

## ON THE CONTROLLABILITY OF THE NEUMANN PROBLEM FOR THE WAVE EQUATION

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*In this paper we prove the controllability of the Neumann problem for the wave equation at  $T > 2\pi$ , using the ontoness approach.*

**Keywords:** wave equation, Neumann boundary conditions, exact controllability.

**MSC2010:** 35L05, 35Q93.

### 1. Introduction

Consider the following wave equation:

$$\frac{\partial^2 u}{\partial t^2}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t), \quad x \in (0, \pi), \quad t > 0, \quad (1)$$

with the initial conditions:

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 0, \quad (2)$$

and with the Neumann boundary conditions:

$$\frac{\partial u}{\partial x}(0, t) = f(t), \quad \frac{\partial u}{\partial x}(\pi, t) = 0, \quad (3)$$

where the control function  $f(\cdot)$  is square integrable for  $t > 0$ .

In our problem,  $u(x, t)$  denotes the transversal displacement of the point of abscissa  $x$  at time  $t$ .

There are some related articles which studied the controllability of the Neumann problem for the wave equation: in [9] Lions proved the exact controllability of the wave equation with Dirichlet boundary conditions, by introducing the Hilbert Uniqueness Method, in [5] Lasiecka and Triggiani study the controllability of the Neumann problem for the wave equation using the ontoness approach, in [2] Chen proved that the exact controllability of the wave equation follows from the Russell "controllability via stabilizability" principle, in [1] Bui showed the exact controllability of the wave equation in a bounded domain of  $\mathbb{R}^n$ , using a combination of the Hilbert Uniqueness Method and the dynamic programming principle. In [10] we proved the exact controllability

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of the wave equation with mixed boundary conditions (Neumann-Dirichlet) at  $T = \pi$ , using the moment problem approach.

In this paper we give an alternative demonstration for the controllability of the Neumann problem for the wave equation, shown in the general case by Lasiecka and Triggiani in [5]. Our proof uses the surjectivity of a suitable adjoint operator.

## 2. Exact controllability in $H_0^1(0, \pi) \times L^2(0, \pi)$

Consider the wave equation:

$$\frac{\partial^2 \varphi}{\partial t^2}(x, t) = \frac{\partial^2 \varphi}{\partial x^2}(x, t), \quad x \in (0, \pi), \quad t > 0, \quad (4)$$

with the following homogeneous Neumann boundary conditions:

$$\frac{\partial \varphi}{\partial x}(0, t) = 0, \quad \frac{\partial \varphi}{\partial x}(\pi, t) = 0, \quad (5)$$

and with the initial conditions:

$$\varphi(x, 0) = \varphi_0(x), \quad \frac{\partial \varphi}{\partial t}(x, 0) = \varphi_1(x), \quad (6)$$

where  $\varphi_0(x) \in H^1(0, \pi)$  and  $\varphi_1(x) \in L^2(0, \pi)$ .

From [4], Proposition 4.4, we know that every solution of the system (4)-(6) satisfies, for  $T > 2\pi$ , the following norm inequality:

$$\|\varphi_0\|_{H^1(0, \pi)}^2 + \|\varphi_1\|_{L^2(0, \pi)}^2 \leq c \|\varphi(0, t)\|_{H^1(0, T)}^2, \quad (7)$$

with  $c$  depending only on  $T$ .

For  $f, g \in H^1(a, b)$ , an equivalent inner product is given by (see [3]):

$$\langle f, g \rangle_{H^1(a, b)} = f(a)g(a) + \int_a^b f'(x)g'(x)dx, \quad (8)$$

and the induced norm is given by:

$$\|f\|_{H^1(a, b)}^2 = f^2(a) + \int_a^b (f'(x))^2 dx. \quad (9)$$

We can also write the inequality (7):

$$\varphi_0^2(0) + \int_0^\pi \left[ (\varphi_0'(x))^2 + \varphi_1^2(x) \right] dx \leq c \left\{ \varphi^2(0, 0) + \int_0^T \left( \frac{\partial \varphi}{\partial t}(0, t) \right)^2 dt \right\}. \quad (10)$$

Define the transformation:

$$\Sigma : H^1(0, \pi) \times L^2(0, \pi) \rightarrow H^1(0, T), \quad (11)$$

with

$$\Sigma(\varphi_0, \varphi_1) = \varphi(0, t). \quad (12)$$

Inequality (7) asserts that the transformation  $\Sigma$  is coercive. With this transformation, the demonstration of the controllability is reduced to proving the surjectivity of the adjoint operator  $\Sigma^*$ .

Proposition 4.4 from [4] implies that the transformation  $\Sigma$  is continuous, so the adjoint can be computed using  $(\varphi_0, \varphi_1)$  in a subspace which is dense in  $H^1(0, \pi) \times L^2(0, \pi)$ . We compute the adjoint of the restriction to the closed, but not dense, subspace  $H_0^1(0, \pi) \times L^2(0, \pi)$ . Let us denote this restriction by  $\tilde{\Sigma}$ . We know that  $\mathcal{D}(0, \pi)$  ( $C^\infty$  functions with compact support) is dense both in  $H_0^1(0, \pi)$  and  $L^2(0, \pi)$  so, in order to compute the adjoint of the restriction, we can assume  $\varphi_0 \in \mathcal{D}(0, \pi)$  and  $\varphi_1 \in \mathcal{D}(0, \pi)$ .

**Theorem 2.1.** *The adjoint operator  $\tilde{\Sigma}^*$  is surjective.*

*Proof.* We need to obtain an expression for the following inner product

$$\langle \tilde{\Sigma} \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}, g \rangle_{H^1(0, T)}, \quad (13)$$

with  $g \in H^1(0, T)$ , in order to find  $\tilde{\Sigma}^* g$ .

Remark on p. 44, from [4], the fact that our data are indefinitely derivable – for  $\varphi$  – and Theorem A, from [6], p. 117 – for  $u$  – justifies the following computations. We know from [6], relation (1.21), that the function  $u$  belongs to  $H^1(0, \pi)$  with respect to the variable  $x$ . In [6] the authors studied the regularity of solutions of general, mixed, second-order, time-dependent, hyperbolic problems of Neumann type with a functional analytic/operator theoretic approach.

If  $\varphi_0 \in \mathcal{D}(0, \pi)$  and  $\varphi_1 \in \mathcal{D}(0, \pi)$ , from (4) we obtain

$$\frac{\partial^3 \varphi}{\partial t^3} = \frac{\partial^3 \varphi}{\partial^2 x \partial t}. \quad (14)$$

Multiply both sides of the above equation with  $u(x, T - t)$ , where  $u$  is the solution of the controlled equation and  $T > 2\pi$ , and formally integrate by parts twice. The left hand side becomes:

$$\begin{aligned} & \int_0^\pi \left[ \frac{\partial^2 \varphi}{\partial t^2}(x, t) u(x, T - t) \Big|_{t=0}^{t=T} + \int_0^T \frac{\partial^2 \varphi}{\partial t^2}(x, t) \frac{\partial u}{\partial t}(x, T - t) dt \right] dx = \\ &= \int_0^\pi \left[ \frac{\partial^2 \varphi}{\partial t^2}(x, T) u(x, 0) - \frac{\partial^2 \varphi}{\partial t^2}(x, 0) u(x, T) + \frac{\partial \varphi}{\partial t}(x, T) \frac{\partial u}{\partial t}(x, 0) - \right. \\ & \quad \left. - \frac{\partial \varphi}{\partial t}(x, 0) \frac{\partial u}{\partial t}(x, T) \right] dx + \int_0^\pi \int_0^T \frac{\partial \varphi}{\partial t}(x, t) \frac{\partial^2 u}{\partial t^2}(x, T - t) dt dx, \end{aligned} \quad (15)$$

and the right hand side becomes:

$$\int_0^T \left[ \frac{\partial^2 \varphi}{\partial x \partial t}(x, t) u(x, T - t) \Big|_{x=0}^{x=\pi} - \int_0^\pi \frac{\partial^2 \varphi}{\partial x \partial t}(x, t) \frac{\partial u}{\partial x}(x, T - t) dx \right] dt =$$

$$\begin{aligned}
&= \int_0^T \left[ \frac{\partial^2 \varphi}{\partial x \partial t}(\pi, t)u(\pi, T-t) - \frac{\partial^2 \varphi}{\partial x \partial t}(0, t)u(0, T-t) - \right. \\
&\quad \left. - \frac{\partial \varphi}{\partial t}(\pi, t)\frac{\partial u}{\partial x}(\pi, T-t) + \frac{\partial \varphi}{\partial t}(0, t)\frac{\partial u}{\partial x}(0, T-t) \right] dt + \\
&\quad + \int_0^T \int_0^\pi \frac{\partial \varphi}{\partial t}(x, t)\frac{\partial^2 u}{\partial x^2}(x, T-t)dxdt. \tag{16}
\end{aligned}$$

Considering the conditions (2)-(3) and (5)-(6), we obtain:

$$-\int_0^\pi \frac{\partial^2 \varphi}{\partial t^2}(x, 0)u(x, T)dx - \int_0^\pi \varphi_1(x)\frac{\partial u}{\partial t}(x, T)dx = \int_0^T \frac{\partial \varphi}{\partial t}(0, t)\frac{\partial u}{\partial x}(0, T-t)dt. \tag{17}$$

We can choose  $g \in H^1(0, T)$  such that

$$\frac{\partial u}{\partial x}(0, T-t) = f(T-t) = g'(t). \tag{18}$$

Since  $\varphi(0, 0) = \varphi_0(0) = 0$ , we infer that the right hand side of (17) is the inner product, mentioned in (8),

$$\langle \tilde{\Sigma} \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}, g \rangle_{H^1(0, T)}.$$

On the other hand,

$$\begin{aligned}
&-\int_0^\pi \frac{\partial^2 \varphi}{\partial t^2}(x, 0)u(x, T)dx = -\int_0^\pi \frac{\partial^2 \varphi}{\partial x^2}(x, 0)u(x, T)dx = \\
&= -\int_0^\pi \varphi_0''(x)u(x, T)dx = \int_0^\pi \varphi_0'(x)\frac{\partial u}{\partial x}(x, T)dx, \tag{19}
\end{aligned}$$

and we obtain the inner product in  $H^1(0; \pi)$ .

Relation (1.21), from [6], p. 121, insures that the last integral in (19) is well defined.

So, (17) can be written in the following way:

$$\langle \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}, \begin{pmatrix} u(x, T) \\ -\frac{\partial u}{\partial t}(x, T) \end{pmatrix} \rangle_{H^1(0, \pi) \times L^2(0, \pi)} = \langle \tilde{\Sigma} \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}, g \rangle_{H^1(0, T)}. \tag{20}$$

Due to the fact that  $\tilde{\Sigma}$  is coercive, it follows that the adjoint  $\tilde{\Sigma}^*$  is surjective.  $\square$

Furthermore, (20) shows that

$$\tilde{\Sigma}^* g = \begin{pmatrix} u(x, T) \\ -\frac{\partial u}{\partial t}(x, T) \end{pmatrix}, \tag{21}$$

where  $u$  solves the problem (1)-(2) with the followings Neumann boundary conditions:

$$\frac{\partial u}{\partial x}(0, t) = -g'(T - t) = f(t), \quad \frac{\partial u}{\partial x}(\pi, t) = 0, \quad (22)$$

where  $f(t)$  is arbitrary in  $L^2(0, T)$ .

In the relation (20) one should take the projection of  $u(x, T)$  onto  $H_0^1(0, \pi)$ ; the same for (21). Remark that, when  $u(x, T)$  is in  $H^1(0, \pi)$ , one can consider its projection

$$v(x, T) = u(x, T) - u(0, T) - \frac{x}{\pi} [u(\pi, T) - u(0, T)].$$

We have  $\square_1 v = \square_1 u$  and  $v(0, T) = v(\pi, T) = 0$ , so  $v(\cdot, T) \in H_0^1(0, \pi)$ .

Then one can consider  $\xi \in H^1(0, \pi)$ ,  $\eta \in L^2(0, \pi)$ ,

$$\tilde{\xi}(x) = \xi(x) - \xi(0) - \frac{x}{\pi} [\xi(\pi) - \xi(0)]$$

is in  $H_0^1(0, \pi)$ , and, if one has a control  $f$  that ensures that  $v(x, T) = \tilde{\xi}(x)$ ,  $\frac{\partial v}{\partial t}(x, T) = \eta(x)$ , it follows that  $u(x, t) = v(x, t) + \xi(0) + \frac{x}{\pi} [\xi(\pi) - \xi(0)]$  verifies  $u(x, T) = \xi(x)$  and  $\frac{\partial u}{\partial t}(x, T) = \eta(x)$ .

Thanks to Theorem 2.1, we obtain the main result of this paper:

**Theorem 2.2.** *For every functions  $\xi \in H_0^1(0, \pi)$ ,  $\eta \in L^2(0, \pi)$  and  $T > 2\pi$ , there exists a control  $f \in L^2(0, T)$  such that for  $u$ , the solution of the problem (1)-(3), we have:*

$$u(x, T) = \xi(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x, T) = \eta(x). \quad (23)$$

This proves that using  $L^2(0, T)$  control it is possible to reach every targets  $\xi \in H_0^1(0, \pi)$  and  $\eta \in L^2(0, \pi)$ , starting from the zero initial conditions.

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