

A NUMERICAL APPROACH FOR HAMMERSTEIN INTEGRAL EQUATIONS OF MIXED TYPE USING OPERATIONAL MATRICES OF HYBRID FUNCTIONS

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Această lucrare prezintă o procedură eficientă numerică pentru rezolvarea ecuațiilor neliniare Hammerstein integrale de tip mixt. Aceste ecuații apar în model dinamic de reactor chimic, unele probleme în teoria controlului și reformulări diferite ale unei ecuații cu derivate parțiale eliptice cu condiții la limită neliniare. Metoda noastră folosește funcția hibrid și unele proprietăți utile dintre aceste funcții pentru a converti o Hammerstein și ecuații Volterra-Hammerstein integrală de tip mixt într-o ecuație algebrică. Fiabilitatea și eficiența sistemului propus sunt demonstrate de unele experimente numerice.

This paper presents an efficient numerical procedure for solving the nonlinear Hammerstein integral equations of mixed type. These equations arise in the dynamic model of chemical reactor, some problems in control theory and various reformulations of an elliptic partial differential equation with nonlinear boundary conditions. Our method uses hybrid function and some useful properties of these functions to convert a Hammerstein and Volterra-Hammerstein integral equations of mixed type into an algebraic equation. The reliability and efficiency of the proposed scheme are demonstrated by some numerical experiments.

Keywords: Hammerstein integral equation of mixed type, Volterra-Hammerstein integral equation of mixed type, block-pulse function, hybrid function, operational matrix, integration of the cross product, coefficient matrix.

1. Introduction

The theory and application of integral equations is an important subject within applied mathematics. Due to the fact that the nonlinear Hammerstein integral equations of mixed types appear in many problems in mathematical physics, the dynamic model of chemical reactor, some problems in control theory, and various reformulations of an elliptic partial differential equation with nonlinear boundary conditions, this paper deals with finding numerical solutions of these problems, see Refs. [1-3].

In this paper we are going to use hybrid Legendre polynomials and Block-Pulse functions as basis for the numerical solution of Hammerstein and Volterra-

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Hammerstein integral equations of mixed type that take the following forms respectively

$$u(x) = f(x) + \sum_{i=1}^d \int_0^1 k_i(x, s) \psi_i(s, u(s)) ds, \quad 0 \leq x \leq 1, \quad (1)$$

$$u(x) = f(x) + \sum_{i=1}^d \int_0^x k_i(x, s) \psi_i(s, u(s)) ds, \quad 0 \leq x \leq 1, \quad (2)$$

where f , u and k are assume to be in L^2 , with $\psi_i(x, u(x))$ nonlinear in u for $i=1, \dots, d$. We assume that (1) and (2) have a unique solution $u(x)$ to be determined. The existence results for these equations had been discussed in [1].

Actually few numerical methods for approximating the solution of Eqs. (1) and (2) are known. Ganesh and Joshi introduced a kind of collocation method in [1] for Hammerstein integral equations of mixed type. Brunner in [4] extended Ganesh and Joshi method for nonlinear Volterra integro-differential equations and Volterra-Hammerstein integral equations of mixed type. Hadizadeh and Azizi in [5] used Taylor expansion approach for solving Eqs.(1) and (2).

In the present paper we introduce a new computational method to solve Hammerstein and Volterra-Hammerstein integral equations of mixed type. This paper consists of reducing these integral equations to an algebraic equation by first expanding the candidate functions as hybrid functions, then some useful properties of hybrid functions such as integration of cross product, a special product matrix and a related coefficient matrix with optimal order are applied to solve these Hammerstein and Volterra-Hammerstein integral equations of mixed type. The main characteristic of this technique is to convert these kinds of integral equations into an algebraic equation and in this way, the solution procedures are either reduced or simplified, accordingly. This method had been used for some Volterra, Fredholm [6], Hammerstein [7], Volterra Hammerstein [8] integral equations and integro-differential [9] and nonlinear Volterra-Fredholm integro-differential [10] equations beforehand.

This paper is organized as follows: first, we introduce hybrid Legendre polynomials and Block-Pulse functions and their properties. In Section 3, we apply these sets of hybrid functions to approximate the solutions of Hammerstein integral equations of mixed type. Using the properties of hybrid functions together with collocation method, we reduce the nonlinear Hammerstein integral equations of mixed type to a system of nonlinear equations. These equations can be solved using Newton's iterative method. In Section 4 we convert Volterra-Hammerstein integral equations of mixed type to an algebraic equation. Section 5 exhibits error estimation for our method. Finally in Section 6, we illustrate some numerical examples to show the convergence and accuracy of this method.

2. Some properties of hybrid functions

2.1. Hybrid of Block-Pulse functions and Legendre polynomials

Consider the Legendre polynomials $L_m(x)$ on the interval $[-1,1]$ as

$$L_0(x) = 1, L_1(x) = x,$$

$$L_{m+1}(x) = \frac{2m+1}{m+1} x L_m(x) - \frac{m}{m+1} L_{m-1}(x), m = 1, 2, 3, \dots$$

The set of $\{L_m(x) : m = 0, 1, \dots\}$ in the Hilbert space $L^2[-1,1]$ is a complete orthogonal set.

A set of Block-Pulse functions $\phi_i(x), i = 1, 2, \dots, n$ on the interval $[0,1)$ are defined as follows

$$\phi_i(x) = \begin{cases} 1, & \frac{i-1}{n} \leq x < \frac{i}{n}, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

The Block-Pulse functions on $[0,1)$ are disjoint, so for $i, j = 1, 2, \dots, n$, we have $\phi_i(x)\phi_j(x) = \delta_{ij}\phi_i(x)$, also these functions have the property of orthogonality on $[0,1)$.

The orthogonal set of hybrid functions $h_{ij}(x)$, $i = 1, 2, \dots, n$ and $j = 0, 1, \dots, m-1$ where i is the order for Block-Pulse functions, j is the order for Legendre polynomials and x is the normalized time, is defined on the interval $[0,1)$ as

$$h_{ij}(x) = \begin{cases} L_j(2nx - 2i + 1), & \frac{i-1}{n} \leq x < \frac{i}{n}, \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

Since $h_{ij}(x)$ is the combination of Legendre polynomials and Block-Pulse functions which are both complete and orthogonal, it follows the set of hybrid functions be complete orthogonal set in $L^2[0,1)$.

2.2. Function approximation

Any function $u(x) \in L^2[0,1)$ can be expanded in a hybrid function as

$$u(x) \approx \sum_{i=1}^n \sum_{j=0}^{m-1} c_{ij} h_{ij}(x) = \mathbf{C}^T \mathbf{B}(x), \quad (5)$$

where

$$c_{ij} = \frac{(u(x), h_{ij}(x))}{(h_{ij}(x), h_{ij}(x))}, \quad (6)$$

for $i = 1, 2, \dots, n, j = 0, 1, \dots, m-1$, such that in Eq.(6), the symbol $(.,.)$ denotes the inner product. Also we have

$$\mathbf{C} = [c_{10}, \dots, c_{1,m-1}, c_{20}, \dots, c_{2,m-1}, \dots, c_{n0}, \dots, c_{n,m-1}]^T, \quad (7)$$

and

$$\mathbf{B}(x) = [h_{10}(x), \dots, h_{1,m-1}(x), h_{20}(x), \dots, h_{2,m-1}(x), \dots, h_{n0}(x), \dots, h_{n,m-1}(x)]^T. \quad (8)$$

We can also approximate the function $k(x, s) \in L^2([0, 1] \times [0, 1])$ as follows

$$k(x, s) \approx \mathbf{B}^T(x) \mathbf{K} \mathbf{B}(s), \quad (9)$$

where \mathbf{K} is an $nm \times nm$ matrix, that is given by

$$\mathbf{K}_{ij} = \frac{(\mathbf{B}_i(x), (k(x, s), \mathbf{B}_j(s)))}{(\mathbf{B}_i(x), \mathbf{B}_i(x))(\mathbf{B}_j(s), \mathbf{B}_j(s))}, \quad (10)$$

for $i, j = 1, 2, \dots, nm$.

2.3. The integration of the cross product

The integration of the cross product of two hybrid function vectors $\mathbf{B}(x)$ in Eq.(8) can be obtained as

$$\mathbf{D} = \int_0^1 \mathbf{B}(x) \mathbf{B}^T(x) dx = \begin{bmatrix} \mathbf{L} & 0 & \dots & 0 \\ 0 & \mathbf{L} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{L} \end{bmatrix}, \quad (11)$$

where \mathbf{L} is a $m \times m$ diagonal matrix that is given by

$$\mathbf{L} = \frac{1}{n} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \frac{1}{3} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{2m-1} \end{bmatrix}. \quad (12)$$

Matrix \mathbf{D} is very important for numerically solving Hammerstein integral equation of mixed type (1), because of its sparsity, it can increase the calculating speed, as well as save the memory storage.

2.4. Operational matrix of integration

The integration of the vector $\mathbf{B}(x)$ defined in Eq.(8) is given by

$$\int_0^x \mathbf{B}(x') dx' \approx \mathbf{P}\mathbf{B}(x), \quad (13)$$

where \mathbf{P} is the $nm \times nm$ operational matrix for integration and is given in [6,11] as

$$\mathbf{P} = \begin{bmatrix} \mathbf{S} & \mathbf{T} & \mathbf{T} & \dots & \mathbf{T} \\ \mathbf{0} & \mathbf{S} & \mathbf{T} & \dots & \mathbf{T} \\ \mathbf{0} & \mathbf{0} & \mathbf{S} & \dots & \mathbf{T} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{S} \end{bmatrix}, \quad (14)$$

that \mathbf{S} and \mathbf{T} are $m \times m$ matrices and they can be seen in [6] in details.

2.5. Product operational matrix

In the study of Volterra-Hammerstein integral equation of mixed type by hybrid functions, it is always necessary to evaluate the product of $\mathbf{B}(x)$ by $\mathbf{B}^T(x)$, that is called the product matrix of hybrid functions. Let

$$\Omega(x) = \mathbf{B}(x)\mathbf{B}^T(x), \quad (15)$$

where $\Omega(x)$ is an $nm \times nm$ matrix. Multiplying the matrix $\Omega(x)$ by \mathbf{C} that defined in Eq.(7) we obtain

$$\Omega(x)\mathbf{C} = \tilde{\mathbf{C}}\mathbf{B}(x), \quad (16)$$

where $\tilde{\mathbf{C}}$ is an $nm \times nm$ matrix and is called the coefficient matrix. Hsiao in [6] shows the method of producing this matrix.

With the powerful properties of Eq.(16) we can convert a Volterra-Hammerstein integral equation of mixed type to an algebraic equation.

3. Numerical solvability of Hammerstein integral equation of mixed type

For solving Hammerstein integral equation of mixed type (1), we let

$$z_i(s) = \psi_i(s, u(s)), \quad 0 \leq s \leq 1, \quad (17)$$

then we get

$$u(x) = f(x) + \sum_{i=1}^d \int_0^1 k_i(x, s) z_i(s) ds. \quad (18)$$

substituting (18) in (17) results,

$$z_i(x) = \psi_i(x, f(x) + \sum_{i=1}^d \int_0^1 k_i(x, s) z_i(s) ds), \quad i = 1, 2, \dots, d. \quad (19)$$

We approximate this equation as

$$z_i(x) \approx \sum_{j=1}^n a_{i,j} \mathbf{B}_j(x) = \mathbf{A}_i^T \mathbf{B}(x), \quad i = 1, 2, \dots, d, \quad (20)$$

which $\mathbf{B}(x)$ is defined in Eq.(8). Using Eqs.(9), (11), (19) and (20) we get

$$\mathbf{A}_i^T \mathbf{B}(x) = \psi_i(x, f(x) + \sum_{j=1}^d \mathbf{B}^T(x) \mathbf{K}_j \mathbf{D} \mathbf{A}_j), \quad i = 1, 2, \dots, d. \quad (21)$$

In order to find \mathbf{A}_i , $i = 1, 2, \dots, d$ we collocate Eq.(21) in nm nodal points of Newton-Cotes as

$$x_p = \frac{2p-1}{2nm}, \quad p = 1, 2, \dots, nm. \quad (22)$$

then we have equation (21) as follows

$$\mathbf{A}_i^T \mathbf{B}(x_p) = \psi_i(x_p, f(x_p) + \sum_{j=1}^d \mathbf{B}^T(x_p) \mathbf{K}_j \mathbf{D} \mathbf{A}_j), \quad p = 1, \dots, nm, i = 1, \dots, d. \quad (23)$$

We can calculate the unknown vectors \mathbf{A}_i , $i = 1, 2, \dots, d$ from the above nonlinear system of equations. The required approximated solution $u(x)$ for our Hammerstein integral equation of mixed type (1), can be obtained by using Eqs.(18) and (20) as follows

$$u(x) = f(x) + \sum_{j=1}^d \mathbf{B}^T(x) \mathbf{K}_j \mathbf{D} \mathbf{A}_j. \quad (24)$$

4. Numerical solvability of Volterra-Hammerstein integral equation of mixed type

Consider the Volterra-Hammerstein integral equation of mixed type given in Eq.(2). For solving these equations like in the previous section, we let $z_i(s) = \psi_i(s, u(s))$ for $0 \leq s \leq 1$. Then from Eq.(2) we get

$$z_i(x) = \psi_i(x, f(x) + \sum_{i=1}^d \int_0^x k_i(x, s) z_i(s) ds), \quad i = 1, 2, \dots, d. \quad (25)$$

After using Eqs.(9), (13), (16) and (20) we get

$$\mathbf{A}_i^T \mathbf{B}(x) = \psi_i(x, f(x) + \sum_{j=1}^d \mathbf{B}^T(x) \mathbf{K}_j \tilde{\mathbf{A}}_j \mathbf{P} \mathbf{B}(x)), \quad i = 1, 2, \dots, d. \quad (26)$$

By collocating Eq.(26) in nm nodal points (22) we have,

$$\mathbf{A}_i^T \mathbf{B}(x_p) = \psi_i(x_p, f(x_p) + \sum_{j=1}^d \mathbf{B}^T(x_p) \mathbf{K}_j \tilde{\mathbf{A}}_j \mathbf{P} \mathbf{B}(x_p)), p = 1, \dots, nm, i = 1, \dots, d. \quad (27)$$

After solving nonlinear system (27) we get \mathbf{A}_i , $i = 1, 2, \dots, d$, then we will have the approximation solution of Volterra-Hammerstein integral equations of mixed type (2) as

$$u(x) = f(x) + \sum_{j=1}^d \mathbf{B}^T(x) \mathbf{K}_j \tilde{\mathbf{A}}_j \mathbf{P} \mathbf{B}(x). \quad (28)$$

5. Error estimation

If we approximate our function with Legendre polynomials, we have the following theorem for its error analysis.

Theorem 1. Let $u(x) \in H^k(-1, 1)$ (Sobolev space), $u_J(x) = \sum_{i=0}^J a_i L_i(x)$ be the best approximation polynomial of $u(x)$ in L^2 , then

$$\|u(x) - u_J(x)\|_{L^2[-1, 1]} \leq C_0 J^{-k} \|u(x)\|_{H^k(-1, 1)}, \quad (29)$$

where C_0 is a positive constant, which depends on the selected norm and is independent of $u(x)$ and J ; see Refs. [12, 13].

If we approximate our function with hybrid Legendre polynomials and Block-Pulse functions, we have the following error bound for it accordingly.

Theorem 2. Let $u(x) \in H^k(0, 1)$, $I_i = [\frac{i-1}{n}, \frac{i}{n}]$ and $u_{nm}(x) = \sum_{i=1}^n \sum_{j=0}^{m-1} c_{ij} h_{ij}(x)$ then

$$\|u(x) - u_{nm}(x)\|_{L^2[0, 1]} \leq C_0 (nm)^{-k} \max_{0 \leq i \leq n} \|u(x)\|_{H^k(I_i)}. \quad (30)$$

Proof. By using Theorem 1 it is obvious.

Now we perform the estimation of the error for the Hammerstein and Volterra-Hammerstein integral equations. If we approximate the answer of Eqs. (1) and (2) by Eqs. (24) and (28) respectively, we have

$$e(x) = |u(x) - u_{nm}(x)| = |u(x) - f(x) - \sum_{j=1}^d \omega_j(x)|,$$

where $e(x)$ is defined as an error function. Also for Hammerstein and Volterra-Hammerstein integral equations of mixed type, the function $\omega_j(x)$ is defined as follows respectively

$$\omega_j(x) = \begin{cases} \mathbf{B}^T(x)\mathbf{K}_j\mathbf{D}\mathbf{A}_j, \\ \mathbf{B}^T(x)\mathbf{K}_j\tilde{\mathbf{A}}_j\mathbf{P}\mathbf{B}(x). \end{cases}$$

When we put $x = x_p$

$$e(x_p) = \left| u(x_p) - f(x_p) - \sum_{j=1}^d \omega_j(x_p) \right|,$$

then our aim is $e(x_p) \leq 10^{-k_p}$ (k_p is any positive integer). If we prescribe, $\max(10^{-k_p}) = 10^{-k}$ and since in our hybrid function n and m are adjustable, for a fixed m we can increase n as far as the following inequality holds at each of the points x_p

$$e(x_p) \leq 10^{-k}.$$

It means that we can find m and n such that the error function $e(x_p)$ approaches 0.

6. Numerical results

6.1. Example 1

Consider a mixed Hammerstein integral equation of the form

$$u(x) = f(x) + \sum_{i=1}^3 \int_0^1 k_i(x, s) \psi_i(s, u(s)) ds, \quad (31)$$

where

$$f(x) = -\frac{x^2}{30} + \frac{5x}{4} - \frac{23}{20},$$

$$k_1(x, s) = x^2 s^2, \quad k_2(x, s) = (x + s), \quad k_3(x, s) = 1,$$

$$\psi_1(s, u(s)) = [u(s)]^2, \quad \psi_2(s, u(s)) = [u(s)]^3, \quad \psi_3(s, u(s)) = [u(s)]^4,$$

and the exact solution is $u(x) = x - 1$. Table 1 exhibits the computational results with $m = 4$, $n = 2$ and $m = 8$, $n = 8$ besides exact solutions.

6.2. Example 2

Consider the Volterra-Hammerstein integral equation of mixed type as follows

$$u(x) = f(x) + \sum_{i=1}^2 \int_0^x k_i(x, s) \psi_i(s, u(s)) ds, \quad (32)$$

where

$$f(x) = 2e^{x/2} - e^x - \frac{1}{4}x^2(1 + 2\ln[x]),$$

$$k_1(x, s) = \ln(x + s), \quad k_2(x, s) = e^{(x-s)},$$

$$\psi_1(s, u(s)) = \ln(u(s)), \quad \psi_2(s, u(s)) = \sqrt{u(s)},$$

and the exact solution is $u(x) = e^x$. Table 2 shows the computational results with $m = 4, n = 2$ and $m = 8, n = 8$ versus the exact solutions.

Table 1

Approximate and exact solutions for Example 1.

x_i	Solution with $m = 8, n = 2$	Solution with $m = 8, n = 8$	Exact solution
0.0	-1.00005	-1.00000	-1
0.1	-0.90006	-0.90000	-0.9
0.2	-0.80007	-0.80001	-0.8
0.3	-0.70008	-0.70001	-0.7
0.4	-0.60008	-0.60002	-0.6
0.5	-0.50009	-0.50001	-0.5
0.6	-0.40009	-0.40002	-0.4
0.7	-0.30010	-0.30006	-0.3
0.8	-0.20010	-0.20005	-0.2
0.9	-0.10011	-0.10008	-0.1
1.0	-0.00011	-0.00008	0

7. Conclusion

As we know nonlinear Hammerstein integral equations of mixed type are some complicated equations that arise in applied science and there are only a few numerical methods for numerically solving them; see Refs [1-5]. In this paper the

Table 2

Approximate and exact solutions for Example 2.

x_i	Solution with $m = 8, n = 2$	Solution with $m = 8, n = 8$	Exact solution
0.0	0.99982	1.00000	1
0.1	1.10517	1.10517	1.10517
0.2	1.22141	1.22140	1.22140
0.3	1.34985	1.34986	1.34986
0.4	1.49183	1.49182	1.49182
0.5	1.64868	1.64872	1.64872
0.6	1.82211	1.82212	1.82212
0.7	2.01376	2.01375	2.01375
0.8	2.22553	2.22554	2.22554
0.9	2.45961	2.45960	2.45960
1.0	2.71835	2.71830	2.71828

hybrid Legendre polynomials and Block-Pulse functions and the associated operational matrices of integration \mathbf{D} , operational matrix \mathbf{P} , product matrix $\mathbf{\Omega}$ and coefficient matrix $\tilde{\mathbf{C}}$ are used to solve Hammerstein and Volterra-

Hammerstein integral equations of mixed type. The method is based on reducing the system into a set of algebraic equations. The main advantage of this method is its efficiency and simple applicability and this truth that the values of n and m for this hybrid function are adjustable as well as being able to yield more accurate numerical solutions. As showed in our examples we can get better errors in Hammerstein and Volterra-Hammerstein integral equations of mixed type when m has a suitable fix value and n is increased. The operational matrices of this hybrid function are sparse hence the implementation of hybrid function method on Hammerstein integral equations of mixed type is much faster than other functions methods and reduces the CPU time and at the same time keeping the accuracy of the solution. The numerical examples support this claim.

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