

APPLICATION OF THE SECOND-KIND CHEBYSHEV POLYNOMIALS FOR THE NONLINEAR AGE-STRUCTURED POPULATION MODELS

S. NEMATI¹, Y. ORDOKHANI², I. MOHAMMADI³

In this paper, we will introduce a method to find a numerical solution of nonlinear age-structured population model using second-kind Chebyshev polynomials. This method convert the nonlinear age-structured population models to an equivalent differential equation. We introduce two variable second-kind Chebyshev polynomials and their basic properties. These properties will be used to reduce the obtained differential equation to the solution of a system of nonlinear algebraic equations. Numerical examples show the accuracy and applicability of our method.

Keywords: Nonlinear age-structured population model, Second-kind Chebyshev polynomials, Operational matrix, Partial differential equations.

1. Introduction

Partial differential equations with integral condition serve as models in many branches of physics and technology. There are many papers that deal with the numerical solution of partial differential equations with integral condition (see for example [2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 19, 20, 21, 22]). The present work focuses on the numerical solution of the nonlinear age-structured population models using second-kind Chebyshev polynomials.

In this paper, we consider the following partial differential equation

$$\frac{\partial p(t, x)}{\partial t} + \frac{\partial p(t, x)}{\partial x} = -[d_1(x) + d_2(x)P(t)]p(t, x), 0 \leq t, 0 \leq x < A, \quad (1)$$

with conditions

$$p(0, x) = p_0(x), 0 \leq x < A, \quad (2)$$

$$p(t, 0) = \int_0^A [b_1(s) - b_2(s)P(t)]p(t, s)ds, \quad 0 \leq t, \quad (3)$$

$$P(t) = \int_0^A p(t, s)ds, 0 \leq t, \quad (4)$$

¹ Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran, e-mail: s.nemati@umz.ac.ir

² Department of Mathematics, Alzahra University, Tehran, Iran

³ Faculty of Basic Sciences, Islamic Azad University, Garmsar Branch, Garmsar, Iran

where t and x denote time and age, respectively, $P(t)$ denotes the total population number at time t , $p(t, x)$ is the age-specific density of individuals of age x at time t , which means that $\int_a^{a+\Delta a} p(t, s) ds$ gives the number of individuals that have age between a and $a + \Delta a$ at time t , $d_1(x)$ is the natural death rate (without considering competition), $d_2(x)P(t)$ is the increase of death rate considering competition, $b_1(x)$ is the natural fertility rate (without considering competition), $b_2(x)P(t)$ is the decrease of fertility rate considering competition and A is the maximum age that an individual of the population may reach.

Several numerical methods were proposed for solving the nonlinear age-structured population models. The authors of [8] presented a reproducing kernel method. In [1] forward difference schemes were proposed based on the Runge–Kutta method. A spline algorithm was introduced for solving age-structured population model in [14]. Kim and Park [15] developed an upwind scheme for this problem. The authors of [17] presented a discontinuous Galerkin method. Xiuying Li [23] used the variational iteration method for the nonlinear age-structured population models and Yousefi et al. [24] used Bernstein polynomials to find the approximate solution of the nonlinear age-structured population models.

Orthogonal functions have been used to solve various problems. The main characteristic of this technique is that it reduces problem to the solution of a system of algebraic equations. In the present paper, the numerical solution of problem (1)–(4) is computed by using two variable shifted second kind Chebyshev orthogonal polynomials.

The paper is organized as follows: In Section 2, basic properties of two variable second kind Chebyshev polynomials are presented and operational matrices of these polynomials are introduced. In Section 3, we give an approximate solution for problem (1)–(4). Numerical examples are given in Section 4 to illustrate the accuracy of our method. Finally, concluding remarks are given in Section 5.

2. Properties of two variable second kind Chebyshev polynomials

2.1. Definition and function approximation

Two variable second kind Chebyshev polynomials are defined on $[0, T] \times [0, A]$ as

$$\psi_{ij}(t, x) = U_i\left(\frac{2}{T}t - 1\right)U_j\left(\frac{2}{A}x - 1\right),$$

here U_i and U_j are the well-known second kind Chebyshev polynomials respectively of order i and j , which are defined on the interval $[-1,1]$ and can be determined with the aid of a recursive formula [18].

Shifted second kind Chebyshev polynomials on the interval $[0,b]$ are defined by:

$$\phi_i(t) = U_i\left(\frac{2}{b}t - 1\right), \quad i = 0, 1, 2, \dots,$$

and some of the main properties of these polynomials are as follows:

$$\int_0^b \phi_n(t) dt = \begin{cases} \frac{b}{n+1}, & n = 2k, \\ 0, & n = 2k+1, \end{cases} \quad (5)$$

$$\int_0^t \phi_n(t') dt' = \frac{b}{2(n+1)} [(-1)^n \phi_0(t) - \frac{1}{2} \phi_{n-1}(t) + \frac{1}{2} \phi_{n+1}(t)], \quad \phi_{-1}(t) = 0, \quad (6)$$

$$\phi'_n(t) = \frac{4}{b} \sum_{i=0}^{\lfloor n/2 \rfloor} (n-2i) \phi_{n-2i-1}(t), \quad (7)$$

$$\phi_n(0) = (-1)^n (n+1), \quad (8)$$

$$\phi_m(t) \phi_n(t) = \sum_{k=0}^{\frac{m+n-|m-n|}{2}} \phi_{m+n-2k}(t). \quad (9)$$

A function $h(t, x)$ defined over $[0, T] \times [0, A]$ can be approximated using the two variable second kind Chebyshev functions as

$$h(t, x) \approx h_{m,n}(t, x) = \sum_{i=0}^m \sum_{j=0}^n c_{ij} \psi_{ij}(t, x) = C^T \psi(t, x), \quad (10)$$

where

$$c_{ij} = \frac{16}{\pi^2 T A} \int_0^T \int_0^A \omega(t, x) h(t, x) \psi_{ij}(t, x) dx dt, \quad (11)$$

$$C = [c_{00}, c_{01}, \dots, c_{0n}, c_{10}, c_{11}, \dots, c_{1n}, \dots, c_{m0}, c_{m1}, \dots, c_{mn}]^T, \quad (12)$$

$$\psi(t, x) = [\psi_{00}(t, x), \psi_{01}(t, x), \dots, \psi_{0n}(t, x), \dots, \psi_{m0}(t, x), \psi_{m1}(t, x), \dots, \psi_{mn}(t, x)]^T, \quad (13)$$

$$\text{and we have } \omega(t, x) = \sqrt{1 - \left(\frac{2}{T}t - 1\right)^2} \sqrt{1 - \left(\frac{2}{A}x - 1\right)^2}.$$

In order to calculate the integral part of (11) we transform the intervals $[0, T]$ and $[0, A]$ into the interval $[-1, 1]$ by means of the transformations

$$t' = \frac{2}{T}t - 1, \quad x' = \frac{2}{A}x - 1,$$

and then use the second kind Gauss-Chebyshev quadrature formula [18].

2.2. Operational matrices

In this section, we give some operational matrices of the two variable second kind Chebyshev functions that will be used to solve the problem (1)–(4) numerically.

The derivation of the vector $\psi(t, x)$ defined by (13) with respect to t can be obtained using equation (7) as:

$$\frac{\partial \psi(t, x)}{\partial t} = M\psi(t, x), \quad (14)$$

where M is the operational matrix of derivation with respect to t and is an $(m+1)(n+1) \times (m+1)(n+1)$ matrix as

$$M = \frac{4}{T} \begin{bmatrix} O & O & O & O & \cdots & O & O \\ I & O & O & O & \cdots & O & O \\ O & 2I & O & O & \cdots & O & O \\ I & O & 3I & O & \cdots & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ M_1 & M_2 & M_3 & M_4 & \cdots & mI & O \end{bmatrix}$$

where M_1, M_2, M_3 and M_4 are $I, O, 3I$ and O , for odd m and $O, 2I, O$ and $4I$, for even m , respectively and I and O are the identity and zero matrix of order $n+1$, respectively.

Also, the operational matrix of integration with respect to t can be approximately obtained using equation (6) as

$$\int_0^t \psi(t', x) dt' \approx P\psi(t, x), \quad (15)$$

where

$$P = \frac{T}{2} \begin{bmatrix} I & \frac{1}{2}I & O & O & O & \cdots & O & O \\ \frac{-3}{4}I & O & \frac{1}{4}I & O & O & \cdots & O & O \\ \frac{1}{3}I & \frac{-1}{6}I & O & \frac{1}{6}I & O & \cdots & O & O \\ \frac{-1}{4}I & O & \frac{-1}{8}I & O & \frac{1}{8}I & \cdots & O & O \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{(-1)^m}{m+1}I & O & O & O & O & \cdots & \frac{-1}{2(m+1)}I & O \end{bmatrix}$$

Analogously, we write

$$\int_0^x \psi(t, x') dx' \approx Q\psi(t, x), \quad (16)$$

where Q is the operational matrix of integration with respect to x as

$$Q = \frac{A}{2} \begin{bmatrix} Q_1 & O & O & \cdots & O \\ O & Q_1 & O & \cdots & O \\ O & O & Q_1 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & Q_1 \end{bmatrix}_{m \times m},$$

such that

$$Q_1 = \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{-3}{4} & 0 & \frac{1}{4} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{3} & \frac{-1}{6} & 0 & \frac{1}{6} & 0 & \cdots & 0 & 0 \\ \frac{-1}{4} & 0 & \frac{-1}{8} & 0 & \frac{1}{8} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{(-1)^n}{n+1} & 0 & 0 & 0 & 0 & \cdots & \frac{-1}{2(n+1)} & 0 \end{bmatrix}$$

Moreover, using equation (5) we obtain

$$\int_0^A \psi(t, x) dx = W \psi(t, x), \quad (17)$$

where W is an $(m+1)(n+1) \times (m+1)(n+1)$ matrix and is given by

$$W = \begin{bmatrix} W_1 & O & O & \cdots & O \\ O & W_1 & O & \cdots & O \\ O & O & W_1 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & W_1 \end{bmatrix},$$

and

$$W_1 = \begin{bmatrix} A & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \frac{A}{3} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha & 0 & 0 & \cdots & 0 \\ \beta & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where α and β are respectively $\frac{A}{n}$ and 0, for odd n and 0 and $\frac{A}{n+1}$, for even n .

The following property of the product of two vectors $\psi(t, x)$ and $\psi^T(t, x)$ will also be used. Let

$$\psi(t, x)\psi^T(t, x)C \approx \tilde{C}\psi(t, x), \quad (18)$$

where C is defined by (13) and \tilde{C} is the product operational matrix of dimension $(m+1)(n+1) \times (m+1)(n+1)$ and is obtained using equation (9) as:

$$\tilde{C} = [C^{(i,j)}], \quad i, j = 0, 1, 2, \dots, m,$$

where

$$C^{(i,j)} = \sum_{k=\max\{0, \lceil \frac{i+j-m}{2} \rceil\}}^{\frac{i+j-|i-j|}{2}} C_{i+j-2k},$$

in which, $\lceil x \rceil$ is the smallest integer not less than x and

$$[C_s]_{hl} = \sum_{k=\max\{0, \lceil \frac{h+l-n}{2} \rceil\}}^{\frac{h+l-|h-l|}{2}} c_{s(h+l-2k)}, \quad h, l = 0, 1, \dots, n.$$

For example with $m = 1$ and $n = 2$, we have

$$\tilde{C} = \begin{bmatrix} c_{00} & c_{01} & c_{02} & c_{10} & c_{11} & c_{12} \\ c_{01} & c_{00} + c_{02} & c_{01} & c_{11} & c_{10} + c_{12} & c_{11} \\ c_{02} & c_{01} & c_{00} + c_{02} & c_{12} & c_{11} & c_{10} + c_{12} \\ c_{10} & c_{11} & c_{12} & c_{00} & c_{01} & c_{02} \\ c_{11} & c_{10} + c_{12} & c_{11} & c_{01} & c_{00} + c_{02} & c_{01} \\ c_{12} & c_{11} & c_{10} + c_{12} & c_{02} & c_{01} & c_{00} + c_{02} \end{bmatrix}.$$

3. Numerical solution of the nonlinear age-structured population model

In this section, we introduce a numerical method for the solution of the nonlinear age-structured population model using the two variable second kind Chebyshev functions.

Integrating both sides of (1) with respect to x yields

$$\int_0^x \frac{\partial p(t, s)}{\partial t} ds + p(t, x) - p(t, 0) = - \int_0^x d_1(s) p(t, s) ds - \int_0^x d_2(s) P(t) p(t, s) ds, \quad (19)$$

therefore, we have

$$p(t, x) = p(t, 0) - \int_0^x \left(\frac{\partial p(t, s)}{\partial t} + d_1(s) p(t, s) + d_2(s) P(t) p(t, s) \right) ds. \quad (20)$$

Differentiating both sides of (20) with respect to t and integrating the result in t yields:

$$p(t, x) - p(0, x) =$$

$$\int_0^t \frac{\partial}{\partial t'} \left[p(t', 0) - \int_0^x \left(\frac{\partial p(t', s)}{\partial t'} + d_1(s) p(t', s) + d_2(s) P(t') p(t', s) \right) ds \right] dt'. \quad (21)$$

Let us define

$$\begin{aligned} p_0(t, x) &= 1 \times p(0, x), \\ b_1(t, x) &= 1 \times b_1(x), \\ b_2(t, x) &= 1 \times b_2(x), \\ d_1(t', x) &= 1 \times d_1(x), \\ d_2(t', x) &= 1 \times d_2(x). \end{aligned} \quad (22)$$

We approximate the functions in (22) and $p(t, x)$ using the method mentioned in Section 2 as

$$p_0(t, x) \approx C_0^T \psi(t, x), \quad (23)$$

$$p(t, x) \approx C^T \psi(t, x), \quad (24)$$

$$b_1(t, x) \approx B_1^T \psi(t, x), \quad (25)$$

$$b_2(t, x) \approx B_2^T \psi(t, x), \quad (26)$$

$$d_1(t, x) \approx D_1^T \psi(t', x), \quad (27)$$

$$d_2(t, x) \approx D_2^T \psi(t, x), \quad (28)$$

where C , C_0 , B_1 , B_2 , D_1 and D_2 are $(m+1)(n+1) \times 1$ vectors so that C is the unknown vector and the other vectors are known. Using (23) - (28) and applying (2) - (4) and (14) - (18) we obtain

$$\begin{aligned} P(t) &= \int_0^A p(t, s) ds \approx C^T \int_0^A \psi(t, s) ds = C^T W \psi(t, x), \\ \int_0^t \frac{\partial}{\partial t'} (p(t', 0)) dt' &= \int_0^t \frac{\partial}{\partial t'} \left(\int_0^A \left[b_1(t', s) - b_2(t', s) \left(\int_0^A p(t', s) ds \right) \right] p(t', s) ds \right) dt' \\ &= \int_0^t \frac{\partial}{\partial t'} \left[\int_0^A \left(B_1^T \psi(t', s) - B_2^T \psi(t', s) \psi^T(t', s) \underbrace{W^T C}_{=\tilde{\Delta}} \right) \psi^T(t', s) C \right] dt' \\ &= \int_0^t \frac{\partial}{\partial t'} \left[\int_0^A \left(B_1^T \psi(t', s) - B_2^T \tilde{\Delta} \psi(t', s) \right) \psi^T(t', s) C \right] dt' \\ &= \int_0^t \frac{\partial}{\partial t'} \left[\int_0^A \left(B_1^T \tilde{C} - B_2^T \tilde{\Delta} \tilde{C} \right) \psi(t', s) \right] dt' \\ &= (B_1^T \tilde{C} - B_2^T \tilde{\Delta} \tilde{C}) W \int_0^t \frac{\partial}{\partial t'} (\psi(t', x)) dt' \end{aligned}$$

$$\begin{aligned}
&= (B_1^T \tilde{C} - B_2^T \tilde{\Delta} \tilde{C}) WM \left(\int_0^t \psi(t', x) dt' \right) \\
&= (B_1^T \tilde{C} WMP - B_2^T \tilde{\Delta} \tilde{C} WMP) \psi(t, x), \tag{29} \\
\int_0^t \frac{\partial}{\partial t'} \left[\int_0^x \left(\frac{\partial p(t', s)}{\partial t'} \right) ds \right] dt' &= \int_0^t \frac{\partial}{\partial t'} \left[\int_0^x (C^T M \psi(t', s)) ds \right] dt' \\
&= \int_0^t \frac{\partial}{\partial t'} [C^T MQ \psi(t', x)] dt' \\
&= \int_0^t C^T MQM \psi(t', x) dt' \\
&= C^T MQMP \psi(t, x), \tag{30} \\
\int_0^t \frac{\partial}{\partial t'} \left[\int_0^x (d_1(t', s) p(t', s) + d_2(t', s) P(t') p(t', s)) ds \right] dt' \\
&= (D_1^T \tilde{C} QMP + D_2^T \tilde{\Delta} \tilde{C} QMP) \psi(t, x). \tag{31}
\end{aligned}$$

Substituting (23), (24) and (29) - (31) into (21) we obtain

$C^T - C_0^T - B_1^T \tilde{C} WMP + B_2^T \tilde{\Delta} \tilde{C} WMP + C^T MQMP + D_1^T \tilde{C} QMP + D_2^T \tilde{\Delta} \tilde{C} QMP = 0$,
 which form a system of $(m+1)(n+1)$ nonlinear equations and can be solved for the elements of C using the well-known Newton's iterative method.

4. Numerical examples

In this section, some examples are given to demonstrate the applicability and accuracy of our method.

Example 1. As the first example, we consider the nonlinear age-structured population model such that [24]

$$\frac{\partial p(t, x)}{\partial t} + \frac{\partial p(t, x)}{\partial x} = -\left(\frac{1}{1-x} + P(t)\right)p(t, x), \quad 0 \leq t, \quad 0 \leq x < A,$$

$$p(0, x) = 5(1-x)e^{-\lambda x}, \quad 0 \leq x < A,$$

$$p(t, 0) = 5P(t), \quad 0 \leq t,$$

$$P(t) = \int_0^A p(t, s) ds, \quad 0 \leq t,$$

where $\lambda = \frac{5+\sqrt{5}}{2}$ and $A = \infty$ and has the exact solution

$$p(t, x) = 5(1-x)e^{-\lambda x} \frac{\lambda}{1 + (\lambda - 1)e^{-\lambda x}}.$$

We solved the problem by the presented method on intervals $[0, 1] \times [0, 50]$ and $[0, 1] \times [0, 100]$ and the numerical results for the absolute error are reported for some points in Tables 1 and 2. Note that our results are more

accurate than the numerical results obtained by the method described in [24], for the same example.

Table 1

Numerical results for Example 1 on $[0,1] \times [0,50]$.

(x,t)	$m=3 \quad n=3$	$m=3 \quad n=6$	$m=5 \quad n=10$
(0,0)	5.0006×10^0	5.0020×10^0	5.0031×10^0
(0.1,5)	2.6457×10^{-4}	4.1539×10^{-5}	8.1220×10^{-5}
(0.2,10)	5.5558×10^{-5}	9.5689×10^{-5}	2.7703×10^{-5}
(0.3,15)	4.5484×10^{-5}	4.7293×10^{-5}	4.1162×10^{-6}
(0.4,20)	6.7508×10^{-5}	4.3631×10^{-5}	9.2406×10^{-6}
(0.5,25)	3.9826×10^{-5}	3.2274×10^{-5}	1.1944×10^{-5}
(0.6,30)	8.2494×10^{-6}	3.7353×10^{-5}	6.0614×10^{-6}
(0.7,35)	4.7405×10^{-5}	2.8539×10^{-5}	1.6998×10^{-6}
(0.8,40)	4.8328×10^{-5}	3.2378×10^{-5}	6.4385×10^{-6}
(0.9,45)	1.8294×10^{-5}	6.0936×10^{-5}	7.7630×10^{-6}
(1,50)	1.8177×10^{-4}	2.6386×10^{-4}	6.4436×10^{-5}

Table 2

Numerical results for Example 1 on $[0,1] \times [0,100]$.

(x,t)	$m=3 \quad n=3$	$m=3 \quad n=6$	$m=5 \quad n=10$
(0,0)	5.0012×10^0	5.0042×10^0	5.0066×10^0
(0.1,10)	5.4191×10^{-4}	7.6811×10^{-5}	1.8137×10^{-4}
(0.2,20)	1.1245×10^{-4}	2.0059×10^{-4}	5.9163×10^{-5}
(0.3,30)	9.4468×10^{-5}	9.9891×10^{-5}	8.7517×10^{-6}
(0.4,40)	1.3912×10^{-4}	9.1301×10^{-5}	1.9614×10^{-5}
(0.5,50)	8.1788×10^{-5}	6.7952×10^{-5}	2.5331×10^{-5}
(0.6,60)	1.7277×10^{-5}	7.8237×10^{-5}	1.2847×10^{-5}
(0.7,70)	9.7802×10^{-5}	6.0001×10^{-5}	3.6017×10^{-6}
(0.8,80)	9.9517×10^{-5}	6.7812×10^{-5}	1.3638×10^{-5}
(0.9,90)	3.7848×10^{-5}	1.2782×10^{-4}	1.6440×10^{-5}
(1,100)	3.7456×10^{-4}	5.5330×10^{-4}	1.3644×10^{-4}

Example 2. Consider the nonlinear age-structured population model [8, 16, 23]

$$\frac{\partial p(t,x)}{\partial t} + \frac{\partial p(t,x)}{\partial x} = -P(t)p(t,x), \quad 0 \leq t, \quad 0 \leq x < A,$$

$$p(0,x) = \frac{e^{-x}}{2}, \quad 0 \leq x < A, \quad p(t,0) = P(t), \quad 0 \leq t,$$

$$P(t) = \int_0^A p(t,s) ds, \quad 0 \leq t,$$

where $A = \infty$ and the exact solution is $p(t,x) = \frac{e^{-x}}{e^{-t} + 1}$.

We approximate $p(t,x)$ on intervals $[0,1] \times [0,20]$ and $[0,1] \times [0,100]$ and present the absolute error for some points in Tables 3 and 4.

Table 3

Numerical results for Example 2 on $[0,1] \times [0,20]$.

(t,x)	$m=3 \quad n=3$	$m=3 \quad n=6$	$m=5 \quad n=10$
(0,0)	2.8550×10^{-1}	7.0906×10^{-2}	3.3142×10^{-3}
(0.1,2)	3.1541×10^{-2}	3.5338×10^{-3}	3.7600×10^{-3}
(0.2,4)	2.2485×10^{-2}	7.6334×10^{-3}	6.8560×10^{-4}
(0.3,6)	5.5794×10^{-3}	1.7086×10^{-3}	3.8607×10^{-4}
(0.4,8)	1.6298×10^{-2}	3.7663×10^{-3}	1.2349×10^{-4}
(0.5,10)	1.1851×10^{-2}	1.7733×10^{-3}	6.1912×10^{-5}
(0.6,12)	1.1153×10^{-4}	3.1816×10^{-3}	5.9603×10^{-5}
(0.7,14)	1.1164×10^{-2}	1.8150×10^{-3}	1.3415×10^{-4}
(0.8,16)	1.2771×10^{-2}	2.8152×10^{-3}	1.6733×10^{-4}
(0.9,18)	3.6156×10^{-3}	4.7402×10^{-3}	2.0422×10^{-4}
(1,20)	4.6550×10^{-2}	2.1177×10^{-2}	1.4430×10^{-3}

Table 4

Numerical results for Example 2 on $[0,1] \times [0,100]$

(t,x)	$m=3 \quad n=3$	$m=3 \quad n=6$	$m=5 \quad n=10$
(0,0)	4.7528×10^{-1}	4.1707×10^{-1}	3.7105×10^{-1}
(0.1,10)	1.0664×10^{-2}	1.5810×10^{-3}	3.4392×10^{-3}
(0.2,20)	2.2299×10^{-3}	3.9070×10^{-3}	1.1415×10^{-3}
(0.3,30)	1.8486×10^{-3}	1.9381×10^{-3}	1.6924×10^{-4}
(0.4,40)	2.7334×10^{-3}	1.7799×10^{-3}	3.7962×10^{-4}
(0.5,50)	1.6098×10^{-3}	1.3205×10^{-3}	4.9050×10^{-4}
(0.6,60)	3.3667×10^{-4}	1.5245×10^{-3}	2.4884×10^{-4}
(0.7,70)	1.9204×10^{-3}	1.1669×10^{-3}	6.9773×10^{-5}
(0.8,80)	1.9560×10^{-3}	1.3214×10^{-3}	2.6424×10^{-4}
(0.9,90)	7.4213×10^{-4}	2.4888×10^{-3}	3.1856×10^{-4}
(1,100)	7.3595×10^{-3}	1.0775×10^{-2}	2.6440×10^{-3}

5. Conclusion

In this article we presented a numerical method to solve the nonlinear age-structured population model numerically based on the second-kind Chebyshev polynomials. The basic properties of these polynomials were employed to reduce the problem to a set of nonlinear algebraic equations. The operational matrices of the two variable second kind Chebyshev polynomials have many zeroes, hence, make these polynomials computationally very attractive. Chebyshev coefficients of the solution are found very easily by using the computer programs without any computational effort and this process is very fast. Numerical examples show that the new described method converges to the exact solution and has good results. Example 1 shows that this method can be used when the mortality function is non-integrable and discontinuous.

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