

## APPROXIMATE FIXED POINTS OF OPERATORS ON G-METRIC SPACES

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*In this paper, we will first introduce the approximate fixed point property and a new class of operators and contraction mapping for a cyclic map  $T$  on  $G$ -metric spaces. Also, we prove two general lemmas regarding approximate fixed Point of cyclic maps on  $G$ -metric spaces. Using these results we prove several approximate fixed point theorems for a new class of operators and contraction mapping on  $G$ -metric spaces. In addition, some examples are presented to illustrate our results for a new class of operators and contraction mapping on  $G$ -metric spaces.*

**Keywords:** Fixed points, Approximate fixed points, Mohseni operator, Mohsenialhosseini operator, Diameter approximate fixed point.

**MSC2000:** 47H10, 54H25, 46B20.

### 1. Introduction

Fixed point theory is a very popular tool in solving existence problems in many branches of Mathematical Analysis and its applications. In physics and engineering fixed point technique has been used in areas like image retrieval, signal processing and the study of existence and uniqueness of solutions for a class of nonlinear integral equations. Some recent work on fixed point theorems of integral type in  $G$ -metric spaces, stability of functional difference equation can be found in [15, 16] and the references therein.

In 1968, Kannan (see [7] ) proved a fixed point theorem for operators which need not be continuous. Further, Chatterjea (see[6]), in 1972, also proved a fixed point theorem for discontinuous mapping, which is actually a kind of dual of Kannan mapping. In 1972, by combining the above three independent contraction conditions above, Zamfirescu (see [18]) obtained another fixed point result for operators which satisfy the following. In 2001, Rus (see [17]) defined  $\alpha$ -contraction. In [3], the author obtained a different contraction condition, also he formulated a corresponding fixed point theorem. In 2006, Berinde (see [4]) obtained some result on  $\alpha$ -contraction for approximate fixed point in metric space. Miandaragh et al. [8, 9]

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obtained some result on approximate fixed points in metric space. In [5], Chandok et al. obtained some result on Coupled common fixed point in partially ordered G-metric spaces,

On the other hand, in 2006, Mustafa and Sims [13, 14] introduced the notion of generalized metric spaces or simply G-metric spaces. Many researchers have obtained fixd point, coupled fixed point, coupled common fixed point results on G-metric spaces (see [2, 5, 15]). Recently, Mohsenalhosseini et al in [12] introduced the approximate fixed point in G-metric spaces for various types of operators.

The aim of this paper is to introduce the new classes of operators and contraction maps regarding approximate fixed point and diameter approximate fixed point for a cyclic map  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  i.e.  $T(A) \subseteq B$ ,  $T(B) \subseteq C$  and  $T(C) \subseteq A$  on  $G$ -metric spaces. Also, we give some illustrative example of our main results.

## 2. Preliminaries

This section recalls the following notations and the ones that will be used in what follows.

**Definition 2.1.** [13] Let  $X$  be a nonempty set and let  $G : X \times X \times X \rightarrow R^+$  be a function satisfying the following properties:

- (G1)  $G(x, y, z) = 0$  if and only if  $x = y = z$ ;
- (G2)  $0 < G(x, x, y)$  for all  $x, y \in X$  with  $x \neq y$ ;
- (G3)  $G(x, x, y) \leq G(x, y, z)$  for all  $x, y, z \in X$  with  $z \neq y$ ;
- (G4)  $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$  (symmetry in all three variables);
- (G5)  $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$  for all  $x, y, z, a \in X$  (rectangle inequality).

Then, the function  $G$  is called generalized metric or, more specifically  $G$  – metric on  $X$ , and the pair  $(X, G)$  is called a  $G$  – metric space.

**Proposition 2.1.** [13] Every  $G$ -metric  $(X, G)$  defines a metric space  $(X, d_G)$  by

- 1)  $d_G(x, y) = G(x, y, y) + G(y, x, x)$ .  
if  $(X, G)$  is a symmetric  $G$ - metric space. Then
- 2)  $d_G(x, y) = 2G(x, y, y)$ .

**Definition 2.2.** [11] Let  $T : X \rightarrow X$ ,  $\epsilon > 0$ ,  $x_0 \in X$ . Then  $x_0 \in X$  is an  $\epsilon$ -fixed point for  $T$  if  $\|Tx_0 - x_0\| < \epsilon$ .

**Remark 2.1.** [11] In this paper we will denote the set of all  $\epsilon$  – fixed points of  $T$ , for a given  $\epsilon$ , by :

$$AF(T) = \{x \in X \mid x \text{ is an } \epsilon \text{ – fixed point of } T\}.$$

**Definition 2.3.** [11] Let  $T : X \rightarrow X$ . Then  $T$  has the approximate fixed point property (a.f.p.p) if

$$\forall \epsilon > 0, \quad AF(T) \neq \emptyset.$$

**Theorem 2.1.** [11] *Let  $(X, \|\cdot\|)$  be a complete norm space,  $T : X \rightarrow X$ ,  $x_0 \in X$  and  $\epsilon > 0$ . If  $\|T^n(x_0) - T^{n+k}(x_0)\| \rightarrow 0$  as  $n \rightarrow \infty$  for some  $k > 0$ , then  $T^k$  has an  $\epsilon$ -fixed point.*

### 3. Main result

We begin with two lemmas which will be used in order to prove all the results given in third section. Let  $(X, G)$  be a  $G$ -metric space.

**Definition 3.1.** [12] *Let  $A, B, C$  are closed subsets of a  $G$ -metric space  $X$  and  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  be a cyclic map. Let  $\epsilon > 0$  and  $x_0 \in A \cup B \cup C$ . Then  $x_0$  is an  $\epsilon$ -fixed point of  $T$  if*

$$[G(x_0, Tx_0, Tx_0) + G(Tx_0, x_0, x_0)] < \epsilon.$$

**Remark 3.1.** *In this paper we will denote the set of all  $\epsilon$ -fixed points of  $T$ , for a given  $\epsilon$ , by:*

$$F_G^\epsilon(T) = \{x \in A \cup B \cup C \mid x \text{ is an } \epsilon\text{-fixed point of } T\}.$$

**Definition 3.2.** *Let  $A, B, C$  are closed subsets of a  $G$ -metric space  $X$ ,  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  be a cyclic map and  $\epsilon > 0$ . We define diameter of the set  $F_G^\epsilon(T)$ , i.e.,*

$$\delta(F_G^\epsilon(T)) = \sup\{G(x, y, z) : x, y, z \in F_G^\epsilon(T)\}.$$

**Definition 3.3.** [12] *Let  $A, B, C$  are closed subsets of a  $G$ -metric space  $X$  and  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  be a cyclic map. Then  $T$  has the approximate fixed point property (a.f.p.p) if  $\forall \epsilon > 0$ ,*

$$F_G^\epsilon(T) \neq \emptyset.$$

**Definition 3.4.** [12] *Let  $A, B, C$  are closed subsets of a  $G$ -metric space  $X$ . A cyclic map  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  is said to be asymptotically regular at a point  $x \in A \cup B \cup C$ , if*

$$\lim_{n \rightarrow \infty} \{G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x)\} = 0,$$

where  $T^n$  denotes the  $n$ th iterate of  $T$  at  $x$ .

**Lemma 3.1.** [12] *Let  $A, B, C$  are closed subsets of a  $G$ -metric space  $X$ . If  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  is asymptotically regular at a point  $x \in A \cup B \cup C$ , Then  $T$  has an approximate fixed point.*

**Lemma 3.2.** [12] *Let  $A, B, C$  are closed subsets of a  $G$ -metric space  $X$ ,  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  a cyclic map and  $\epsilon > 0$ . We assume that:*

- i)  $F_G^\epsilon(T) \neq \emptyset$ ;
- ii)  $\forall \xi > 0 \exists \psi(\xi) > 0$  such that

$$[G(x, y, y) + G(y, x, x)] - [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] < \xi \implies$$

$$G(x, y, y) + G(y, x, x) \leq \psi(\xi), \quad \forall x, y \in F_G^\epsilon(T).$$

Then:

$$\delta(F_G^\epsilon(T)) \leq \psi(2\epsilon).$$

#### 4. Diameter approximate fixed point for a new class of operators

In this section a series of qualitative and quantitative results will be obtained regarding the diameter approximate fixed point. Also, we prove diameter approximate fixed point theorems for a new class of operators on metric spaces.

**Definition 4.1.** [12] Let  $(X, d)$  be a metric space. A mapping  $T : X \rightarrow X$  is a **Mohseni operator** if there exists  $\alpha \in (0, \frac{1}{2})$  such that

$$d(Tx, Ty) \leq \alpha[d(x, y) + d(Tx, Ty)].$$

**Definition 4.2.** [12] Let  $A, B$  and  $C$  be non-empty subsets of a  $G$ -metric space  $X$ . The cyclic mapping  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  is a  $G$ -Mohseni operator if there exists  $\alpha \in (0, \frac{1}{2})$  such that

$$\begin{aligned} G(Tx, Ty, Ty) + G(Ty, Tx, Tx) &\leq \alpha[G(x, y, y) + G(y, x, x) \\ &\quad + G(Tx, Ty, Ty) + G(Ty, Tx, Tx)]. \end{aligned}$$

**Theorem 4.1.** [12] Let  $A, B$  and  $C$  be non-empty subsets of a  $G$ -metric space  $X$ . Suppose that the cyclic mapping  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  is a  $G$ -Mohseni operator. Then for every  $\epsilon > 0$ ,

$$F_G^\epsilon(T) \neq \emptyset.$$

**Theorem 4.2.** Let  $(X, G)$  be a  $G$ -metric space. Suppose that the cyclic mapping  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  is a  $G$ -Mohseni operator. Then for every  $\epsilon > 0$ ,

$$\delta(F_G^\epsilon(T)) \leq \frac{2\epsilon(1 + \alpha)}{1 - 2\alpha}.$$

**Proof:** Let  $\epsilon > 0$ . Condition i) in Lemma 3.2 is satisfied, as one can see in the proof of Theorem 4.1 we only verify that condition ii) in Lemma 3.2 holds. Let  $\theta > 0$  and  $x, y \in F_G^\epsilon(T)$  and assume that Then:

$$[G(x, y, y) + G(y, x, x)] - [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] < \theta.$$

Then:

$$\begin{aligned} [G(x, y, y) + G(y, x, x)] &\leq \alpha[G(x, y, y) + G(y, x, x)] \\ &\quad + [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] + \theta. \end{aligned}$$

Therefore As  $x, y \in F_G^\epsilon(T)$ , we know that

$$G(x, Tx, Tx) + G(Tx, x, x) \leq \epsilon, G(y, Ty, Ty) + G(Ty, y, y) \leq \epsilon.$$

Therfore,  $G(x, y, y) + G(y, x, x) \leq \frac{2\alpha\epsilon + \theta}{1 - 2\alpha}$ . So for every  $\theta > 0$  there exists  $\phi(\theta) = \frac{2\alpha\epsilon + \theta}{1 - 2\alpha} > 0$  such that

$$[G(x, y, y) + G(y, x, x)] - [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] < \theta \Rightarrow$$

$$G(x, y, y) + G(y, x, x) \leq \phi(\theta).$$

Now by Lemma 3.2, it follows that

$$\delta(F_G^\epsilon(T)) \leq \phi(2\epsilon), \forall \epsilon > 0,$$

which means exactly that

$$\delta(F_G^\epsilon(T)) \leq \frac{2\epsilon(1+\alpha)}{1-2\alpha}. \blacksquare$$

**Example 4.1.** Let  $X = \{0, 1, 2, \dots, 18\}$  and  $G : X \times X \times X \rightarrow \mathbb{R}^+$  be defined as follows:

$$G(x, y, z) = \begin{cases} x + y + z & \text{if } x \neq y \neq z \neq 0, \\ x + y & \text{if } x = y \neq z, x, y, z \neq 0, \\ y + z + 1 & \text{if } x = 0, y \neq z, y, z \neq 0, \\ y + 2 & \text{if } x = 0, y = z \neq 0, \\ z + 1 & \text{if } x = 0, y = 0, z \neq 0, \\ 0 & \text{if } x = y = z. \end{cases}$$

Let  $A = \{4, 18\}$ ,  $B = \{3, 7, 17\}$  and  $C = \{0\}$ . Obviously  $A, B, C$  are closed subsets of  $G$ -metric space  $X$ . Define the mapping  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  by

$$Tx = \begin{cases} x - 1 & \text{if } x \in \{4, 18\} \\ 0 & \text{if } x \in \{3, 7, 17\} \\ 4 & \text{if } x = 0. \end{cases}$$

It is easily to be checked that  $T(A) \subseteq B$ ,  $T(B) \subseteq A$  and  $T(C) \subseteq A$ . For any  $x, y \in A \cup B \cup C$  we have the chain of inequalities

$$\begin{aligned} G(Tx, Ty, Ty) + G(Ty, Tx, Tx) &\leq \alpha[G(x, y, y) + G(y, x, x) \\ &\quad + G(Tx, Ty, Ty) + G(Ty, Tx, Tx)]. \end{aligned}$$

So  $T$  satisfies all the conditions of Theorems 4.1 and 4.2 and thus for every  $\epsilon > 0$ ,  $F_G^\epsilon(T) \neq \emptyset$  and  $\delta(F_G^\epsilon(T)) \leq \frac{2\epsilon(1+\alpha)}{1-2\alpha}$  respectively.

**Definition 4.3.** [12] Let  $A, B$  and  $C$  be non-empty subsets of a  $G$ -metric space  $X$ . The cyclic mapping  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  is a  $G_\alpha$ -contraction if there exists  $\alpha \in (0, 1)$

$$G(Tx, Ty, Ty) + G(Ty, Tx, Tx) \leq \alpha[G(x, y, y) + G(y, x, x)], \forall x, y \in A \cup B \cup C.$$

**Definition 4.4.** [12] Let  $A, B$  and  $C$  be non-empty subsets of a  $G$ -metric space  $X$ . The cyclic mapping  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  is a  $G$ -Chatterjea operator if there exists  $\alpha \in (0, \frac{1}{2})$  such that

$$\begin{aligned} G(Tx, Ty, Ty) + G(Ty, Tx, Tx) &\leq \alpha[G(x, Ty, Ty) + G(Ty, x, x) \\ &\quad + G(y, Tx, Tx) + G(Tx, y, y)], \forall x, y \in A \cup B \cup C. \end{aligned}$$

By combining the three independent contraction conditions:  $G_\alpha$ -contraction, Mohseni, and Chatterjea operators we will be obtained another approximate fixed point result for operators which satisfy the followings:

**Definition 4.5.** Let  $A, B$  and  $C$  be non-empty subsets of a  $G$ -metric space  $X$ . The cyclic mapping  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  is a  $G$ -Mohsenialhosseini operator if there exists  $\alpha \in [0, 1)$ ,  $\beta \in [0, \frac{1}{2})$ ,  $\gamma \in [0, \frac{1}{2})$  such that for all  $x, y \in A \cup B \cup C$  at least one of the following is true:

- i)  $[G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] \leq \alpha[G(x, y, y) + G(y, x, x)];$
- ii)  $[G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] \leq \beta[G(x, y, y) + G(y, x, x) + G(Tx, Ty, Ty) + G(Ty, Tx, Tx)];$
- iii)  $[G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] \leq \gamma[G(x, Ty, Ty) + G(Ty, x, x) + G(y, Tx, Tx) + G(Tx, y, y)].$

**Theorem 4.3.** Let  $A, B$  and  $C$  be non-empty subsets of a  $G$ -metric space  $X$ . Suppose that the cyclic mapping  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  is a  $G$ -Mohsenialhosseini operator. Then for every  $\epsilon > 0$ ,  $F_G^\epsilon(T) \neq \emptyset$ .

*Proof.* Let  $\epsilon > 0$  and  $x \in A \cup B \cup C$ . Supposing (ii) holds, we have that:

$$\begin{aligned}
 [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] &\leq \beta[G(x, y, y) + G(y, x, x) \\
 &\quad + G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] \\
 &\leq \beta[G(x, Tx, Tx) + G(Tx, x, x) \\
 &\quad + G(Tx, y, y) + G(y, Tx, Tx) \\
 &\quad + G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] \\
 &= 2\beta[G(x, Tx, Tx) + G(Tx, x, x)] \\
 &\quad + \beta[G(x, y, y) + G(y, x, x)] \\
 &\quad + \beta[G(Tx, Ty, Ty) + G(Ty, Tx, Tx)].
 \end{aligned}$$

Thus,

$$\begin{aligned}
 [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] &\leq 2\beta[G(x, Tx, Tx) + G(Tx, x, x)] \\
 &\quad + \left(\frac{\beta}{1-\beta}\right)[G(x, y, y) + G(y, x, x)]. \quad (3.1)
 \end{aligned}$$

Supposing (iii) holds, we have that:

$$\begin{aligned}
 [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] &\leq \gamma[G(x, Ty, Ty) + G(Ty, x, x) \\
 &\quad + G(y, Tx, Tx) + G(Tx, y, y)] \\
 &\leq \gamma[G(x, y, y) + G(y, x, x) \\
 &\quad + G(y, Ty, Ty) + G(Ty, y, y)] \\
 &\quad + \gamma[G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] \\
 &= \gamma[G(x, y, y) + G(y, x, x)] \\
 &\quad + 2\gamma[G(y, Ty, Ty) + G(Ty, y, y)] \\
 &\quad + \gamma[G(Tx, Ty, Ty) + G(Ty, Tx, Tx)].
 \end{aligned}$$

Thus,

$$\begin{aligned} [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] &\leq \frac{2\gamma}{1-\gamma} [G(y, Ty, Ty) + G(Ty, y, y)] \\ &+ \left(\frac{\gamma}{1-\gamma}\right) [G(x, y, y) + G(y, x, x)]. \end{aligned} \quad (3.2)$$

Similarly,

$$\begin{aligned} [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] &\leq \gamma [G(x, Ty, Ty) + G(Ty, x, x)] \\ &+ G(y, Tx, Tx) + G(Tx, y, y). \\ &\leq \gamma [G(x, Tx, Tx) + G(Tx, x, x)] \\ &+ G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] \\ &+ \gamma [G(x, y, y) + G(y, x, x)] \\ &+ G(x, Tx, Tx) + G(Tx, x, x)] \\ &= 2\gamma [G(x, Tx, Tx) + G(Tx, x, x)] \\ &+ \gamma [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] \\ &+ \gamma [G(x, y, y) + G(y, x, x)]. \end{aligned}$$

Then

$$\begin{aligned} [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] &\leq \left(\frac{2\gamma}{1-\gamma}\right) [G(x, Tx, Tx) + G(Tx, x, x)] \\ &+ \left(\frac{\gamma}{1-\gamma}\right) [G(x, y, y) + G(y, x, x)]. \end{aligned} \quad (3.3)$$

In view of (i), (3.1), (3.2) and (3.3), we have,  $\xi = \max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$ , and it is easy to see that  $\xi \in [0, 1]$  for  $T$  satisfying at least one of the condition (i), (ii) and (iii) we have that.

$$\begin{aligned} [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] &\leq 2\xi [G(x, Tx, Tx) + G(Tx, x, x)] \\ &+ \xi [G(x, y, y) + G(y, x, x)] \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] &\leq 2\xi [G(y, Ty, Ty) + G(Ty, y, y)] \\ &+ \xi [G(x, y, y) + G(y, x, x)] \end{aligned}$$

hold. Using these conditions implied by i) -iii) and taking  $x \in A \cup B \cup C$ , we have

$$\begin{aligned}
G(T^n x, T^{n+1} x, T^{n+1} x) &+ G(T^{n+1} x, T^n x, T^n x) \\
&= G(T(T^{n-1} x), T(T^n x), T(T^n x)) \\
&+ G(T(T^n x), T(T^{n-1} x), T(T^{n-1} x)) \\
&\stackrel{(3.4)}{\leq} 2\xi[G(T^{n-1} x, T(T^{n-1} x), T(T^{n-1} x)) \\
&+ G(T(T^{n-1} x), T^{n-1} x, T^{n-1} x) \\
&+ \xi G(T^{n-1} x, T^n x, T^n x) \\
&+ G(T^n x, T^{n-1} x, T^{n-1} x)] \\
&= 3\xi[G(T^{n-1} x, T^n x, T^n x) \\
&+ G(T^n x, T^{n-1} x, T^{n-1} x)] \\
&\vdots \\
&\leq (3\xi)^n[G(x, Tx, Tx) + G(Tx, x, x)].
\end{aligned}$$

Therefore,

$$G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x) \leq (3\xi)^n[G(x, Tx, Tx) + G(Tx, x, x)].$$

Then, we have

$$\lim_{n \rightarrow \infty} [G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x)] = 0, \quad \forall x \in A \cup B \cup C.$$

Using Lemma 3.1, we find that  $F_G^\epsilon(T) \neq \emptyset$  for all  $\epsilon > 0$ .  $\square$

**Theorem 4.4.** *Let  $(X, G)$  be a G-metric space. Suppose that the cyclic mapping  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  is a G-Mohsenialhosseini operator. Then for every  $\epsilon > 0$ ,*

$$\delta(F_G^\epsilon(T)) \leq 2\epsilon \frac{1+\eta}{1-\eta},$$

where  $\eta = \max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$ , and  $\alpha, \beta, \gamma$  as in Definition 4.5

**Proof:** In the proof of Theorem 4.3, we have already shown that if  $T$  satisfies at least one of the conditions i), ii), iii) from Definition 4.5, then

$$\begin{aligned}
[G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] &\leq 2\eta[G(x, Tx, Tx) + G(Tx, x, x)] \\
&+ \eta[G(x, y, y) + G(y, x, x)]
\end{aligned}$$

and

$$\begin{aligned}
[G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] &\leq 2\eta[G(y, Ty, Ty) + G(Ty, y, y)] \\
&+ \eta[G(x, y, y) + G(y, x, x)]
\end{aligned}$$

hold.

Let  $\epsilon > 0$ . We will only verify that condition ii) in Lemma 3.2 is satisfied, as i) holds, see the Proof of Theorem 4.3. Let  $\theta > 0$  and  $x, y \in F_G^\epsilon(T)$  and assume that

$$\begin{aligned} G(x, y, y) + G(y, x, x) - G(Tx, Ty, Ty) + G(Ty, Tx, Tx) &\leq \theta \Rightarrow \\ G(x, y, y) + G(y, x, x) &\leq G(Tx, Ty, Ty) + G(Ty, Tx, Tx) + \theta \Rightarrow \end{aligned}$$

$$\begin{aligned} G(x, y, y) + G(y, x, x) &\leq 2\eta[G(x, Tx, Tx) + G(Tx, x, x)] \\ &\quad + \eta[G(x, y, y) + G(y, x, x)] + \theta \Rightarrow \end{aligned}$$

$$(1 - \eta)[G(x, y, y) + G(y, x, x)] \leq 2\eta\epsilon + \theta$$

$$[G(x, y, y) + G(y, x, x)] \leq \frac{2\eta\epsilon + \theta}{1 - \eta}.$$

So for every  $\theta > 0$  there exists  $\phi(\theta) = \frac{2\eta\epsilon + \theta}{1 - \eta} > 0$  such that

$$\begin{aligned} G(x, y, y) + G(y, x, x) - [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] &\leq \theta \\ \Rightarrow G(x, y, y) + G(y, x, x) &\leq \phi(\theta). \end{aligned}$$

Now by Lemma 3.2, it follows that

$$\delta(F_G^\epsilon(T)) \leq \phi(2\epsilon), \forall \epsilon > 0,$$

which means exactly that

$$\delta(F_G^\epsilon(T)) \leq 2\epsilon \frac{1 + \eta}{1 - \eta}, \forall \epsilon > 0. \blacksquare$$

**Remark 4.1.** Example 4.1 shows that  $T$  satisfies all the conditions of Theorems 4.3 and 4.4 and thus for every  $\epsilon > 0$ ,  $F_G^\epsilon(T) \neq \emptyset$  and  $\delta(F_G^\epsilon(T)) \leq 2\epsilon \frac{1 + \eta}{1 - \eta}$  respectively.

**Definition 4.6.** Let  $A, B$  and  $C$  be non-empty subsets of a  $G$ -metric space  $X$ . The cyclic mapping  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  is a  $G$ -Mohseni-semi contraction if there exists  $\alpha \in ]0, \frac{1}{2}[$  such that

$$\begin{aligned} [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] &\leq \alpha[G(x, y, y) + G(y, x, x)] \\ &\quad + G(x, Tx, Tx) + G(Tx, x, x). \end{aligned}$$

**Theorem 4.5.** Let  $A, B$  and  $C$  be non-empty subsets of a  $G$ -metric space  $X$ . Suppose that the cyclic mapping  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  is a  $G$ -Mohseni-semi contraction. Then for every  $\epsilon > 0$ ,  $F_G^\epsilon(T) \neq \emptyset$ .

**Proof:** Let  $\epsilon > 0$  and  $x \in A \cup B \cup C$ .

$$\begin{aligned}
[G(T^n x, T^{n+1} x, T^{n+1} x) &+ G(T^{n+1} x, T^n x, T^n x)] \\
&= [G(T(T^{n-1}) x, T(T^n) x, T(T^n) x) \\
&+ G(T(T^n) x, T(T^{n-1}) x, T(T^{n-1}) x)] \\
&\leq 2\alpha[G(T^{n-1} x, T^n x, T^n x) \\
&+ G(T^n x, T^{n-1} x, T^{n-1} x)] \\
&\leq \\
&\vdots \\
&\leq (2\alpha)^n [G(x, Tx, Tx) + G(Tx, x, x)].
\end{aligned}$$

But  $\alpha \in ]0, \frac{1}{2}[$ . Therfore

$$\lim_{n \rightarrow \infty} [G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x)] = 0, \quad \forall x \in A \cup B \cup C.$$

Now by Lemma 3.1, it follows that  $F_G^\epsilon(T) \neq \emptyset, \forall \epsilon > 0$ . ■

**Theorem 4.6.** *Let  $(X, G)$  be a  $G$ -metric space. Suppose that the cyclic mapping  $T : A \cup B \cup C \rightarrow A \cup B \cup C$  is a  $G$ -Mohseni-semi contraction. Then for every  $\epsilon > 0$ ,*

$$\delta(F_G^\epsilon(T)) \leq \epsilon \frac{2 + \alpha}{1 - \alpha}.$$

**Proof:**

Let  $\epsilon > 0$ . We will only verify that condition ii) in Lemma 3.2 is satisfied. Let  $\theta > 0$  and  $x, y \in F_G^\epsilon(T)$  and assume that

$$\begin{aligned}
G(x, y, y) + G(y, x, x) - G(Tx, Ty, Ty) - G(Ty, Tx, Tx) &\leq \theta \Rightarrow \\
G(x, y, y) + G(y, x, x) &\leq G(Tx, Ty, Ty) + G(Ty, Tx, Tx) + \theta \Rightarrow \\
G(x, y, y) + G(y, x, x) &\leq \alpha[G(x, y, y) + G(y, x, x)] \\
&+ G(x, Tx, Tx) + G(Tx, x, x)] + \theta \Rightarrow \\
(1 - \alpha)[G(x, y, y) + G(y, x, x)] &\leq 2\alpha\epsilon + \theta \\
[G(x, y, y) + G(y, x, x)] &\leq \frac{2\alpha\epsilon + \theta}{1 - \alpha}.
\end{aligned}$$

So for every  $\theta > 0$  there exists  $\phi(\theta) = \frac{2\alpha\epsilon + \theta}{1 - \alpha} > 0$  such that

$$\begin{aligned}
G(x, y, y) + G(y, x, x) - [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] &\leq \theta \Rightarrow \\
G(x, y, y) + G(y, x, x) &\leq \phi(\theta)
\end{aligned}$$

Now by Lemma 3.2, it follows that

$$\delta(F_G^\epsilon(T)) \leq \phi(2\epsilon), \quad \forall \epsilon > 0,$$

which means exactly that

$$\delta(F_G^\epsilon(T)) \leq \epsilon \frac{2 + \alpha}{1 - \alpha}, \quad \forall \epsilon > 0. \quad \blacksquare$$

**Remark 4.2.** Example 4.1 shows that  $T$  satisfies all the conditions of Theorems 4.5 and 4.6 and thus for every  $\epsilon > 0$ ,  $F_G^\epsilon(T) \neq \emptyset$  and  $\delta(F_G^\epsilon(T)) \leq \epsilon \frac{2+\alpha}{1-\alpha}$  respectively.

## 5. Conclusions

Nowadays, fixed points and approximate fixed points in the both metric and  $G$ -metric spaces play an important role in different areas of mathematics and its applications, particularly in mathematics, physics, differential equation, game theory, and dynamic programming.

In this work, we introduced the new classes of operators and contraction maps and gave results about approximate fixed points and diameter approximate fixed point on  $G$ -metric spaces. Also, by using two general lemmas regarding approximate fixed Point of cyclic maps on  $G$ -metric spaces we proved several approximate fixed point theorems and diameter approximate fixed point for a new class of operators and contraction mapping on  $G$ -metric spaces. We accompanied our theoretical results by some applied examples.

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