_\rho \rho(a) V_\rho^* = V_\rho V_\rho^* \Phi_\rho(a) V_\rho V_\rho^*,$$

where the third equality results from the u -covariance of ρ . \square

Remark 2.1. If V_ρ is an unitary operator, then the triplet $(\Phi_\rho, v^\rho, E_\rho)$ becomes a projective covariant representation of (G, A, α) into $\mathcal{L}_B(E_\rho)$.

Proof. Let $g \in G$ and $a \in A$. Then

$$v_g^\rho \Phi_\rho(a) (v_g^\rho)^* = V_\rho u_g V_\rho^* \Phi_\rho(a) (V_\rho u_g V_\rho^*)^* = V_\rho u_g V_\rho^* \Phi_\rho(a) V_\rho u_g^* V_\rho^* = V_\rho u_g \rho(a) u_g^* V_\rho^* =$$

$$V_\rho \rho(\alpha_g(a)) V_\rho^* = V_\rho V_\rho^* \Phi_\rho(\alpha_g(a)) V_\rho V_\rho^* = \Phi_\rho(\alpha_g(a)).$$

\square

3. Projective u -covariant completely positive linear maps extended on the twisted crossed product $A \times_\alpha^\omega G$

Busby and Smith [2] introduced the twisted actions and constructed in Theorem 2.2, [2] a Banach $*$ -algebra denoted by $L^1(G, A, \alpha, \omega)$ as a generalization of the group algebra $L^1(G)$ (the algebra of all integrable functions on G with scalar values), now the integrable functions on G taking values in a C^* -algebra A . They studied the theory of covariant representations of the twisted dynamical system (G, A, α, ω) and proved that the representations of the twisted group algebra $L^1(G, A, \alpha, \omega)$ are in one-to-one correspondence with the covariant representations of the twisted dynamical system (G, A, α, ω) (Theorem 3.3,[2]) and that the enveloping C^* -algebra of $L^1(G, A, \alpha, \omega)$ is the twisted crossed product of A by G under the action α relative to the multiplier ω denoted by $A \times_\alpha^\omega G$.

The following theorem is a generalization to projective representations of Proposition 2, [12].

Theorem 3.1. Let (G, A, α) be a unital C^* -dynamical system and let E be a Hilbert module over a unital C^* -algebra B . If $u: G \rightarrow \mathcal{L}_B(E)$ is a projective unitary representation of G on E with the multiplier ω and $\rho: A \rightarrow \mathcal{L}_B(E)$ is a projective u -covariant completely positive linear map of (G, A, α) , then there is a completely positive linear map $\varphi: A \times_\alpha^\omega G \rightarrow \mathcal{L}_B(E)$ uniquely defined by

$$\varphi(f) = \int_G \rho(f(g)) u_g d\mu, \text{ for all } f \in C_c(G, A)$$

where $C_c(G, A)$ is the set of continuous functions from G to A with compact supports.

Proof. By Theorem 2.1, there is a projective covariant representation $(\Phi_\rho, v^\rho, E_\rho)$ of (G, A, α) in $\mathcal{L}_B(E, E_\rho)$ such that $\rho(a) = V_\rho^* \Phi_\rho(a) V_\rho$ and $V_\rho^* v_g^\rho V_\rho = u_g$, for all $a \in A$ and $g \in G$.

Let $\Phi_\rho \times v^\rho$ be the representation of $A \times_\alpha^\omega G$ associated with $(\Phi_\rho, v^\rho, E_\rho)$ (Theorem 3.3, [2]). We define $\varphi: A \times_\alpha^\omega G \rightarrow \mathcal{L}_B(E)$ by

$$\varphi(f) = V_\rho^* (\Phi_\rho \times v^\rho)(f) V_\rho.$$

It is clear that φ is a completely positive linear map from $A \times_\alpha^\omega G$ into $\mathcal{L}_B(E)$.

If $f \in C_c(G, A)$, then

$$\begin{aligned} \varphi(f) &= V_\rho^* (\Phi_\rho \times v^\rho)(f) V_\rho = \int_G V_\rho^* \Phi_\rho(f(g)) v_g^\rho V_\rho dg = \\ &= \int_G V_\rho^* \Phi_\rho(f(g)) V_\rho u_g dg = \int_G \rho(f(g)) u_g dg \end{aligned}$$

and since $C_c(G, A)$ is dense in $A \times_\alpha^\omega G$, φ is unique. \square

Remark 3.1. *Theorem 3.1 can be written in the case of discrete groups:*

Theorem 3.2. *Let (G, A, α) be a unital C^* -dynamical system with G a discrete group and let E be a Hilbert module over a unital C^* -algebra B . If $u: G \rightarrow \mathcal{L}_B(E)$ is a projective unitary representation of G on E with the multiplier ω and $\rho: A \rightarrow \mathcal{L}_B(E)$ is a projective u -covariant completely positive linear map of (G, A, α) , then there is a completely positive linear map $\psi: A \times_\alpha^\omega G \rightarrow \mathcal{L}_B(E)$ uniquely defined by*

$$\psi(f) = \sum_g \rho(f(g)) u_g, \text{ for all } f \in K(G, A)$$

where $K(G, A)$ is the dense subalgebra of $l^1(G, A)$ of all functions from G to A with finite supports.

Proof. By Theorem 2.1, there are a Hilbert B -module E_ρ , a representation Φ of A into $\mathcal{L}_B(E_\rho)$, a projective unitary representation v^ρ from G into $\mathcal{L}_B(E_\rho)$ with the multiplier ω and an isometry $V_\rho \in \mathcal{L}_B(E, E_\rho)$ which satisfy (a) and (b) from Theorem 2.1. The representation (Φ_ρ, v^ρ) gives rise to a homomorphism $\Phi_\rho \times v^\rho: l^1(G, A) \rightarrow \mathcal{L}_B(E_\rho)$ uniquely defined by $(\Phi_\rho \times v^\rho)(f) = \sum_g \Phi_\rho(f(g)) v_g^\rho$, $f \in K(G, A)$ and from

Theorem 3.3, [2] results that $\Phi_\rho \times v^\rho$ extends to a representation $\Phi_\rho \times v^\rho: A \times_\alpha^\omega G \rightarrow \mathcal{L}_B(E_\rho)$. We consider a completely positive map $\psi: A \times_\alpha^\omega G \rightarrow \mathcal{L}_B(E)$ given by $\psi(f) = V_\rho^* (\Phi_\rho \times v^\rho)(f) V_\rho$, $f \in A \times_\alpha^\omega G$.

By Theorem 2.1 (a), we have

$$\psi(f)(\xi) = \sum_g V_\rho^* \Phi_\rho(f(g)) v_g^\rho V_\rho \xi = \sum_g V_\rho^* \Phi_\rho(f(g)) V_\rho u_g \xi = \sum_g \rho(f(g)) u_g \xi$$

for all $f \in K(G, A)$ and $\xi \in E$. \square

4. The KSGNS construction associated with a projective u -covariant completely positive linear map

We present a projective generalization of the construction proved by Joița in Theorem 4.3, [9]. Given a C^* -dynamical system (G, A, α) , a projective u -covariant

non-degenerate completely positive linear map from A to $\mathcal{L}_B(E)$ induces a projective covariant non-degenerate representation of (G, A, α) on a Hilbert B -module, uniquely up to unitary equivalence.

Definition 4.1. ([15]) Let B a C^* -algebra and let E and F be two Hilbert B -modules. The **strict topology** on $\mathcal{L}_B(E, F)$ is the one given by the seminorms $V \mapsto \|Vx\|$ ($x \in E$) and $V \mapsto \|V^*y\|$ ($y \in F$). Let A a C^* -algebra. A completely positive linear map $\rho: A \rightarrow \mathcal{L}_B(E)$ is said to be **strict** if for some approximate unit $\{e_\lambda\}_\lambda$ of B , the net $\{\rho(e_\lambda)\}_\lambda$ satisfies the Cauchy condition for the strict topology in $\mathcal{L}_B(E)$. If A is unital then the condition of strictness is automatically satisfied.

Definition 4.2. ([15]) Let A be a C^* -algebra, let E be a Hilbert C^* -module over a C^* -algebra B and let $\rho: A \rightarrow \mathcal{L}_B(E)$ be a completely positive map. ρ is called **non-degenerate** if $\rho(e_\lambda) \rightarrow 1$ strictly in $\mathcal{L}_B(E)$, for $(e_\lambda)_\lambda$ an approximate unit of A .

Definition 4.3. ([10]) A representation Φ of a C^* -algebra A on a Hilbert module E is **non-degenerate** if $\Phi(A)E$ is dense in E .

Theorem 4.1. Let (G, A, α) be a C^* -dynamical system, let u be a projective unitary representation of G on a Hilbert module E over a C^* -algebra B with the multiplier ω and let ρ be a projective u -covariant, non-degenerate, completely positive linear map from A to $\mathcal{L}_B(E)$.

1. Then there is a projective covariant non-degenerate representation $(\Phi_\rho, v^\rho, E_\rho)$ of (G, A, α) , where v^ρ is a projective unitary representation with the multiplier ω and V_ρ in $\mathcal{L}_B(E, E_\rho)$ such that

- (a) $\rho(a) = V_\rho^* \Phi_\rho(a) V_\rho$, for all $a \in A$;
- (b) $\{\Phi_\rho(a) V_\rho \xi; a \in A, \xi \in E\}$ spans a dense submodule of E_ρ ;
- (c) $v_g^\rho V_\rho = V_\rho u_g$ for all $g \in G$.

2. If F is a Hilbert B -module, (Φ, v, F) is a projective covariant non-degenerate representation of (G, A, α) , where v is a projective unitary representation with multiplier ω and W is an element in $\mathcal{L}_B(E, F)$ such that

- (a) $\rho(a) = W^* \Phi(a) W$, for all $a \in A$;
- (b) $\{\Phi(a) W \xi; a \in A, \xi \in F\}$ spans a dense submodule of F ;
- (c) $v_g W = W u_g$ for all $g \in G$,

then there is a unitary operator U in $\mathcal{L}_B(E_\rho, F)$ such that

- (i) $\Phi(a)U = U\Phi_\rho(a)$, for all $a \in A$;
- (ii) $v_g U = U v_g^\rho$, for all $g \in G$;
- (iii) $W = U V_\rho$.

Proof. 1. Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit of A such that the net $\{\rho(e_\lambda)\}_{\lambda \in \Lambda}$ is strictly Cauchy in $\mathcal{L}_B(E)$.

Let $a \in A$. It is not difficult to check that the linear map $\Phi_\rho(a)$ from $A \otimes_{alg} E$ to $A \otimes_{alg} E$ defined by

$$\Phi_\rho(a)(b \otimes \xi) = ab \otimes \xi,$$

$a, b \in A, \xi \in E$ extends to a bounded linear operator Φ_ρ from E_ρ to E_ρ . Moreover, $\Phi_\rho(a)$ is adjointable and $(\Phi_\rho(a))^* = \Phi_\rho(a^*)$. Thus we have obtained a map Φ_ρ from A to $\mathcal{L}_B(E_\rho)$. It is easy to verify that Φ_ρ is a $*$ -representation of A on E_ρ .

Let $a_1, \dots, a_n \in A$ and $\xi_1, \dots, \xi_n \in E_\rho$. Since

$$\begin{aligned} \left\| \sum_i \rho(a_i) \xi_i \right\|^2 &= \left\| \sum_{i,j} \langle \xi_i, \rho(a_i^*) \rho(a_j) \xi_j \rangle \right\| \leq \\ \|\rho\| \cdot \left\| \sum_{i,j} \langle \xi_i, \rho(a_i^* a_j) \xi_j \rangle \right\| &= \|\rho\| \cdot \left\| \sum_i a_i \otimes \xi_i \right\|^2 \end{aligned}$$

(the inequality is obtained from Lemma 5.4. [15]), the linear map $a \otimes \xi + \mathcal{N} \mapsto \rho(a)\xi$ from $A \otimes_{alg} E/\mathcal{N}$ to E extends by linearity and continuity to a bounded linear operator \tilde{V} from E_ρ to E , where $\mathcal{N} = \{z \in A \otimes_{alg} E; \langle z, z \rangle = 0\}$.

Let $\lambda \in \Lambda$ and $\xi \in E$. We denote by ξ^λ the element $e_\lambda \otimes \xi$ in $A \otimes_{alg} E$. Since the net $\{\rho(e_\lambda)\xi\}_\lambda$ is convergent in E , the net $\{\xi^\lambda + \mathcal{N}\}_\lambda$ is convergent in E_ρ . Define a map V_ρ from E to E_ρ by $V_\rho \xi = \lim_{\lambda \in \Lambda} (\xi^\lambda + \mathcal{N})$. To show that V_ρ is an element in $\mathcal{L}_B(E, E_\rho)$ it is sufficient to show that

$$\langle V_\rho \xi, a \otimes \eta + \mathcal{N} \rangle = \langle \xi, \tilde{V}(a \otimes \eta + \mathcal{N}) \rangle$$

for all $\xi \in E$ and $a \otimes \eta \in A \otimes_{alg} E$.

Let $\xi \in E$ and $a \otimes \eta \in A \otimes_{alg} E$. Then we have

$$\begin{aligned} \langle V_\rho \xi, a \otimes \eta + \mathcal{N} \rangle &= \lim_{\lambda \in \Lambda} \langle \xi^\lambda + \mathcal{N}, a \otimes \eta + \mathcal{N} \rangle = \lim_{\lambda \in \Lambda} \langle \xi, \rho(e_\lambda a) \eta \rangle = \\ \langle \xi, \rho(a) \eta \rangle &= \langle \xi, \tilde{V}(a \otimes \eta + \mathcal{N}) \rangle. \end{aligned}$$

Hence, $V_\rho \in \mathcal{L}_B(E, E_\rho)$.

Let $a \in A$ and $\xi \in E$. We denote by ξ_a the element $a \otimes \xi$ in $A \otimes_{alg} E$. It is not difficult to check that $\Phi_\rho(a)V_\rho \xi = \xi_a + \mathcal{N}$. Therefore the submodule of E_ρ generated by $\{\Phi_\rho(a)V_\rho \xi; a \in A, \xi \in E\}$ is exactly $A \otimes_{alg} E/\mathcal{N}$ and thus the condition (b) is verified.

Let $a \in A$. Then we have

$$V_\rho^* \Phi_\rho(a) V_\rho \xi = V_\rho^* (\xi_a + \mathcal{N}) = \rho(a) \xi,$$

for all $\xi \in E$ and so the condition 1(a) is also verified.

For each $g \in G$, we define a linear map v_g^ρ from $A \otimes_{alg} E$ to $A \otimes_{alg} E$ by

$$v_g^\rho(a \otimes \xi) = \alpha_g(a) \otimes u_g \xi$$

Using the fact that ρ is projective u -covariant, it is not difficult to check that v_g^ρ extends to a bounded linear map v_g^ρ from E_ρ to E_ρ and since

$$\langle v_g^\rho(a \otimes \xi + \mathcal{N}), b \otimes \eta + \mathcal{N} \rangle = \langle a \otimes \xi + \mathcal{N}, v_{g^{-1}}^\rho(b \otimes \eta + \mathcal{N}) \rangle$$

for all $a \otimes \xi, b \otimes \eta \in A \otimes_{alg} E$, $v_g^\rho \in \mathcal{L}_B(E_\rho)$ and moreover, $(v_g^\rho)^* = v_{g^{-1}}^\rho$. Also it is easy to check that the map $g \mapsto v_g^\rho$ is a unitary representation of G on E_ρ .

To show that $(\Phi_\rho, v_\rho, E_\rho)$ is a covariant projective representation of (G, A, α) it remains to prove that $\Phi_\rho(\alpha_g(a)) = v_g^\rho \Phi_\rho(a) v_{g^{-1}}^\rho$ for all $g \in G$ and $a \in A$ and that v^ρ is a projective representation with the multiplier ω .

Let $g \in G$ and $a \in A$. We have

$$(v_g^\rho \Phi_\rho(a) v_{g^{-1}}^\rho)(a \otimes \xi + \mathcal{N}) = (v_g^\rho \Phi_\rho(a))(\alpha_{g^{-1}}(a) \otimes u_{g^{-1}} \xi + \mathcal{N}) =$$

$$\begin{aligned}
v_g^\rho(a\alpha_{g^{-1}}(a) \otimes u_{g^{-1}}\xi + \mathcal{N}) &= \alpha_g(a\alpha_{g^{-1}}(a)) \otimes u_g u_{g^{-1}}\xi + \mathcal{N} = \\
\alpha_g(a)\alpha_g(\alpha_{g^{-1}}(a)) \otimes u_g u_{g^*}\xi + \mathcal{N} &= \alpha_g(a)a \otimes \xi + \mathcal{N} = \\
\Phi_\rho(\alpha_g(a))(a \otimes \xi + \mathcal{N})
\end{aligned}$$

for all $a \otimes \xi \in A \otimes_{alg} E$. Hence $\Phi_\rho(\alpha_g(a)) = v_g^\rho \Phi_\rho(a) v_{g^{-1}}^\rho = v_g^\rho \Phi_\rho(a) (v_g^\rho)^*$, so $(\Phi_\rho, v^\rho, E_\rho)$ is a covariant representation.

To show that condition (c) is verified, let $\xi \in E$ and $g \in G$. Then we have

$$\begin{aligned}
\|v_g^\rho V_\rho \xi - V_\rho u_g \xi\|^2 &= \lim_\lambda \|v_g^\rho \xi^\lambda - V_\rho u_g \xi\|^2 = \\
\lim_\lambda \|\langle \xi, \rho(e_\lambda^2) \xi \rangle + \langle \xi, \xi \rangle - \langle \rho(\alpha_g(e_\lambda)) u_g \xi, u_g \xi \rangle - \langle u_g \xi, \rho(\alpha_g(e_\lambda)) u_g \xi \rangle\| &\leq \\
\lim_\lambda \|\langle \xi, \rho(e_\lambda) \xi \rangle + \langle \xi, \xi \rangle - \langle \rho(e_\lambda) \xi, \xi \rangle - \langle \xi, \rho(e_\lambda) \xi \rangle\| &= \lim_\lambda \|\langle \xi - \rho(e_\lambda) \xi, \xi \rangle\| = 0
\end{aligned}$$

Hence condition (c) is also verified.

Now we show that v^ρ is a projective representation with the multiplier ω .

Let $g_1, g_2 \in G$. By condition (c) and the fact that u is a projective representation with the multiplier ω , we have:

$$\begin{aligned}
v_{g_1 g_2}^\rho V_\rho &= V_\rho u_{g_1 g_2} = V_\rho \omega(g_1, g_2) u_{g_1} u_{g_2} = \\
\omega(g_1, g_2) V_\rho u_{g_1} u_{g_2} &= \omega(g_1, g_2) v_{g_1}^\rho V_\rho u_{g_2} = \omega(g_1, g_2) v_{g_1}^\rho v_{g_2}^\rho V_\rho.
\end{aligned}$$

Therefore, v^ρ is a projective representation.

2. Using the fact that $\rho(a) = V_\rho^* \Phi_\rho(a) V_\rho = W^* \Phi(a) W$ for all $a \in A$, it is not difficult to check that $\|\sum_{s=1}^m \beta \Phi_\rho(a_s) V_\rho \xi_s\| = \|\sum_{s=1}^m \beta \Phi(a_s) W \xi_s\|$ for all $\beta \in \mathbb{C}$, $a_1, \dots, a_m \in A$ and $\xi_1, \dots, \xi_m \in E$. Therefore the linear map $\Phi_\rho(a) V_\rho \xi \mapsto \Phi(a) W \xi$ from the submodule of E_ρ generated by $\{\Phi_\rho(a) V_\rho \xi; a \in A, \xi \in E\}$ to the submodule of F generated by $\{\Phi(a) W \xi; a \in A, \xi \in E\}$ extends to a surjective isometric B -linear map U from E_ρ onto F . Then, by Theorem 3.5, [15], U is unitary. We define this unitary operator U in $\mathcal{L}(E_\rho, E)$ by

$$U\left(\sum_{s=1}^m \beta \Phi_\rho(a_s) V_\rho \xi_s\right) = \sum_{s=1}^m \beta \Phi(a_s) W \xi_s, \forall a_1, \dots, a_m \in A, \forall \xi_1, \dots, \xi_m \in E$$

Let $a \in A$. From

$$\begin{aligned}
\Phi(a) U(\Phi_\rho(b) V_\rho \xi) &= \Phi(a) \Phi(b) W \xi = \Phi(ab) W \xi = \\
U(\Phi_\rho(ab) V_\rho \xi) &= U \Phi_\rho(a)(\Phi_\rho(b) V_\rho \xi)
\end{aligned}$$

for all $b \in A$, $\xi \in E$, we conclude that $\Phi(a)U = U\Phi_\rho(b)$.

Since Φ and Φ_ρ are non-degenerate, by Proposition 4.2, [11], we have $UV_\rho \xi = \lim_\lambda U\Phi_\rho(e_\lambda) V_\rho \xi = \lim_\lambda \Phi(e_\lambda) W \xi = W \xi$ for all $\xi \in E$. Therefore, $W = UV_\rho$.

Let $g \in G, a \in A, \xi \in E$. We have

$$\begin{aligned}
(v_g U)(\Phi_\rho(a) V_\rho \xi) &= v_g(\Phi(a) U V_\rho \xi) v_g(\Phi(a) W \xi) = \Phi(a) v_g W \xi = \\
\Phi(a) W u_g \xi &= U(\Phi_\rho(a) V_\rho u_g \xi) = U(\Phi_\rho(a) v_g^\rho V_\rho \xi) = (U v_g^\rho)(\Phi_\rho(a) V_\rho \xi)
\end{aligned}$$

This implies that $v_g U = U v_g^\rho$ and thus the assertion 2 is proved. \square

Remark 4.1. The representation $(\Phi_\rho, V_\rho, E_\rho)$ constructed in the theorem above is called KSGNS construction associated with ρ .

REFERENCES

- [1] *W. Arveson*, Subalgebra of C^* -algebras, *Acta Math.* **123** (1969), 141-224.
- [2] *R. C. Busby, H. A. Smith*, Representations of twisted group algebras, *Trans. Amer. Math. Soc.*, Vol. **149** (1970), 503-537.
- [3] *T.-L. Costache*, Projective version of Stinespring type theorems, *Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie*, Vol. **53**, No.2 (2010), 109-124.
- [4] *T.-L. Costache*, On the projective covariant representations of C^* - dynamical systems associated with completely multi-positive projective u -covariant maps, *UPB Sci. Bull., Series A*, Vol. **72**, Iss.4 (2010), 185-196.
- [5] *T.-L. Costache*, Multilinear Completely Bounded Projective u -Covariant Maps Extended on Twisted Crossed Products , Recent Advances in Mathematics and Computer in Business, Economics, Biology and Chemistry, Proceedings of "the 11th WSEAS International Conference on Mathematics and Computers in Biology and Chemistry (MCBC'10), "G. Enescu" University, Iasi, Romania, June 13-15, 2010" (2010), 54-61.
- [6] *J. Heo*, Completely multi-positive linear maps and representations on Hilbert C^* -modules, *J. Operator Theory* **41** (1999), 3-22.
- [7] *J. Heo*, Hilbert C^* -modules and projective representations associated with multipliers, *J. Math. Anal. Appl.* **331** (2007), 499-505.
- [8] *M. Joița*, Strict completely positive maps between locally C^* -algebras and representations on Hilbert modules, *J. London Math. Soc.* (2), **66** (2002), 421-432.
- [9] *M. Joița*, Completely multi-positive linear maps between locally C^* -algebras and representations on Hilbert modules, *Studia Math.* **172** (2006), 181-196.
- [10] *M. Joița, T.-L. Costache, M. Zamfir*, Covariant representations associated with covariant completely n -positive linear maps between C^* -algebras, III Workshop on Coverings, Selections and Games in Topology, April 25-29, 2007, Vrnjacka Banja, Serbia.
- [11] *M. Joița, T.-L. Costache, M. Zamfir*, Representations associated with completely n -positive linear maps between C^* -algebras, *Stud. Cercet. Stiint. Ser. Mat.* **16** (2006), Supplement Proceedings of ICMI 45, Bacău, Sept. 18-20, 2006, 111-122.
- [12] *A. Kaplan*, Covariant completely positive maps and liftings, *Rocky Mountain J. Math.*, vol.23, No. 3 (1993), 939-946.
- [13] *I. Kaplansky*, Modules over operator algebras, *Amer. J. Math* **75** (1953), 839-853.
- [14] *G. G. Kasparov*, Hilbert C^* -modules: Theorem of Stinespring and Voiculescu, *J.Operator Theory* **4** (1980), 133-150.
- [15] *E. C. Lance*, Hilbert C^* -modules. A toolkit for operator algebraists, *London Math. Soc. Lect. Notes Ser.* 210, Cambridge University Press, Cambridge (1995).
- [16] *G. W. Mackey*, On induced representations of groups, *Amer. J. Math.* **73** (1951), 576-592.
- [17] *G. J. Murphy*, Positive definite kernels and Hilbert C^* -modules, *Proc. of the Edinburgh Math. Soc.* **40** (1997), 367-374.
- [18] *W. L. Paschke*, Inner product modules over B^* -algebras, *Trans. Amer. Math. Soc.*, **182** (1973), 443-468.
- [19] *H. Scutaru*, Some remarks on covariant completely positive linear maps on C^* -algebras, *Rep. Math. Phys.* **16**, No. 1 (1979), 79-87.
- [20] *I. Schur*, Untersuchungen über die Darstellung der endlichen Gruppe durch gebrochene lineare Substitutionen, *J. fur Math.* **132** (1907).
- [21] *W. F. Stinespring*, Positive functions on C^* -algebras, *Proc. Amer. Math. Soc.* **6** (1955), 211-216.
- [22] *T. Xu*, Covariant completely positive maps, *Indian J. Pure Appl. Math.* **34** (8) (2003), 1131-1140.
- [23] *D. Williams*, Crossed Products of C^* - algebras, *Mathematical Surveys and Monographs*, Vol.134, Amer. Math. Soc., Providence, RI, 2007.