

## ARENS REGULARITY AND MODULE ARENS REGULARITY OF MODULE ACTIONS

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*In this paper, we extend the notion of Arens regularity and module Arens regularity of Banach algebras to Arens regularity of module actions. We also investigate the more general notion of Arens regularity for bilinear maps. Finally we find necessary and sufficient conditions for module Arens regularity of semigroup algebra of an inverse semigroup.*

**Keywords:** Arens regularity; Banach module; Module Arens regularity; Module topological center.

**MSC2010:** 46H25.

### 1. Introduction

In [1], for Banach algebras  $\mathcal{A}$  and  $\mathcal{O}$ , authors extended the concept of Arens regularity of  $\mathcal{A}$  to the case that there is an  $\mathcal{O}$ -module structure on  $\mathcal{A}$  and called it the module Arens regularity of  $\mathcal{A}$  as an  $\mathcal{O}$ -module. In this paper, we study this problem for left and right module actions  $\pi_\ell : \mathcal{A} \times X \rightarrow X$  and  $\pi_r : X \times \mathcal{A} \rightarrow X$  where  $X$  is a Banach  $\mathcal{A}$ -bimodule and extend the notion of Arens regularity of Banach algebras to that of module actions. More generally, we define the concept of module Arens regularity for general bilinear maps and find necessary and sufficient conditions for a module map to be module Arens regular. Finally, we study module Arens regularity for inverse semigroup algebras.

Let  $X, Y, Z$  be normed spaces and  $m : X \times Y \rightarrow Z$  be a bounded bilinear mapping. Arens defines two natural extensions  $m^{***}$  and  $m^{t***}$  of  $m$  from  $X^{**} \times Y^{**}$  into  $Z^{**}$  as follows:

1.  $m^* : Z^* \times X \rightarrow Y^*$ , given by  $\langle m^*(z', x), y \rangle = \langle z', m(x, y) \rangle$  where  $x \in X$ ,  $y \in Y$ ,  $z' \in Z^*$ ,
2.  $m^{**} : Y^{**} \times Z^* \rightarrow X^*$ , given by  $\langle m^{**}(y'', z'), x \rangle = \langle y'', m^*(z', x) \rangle$  where  $x \in X$ ,  $y'' \in Y^{**}$ ,  $z' \in Z^*$ ,
3.  $m^{***} : X^{**} \times Y^{**} \rightarrow Z^{**}$ , given by  $\langle m^{***}(x'', y''), z' \rangle = \langle x'', m^{**}(y'', z') \rangle$  where  $x'' \in X^{**}$ ,  $y'' \in Y^{**}$ ,  $z' \in Z^*$ .

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The mapping  $m^{***}$  is the unique extension of  $m$  so that for each  $y'' \in Y^{**}$ ,  $x'' \rightarrow m^{***}(x'', y'')$  from  $X^{**}$  into  $Z^{**}$  is weak\*-weak\* continuous, whereas the mapping  $y'' \rightarrow m^{***}(x'', y'')$  is not in general weak\*-weak\* continuous from  $Y^{**}$  into  $Z^{**}$  unless  $x'' \in X$ . Hence the first topological center of  $m$  may be defined as follows:

$$Z_1(m) = \{x'' \in X^{**} : y'' \rightarrow m^{***}(x'', y'') \text{ is weak*-weak* continuous}\}.$$

Let  $m^t : Y \times X \rightarrow Z$  be the transpose of  $m$  defined by  $m^t(y, x) = m(x, y)$  for every  $x \in X$  and  $y \in Y$ . Then  $m^t$  is a continuous bilinear map from  $Y \times X$  to  $Z$ , and so it may be extended as above to  $m^{t***} : Y^{**} \times X^{**} \rightarrow Z^{**}$ . The mapping  $m^{t***} : X^{**} \times Y^{**} \rightarrow Z^{**}$  in general is not equal to  $m^{***}$  (see [2] and [3]), if  $m^{***} = m^{t***}$ , then  $m$  is called Arens regular. The mapping  $y'' \rightarrow m^{t***}(x'', y'')$  is weak\*-weak\* continuous for every  $x'' \in X^{**}$ , but the mapping  $x'' \rightarrow m^{t***}(x'', y'')$  from  $X^{**}$  into  $Z^{**}$  is not in general weak\*-weak\* continuous for every  $y'' \in Y^{**}$ . So we define the second topological center of  $m$  as

$$Z_2(m) = \{y'' \in Y^{**} : x'' \rightarrow m^{t***}(x'', y'') \text{ is weak*-weak* continuous}\}.$$

It is clear that  $m$  is Arens regular if and only if  $Z_1(m) = X^{**}$  or  $Z_2(m) = Y^{**}$ . Arens regularity of  $m$  is equivalent to the following

$$\lim_i \lim_j \langle z', m(x_i, y_j) \rangle = \lim_j \lim_i \langle z', m(x_i, y_j) \rangle,$$

whenever both limits exist for all bounded sequences  $(x_i)_i \subset X$ ,  $(y_j)_j \subset Y$  and  $z' \in Z^*$  (for more details see [4, 5]). The mapping  $m$  is left strongly Arens irregular if  $Z_1(m) = X$  and  $m$  is right strongly Arens irregular if  $Z_2(m) = Y$ .

The regularity of a Banach algebra  $\mathcal{A}$  is defined as the regularity of its algebra multiplication when considered as a bilinear mapping  $\pi : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  so that  $\pi(a, b) = ab$  ( $a, b \in \mathcal{A}$ ). For  $a''$  and  $b''$  in  $\mathcal{A}^{**}$ , we denote  $\pi^{***}(a'', b'')$  and  $\pi^{t***}(a'', b'')$  by symbols  $a'' \square b''$  and  $a'' \diamond b''$ , respectively. These are called the *first and second Arens products* on  $\mathcal{A}^{**}$ . When these two products coincide on  $\mathcal{A}^{**}$ , we say that  $\mathcal{A}$  is *Arens regular* (see [5, 6]). Let  $a''$  and  $b''$  be elements of  $\mathcal{A}^{**}$ , the second dual of  $\mathcal{A}$ . By Goldstine's Theorem [7, P.424-425], there are nets  $(a_j)_j$  and  $(b_k)_k$  in  $\mathcal{A}$  such that  $a'' = \text{weak}^* - \lim_j a_j$  and  $b'' = \text{weak}^* - \lim_k b_k$ . Hence it is easy to see that for all  $a' \in \mathcal{A}^*$ , we have

$$\lim_j \lim_k \langle a', m(a_j, b_k) \rangle = \langle a'' \square b'', a' \rangle$$

and

$$\lim_k \lim_j \langle a', m(a_j, b_k) \rangle = \langle a'' \diamond b'', a' \rangle.$$

## 2. The topological centers of module actions

Let  $X$  be a Banach  $\mathcal{A}$ -bimodule, and let

$$\pi_\ell : \mathcal{A} \times X \rightarrow X \text{ and } \pi_r : X \times \mathcal{A} \rightarrow X.$$

be the left and right module actions of  $\mathcal{A}$  on  $X$ . Then  $X^{**}$  is a Banach  $\mathcal{A}^{**}$ -bimodule with module actions

$$\pi_\ell^{***} : \mathcal{A}^{**} \times X^{**} \rightarrow X^{**} \text{ and } \pi_r^{***} : X^{**} \times \mathcal{A}^{**} \rightarrow X^{**}.$$

Similarly,  $X^{**}$  is a Banach  $\mathcal{A}^{**}$ -bimodule with module actions

$$\pi_\ell^{t***} : \mathcal{A}^{**} \times X^{**} \rightarrow X^{**} \text{ and } \pi_r^{t***} : X^{**} \times \mathcal{A}^{**} \rightarrow X^{**}.$$

We may therefore define the topological centers of the right and left module actions of  $\mathcal{A}$  on  $X$  as follows:

$$\begin{aligned} Z_{\mathcal{A}^{**}}(X^{**}) &= Z(\pi_r) = \{x'' \in X^{**} : \text{the map } a'' \rightarrow \pi_r^{***}(x'', a'') : \mathcal{A}^{**} \rightarrow X^{**} \\ &\hspace{15em} \text{is weak}^*\text{-weak}^* \text{ continuous}\} \\ Z_{X^{**}}(\mathcal{A}^{**}) &= Z(\pi_\ell) = \{a'' \in \mathcal{A}^{**} : \text{the map } x'' \rightarrow \pi_\ell^{***}(a'', x'') : X^{**} \rightarrow X^{**} \\ &\hspace{15em} \text{is weak}^*\text{-weak}^* \text{ continuous}\} \\ Z_{\mathcal{A}^{**}}^t(X^{**}) &= Z(\pi_\ell^t) = \{x'' \in X^{**} : \text{the map } a'' \rightarrow \pi_\ell^{t***}(x'', a'') : \mathcal{A}^{**} \rightarrow X^{**} \\ &\hspace{15em} \text{is weak}^*\text{-weak}^* \text{ continuous}\} \\ Z_{X^{**}}^t(\mathcal{A}^{**}) &= Z(\pi_r^t) = \{a'' \in \mathcal{A}^{**} : \text{the map } x'' \rightarrow \pi_r^{t***}(a'', x'') : X^{**} \rightarrow X^{**} \\ &\hspace{15em} \text{is weak}^*\text{-weak}^* \text{ continuous}\} \end{aligned}$$

We note also that if  $X$  is a left (resp. right) Banach  $\mathcal{A}$ -module and  $\pi_\ell : \mathcal{A} \times X \rightarrow X$  (resp.  $\pi_r : X \times \mathcal{A} \rightarrow X$ ) is left (resp. right) module action of  $\mathcal{A}$  on  $X$ , then  $X^*$  is a right (resp. left) Banach  $\mathcal{A}$ -module. We write  $ax = \pi_\ell(a, x)$ ,  $xa = \pi_r(x, a)$ ,  $\pi_\ell(a_1 a_2, x) = \pi_\ell(a_1, a_2 x)$ ,  $\pi_r(x, a_1 a_2) = \pi_r(x a_1, a_2)$ ,  $\pi_\ell^*(a_1 x', a_2) = \pi_\ell^*(x', a_2 a_1)$ ,  $\pi_r^*(x' a, x) = \pi_r^*(x', a x)$ , for all  $a_1, a_2, a \in \mathcal{A}$ ,  $x \in X$  and  $x' \in X^*$  when there is no confusion.

A functional  $a'$  in  $\mathcal{A}^*$  is said to be *wap* (weakly almost periodic) on  $\mathcal{A}$  if the mapping  $a \rightarrow a'a$  from  $\mathcal{A}$  into  $\mathcal{A}^*$  is weakly compact. In [5], Pym showed that this definition is equivalent to the following condition:

For any two nets  $(a_j)_j$  and  $(b_k)_k$  in  $\{a \in \mathcal{A} : \|a\| \leq 1\}$ , we have

$$\lim_j \lim_k \langle a', a_j b_k \rangle = \lim_k \lim_j \langle a', a_j b_k \rangle,$$

whenever both iterated limits exist. The collection of all *wap* functionals on  $\mathcal{A}$  is denoted by  $wap(\mathcal{A})$ . Also we have  $a' \in wap(\mathcal{A})$  if and only if  $\langle a'' \square b'', a' \rangle = \langle a'' \diamond b'', a' \rangle$  for every  $a'', b'' \in \mathcal{A}^{**}$ .

Let  $X$  be a Banach left  $\mathcal{A}$ -module. Then,  $x' \in X^*$  is said to be left weakly almost periodic functional if the set  $\{\pi_\ell^*(x', a) : a \in \mathcal{A}, \|a\| \leq 1\}$  is relatively weakly compact. We denote by  $wap_\ell(X)$  the closed subspace of  $X^*$  consisting of all the left weakly almost periodic functionals in  $X^*$ . The definition of the right weakly almost periodic functional ( $= wap_r(X)$ ) is similarly. By [5],  $x' \in wap_\ell(X)$  is equivalent to the following

$$\langle \pi_\ell^{***}(a'', x''), x' \rangle = \langle \pi_\ell^{t***}(a'', x''), x' \rangle$$

for all  $a'' \in \mathcal{A}^{**}$  and  $x'' \in X^{**}$ . Thus, we can write

$$wap_\ell(X) = \{x' \in X^* : \langle \pi_\ell^{***}(a'', x''), x' \rangle = \langle \pi_\ell^{t***}(a'', x''), x' \rangle \text{ for all } a'' \in \mathcal{A}^{**}, x'' \in X^{**}\}.$$

**Theorem 2.1.** *Suppose that  $X$  is a left Banach  $\mathcal{A}$ -module. Then the following assertions are equivalent.*

- (i) *The mapping  $a \rightarrow \pi_\ell^*(x', a)$  from  $\mathcal{A}$  into  $X^*$  is weakly compact;*
- (ii)  $Z_{X^{**}}(\mathcal{A}^{**}) = \mathcal{A}^{**}$ ;
- (iii) *There is a subset  $E$  of  $X^*$  with  $\overline{\text{lin}E} = X^*$  such that for each sequence  $(a_n)_n \subset \mathcal{A}$  and  $(x_m)_m \subset X$  and each  $x' \in X^*$ , we have*

$$\lim_m \lim_n \langle x', a_n x_m \rangle = \lim_n \lim_m \langle x', a_n x_m \rangle,$$

*whenever both the iterated limits exist;*

(iv) Suppose that  $a'' \in \mathcal{A}^{**}$  and  $(a_j)_j \subset \mathcal{A}$  such that  $a_j \xrightarrow{w^*} a''$ . Then we have

$$\pi_\ell^*(x', a_j) \xrightarrow{w} \pi_\ell^{t^{***t}}(x', a''),$$

for each  $x' \in X^*$ .

*Proof.* (i) $\Rightarrow$ (ii): Suppose that  $x' \in X^*$ . Take  $T(a) = \pi_\ell^*(x', a)$  where  $a \in \mathcal{A}$ . By easy calculation, we have  $T^{**}(a'') = \pi_\ell^{t^{***t}}(x', a'')$  for each  $a'' \in \mathcal{A}^{**}$ . Now let  $T$  be a weakly compact mapping. Then by using Theorem VI 4.2 and VI 4.8 of [7], we have  $\pi_\ell^{t^{***t}}(x', a'') \in X^*$  for each  $a'' \in \mathcal{A}^{**}$ . Suppose that  $(x'_j)_j \subset X^{**}$  such that  $x'_j \xrightarrow{w^*} x''$  on  $X^{**}$ . Then for every  $a'' \in \mathcal{A}^{**}$ , we have

$$\begin{aligned} \langle \pi_\ell^{t^{***t}}(x', a''), x'_j \rangle &= \langle a'', \pi_\ell^{**}(x'_j, x') \rangle = \langle \pi_\ell^{t^{***t}}(x'_j, x'), a'' \rangle = \langle x'_j, \pi_\ell^{t^{***t}}(x', a'') \rangle \\ &\rightarrow \langle x'', \pi_\ell^{t^{***t}}(x', a'') \rangle = \langle \pi_\ell^{**}(a'', x''), x'_j \rangle. \end{aligned}$$

It follows that  $\pi_\ell^{t^{***t}}(x', a'') \xrightarrow{w^*} \pi_\ell^{**}(a'', x'')$ , and so  $a'' \in Z_{X^{**}}(\mathcal{A}^{**})$ .

(ii) $\Rightarrow$ (i): Let  $Z_{X^{**}}(\mathcal{A}^{**}) = \mathcal{A}^{**}$ . Suppose that  $(x'_j)_j \subset X^{**}$  such that  $x'_j \xrightarrow{w^*} x''$  in  $X^{**}$ . Then for every  $a'' \in \mathcal{A}^{**}$ , we have  $\pi_\ell^{t^{***t}}(x', a'') \xrightarrow{w^*} \pi_\ell^{**}(a'', x'')$ . It follows that

$$\langle T^{**}(a''), x'_j \rangle \rightarrow \langle T^{**}(a''), x'' \rangle,$$

for each  $a'' \in \mathcal{A}^{**}$ . Consequently,  $T^{**}(a'') \in X^*$  for each  $a'' \in \mathcal{A}^{**}$ , and so  $T^{**}(\mathcal{A}^{**}) \subseteq X^*$ . Again by Theorem VI 4.2 and VI 4.8 of [7], we conclude that the mapping  $a \rightarrow \pi_\ell(x', a)$  from  $\mathcal{A}$  into  $X^*$  is weakly compact.

(ii) $\Rightarrow$ (iii): By definition of  $Z_{X^{**}}(\mathcal{A}^{**})$ , since  $Z_{X^{**}}(\mathcal{A}^{**}) = \mathcal{A}^{**}$ , proof hold.

(iii) $\Rightarrow$ (i): The proof is similar to that of Theorem 2.6.17 in [4].

(i) $\Rightarrow$ (iv): Let  $a'' \in \mathcal{A}^{**}$  and  $(a_j)_j \subset \mathcal{A}$  such that  $a_j \xrightarrow{w^*} a''$ . Then for each  $x'' \in X^{**}$ , we have

$$\begin{aligned} \lim_j \langle x'', \pi_\ell^*(x', a_j) \rangle &= \lim_\alpha \langle \pi_\ell^{**}(x'', x'), a_j \rangle = \langle \pi_\ell^{t^{***t}}(x'', x'), a'' \rangle \\ &= \langle \pi_\ell^{t^{***t}}(x'', x'), a'' \rangle = \langle x'', \pi_\ell^{t^{***t}}(x', a'') \rangle. \end{aligned}$$

It follows that  $\pi_\ell^*(x', a_j) \xrightarrow{w} \pi_\ell^{t^{***t}}(x', a'')$ , and this completes the proof.

(iv) $\Rightarrow$ (ii): Let  $x' \in X^*$  and suppose that  $a'' \in \mathcal{A}^{**}$  and  $x'' \in X^{**}$ . Let  $(a_j)_j \subset \mathcal{A}$  such that  $a_j \xrightarrow{w^*} a''$ . Since

$$\pi_\ell^*(x', a_\alpha) \xrightarrow{w} \pi_\ell^{t^{***t}}(x', a''),$$

for each  $x' \in X^*$ , we have the following equality

$$\begin{aligned} \langle \pi_\ell^{t^{***t}}(x', a''), x'' \rangle &= \langle a'', \pi_\ell^{**}(x'', x') \rangle = \lim_j \langle \pi_\ell^{**}(x'', x'), a_j \rangle \\ &= \lim_j \langle x'', \pi_\ell^*(x', a_j) \rangle = \langle x'', \pi_\ell^{t^{***t}}(x', a'') \rangle \\ &= \langle \pi_\ell^{t^{***t}}(x', a''), x'' \rangle. \end{aligned}$$

It follows that  $x'' \in \text{wap}_\ell(X)$ , and so  $Z_{X^{**}}(\mathcal{A}^{**}) = \mathcal{A}^{**}$ .  $\square$

**Corollary 2.1.** Suppose that  $X$  is a left Banach  $\mathcal{A}$ -module. Then  $X^* \mathcal{A}^{**} \subseteq X^*$  if and only if  $Z_{X^{**}}(\mathcal{A}^{**}) = \mathcal{A}^{**}$ .

**Example 2.1.** Suppose that  $G$  is a locally compact group. In the preceding corollary, take  $\mathcal{A} = X = c_0(G)$ . Therefore we conclude that  $Z_1(\ell^1(G)^{**}) = \ell^1(G)^{**}$ , see [4, Example 2.6.22(iii)].

**Theorem 2.2.** Suppose that  $X$  is a right Banach  $\mathcal{A}$ -module. Then the following assertions are equivalent.

- (i)  $Z_{\mathcal{A}^{**}}(X^{**}) = X^{**}$ ;
- (ii) The mapping  $x \rightarrow \pi_r^*(x', b)$  from  $X$  into  $\mathcal{A}^*$  is weakly compact;
- (iii) There are a subset  $E$  of  $X^*$  with  $\overline{\text{lin}E} = X^*$  such that for each sequence  $(a_n)_n \subset \mathcal{A}$  and  $(b_m)_m \subset X$  and each  $x' \in B^*$ ,

$$\lim_m \lim_n \langle x', x_m a_n \rangle = \lim_n \lim_m \langle x', x_m a_n \rangle$$

whenever both the iterated limits exist;

- (iv) Suppose that  $x'' \in X^{**}$  and  $(x_j)_j \subset X$  such that  $x_j \xrightarrow{w^*} x''$  on  $X^{**}$ . Then

$$\pi_r^*(x', x_j) \xrightarrow{w} \pi_r^{t^{**t}}(x', x''),$$

for all  $x' \in X^*$ .

*Proof.* Proof is similar to that of Theorem 2.1. □

**Corollary 2.2.** Suppose that  $X$  is a right Banach  $\mathcal{A}$ -module. Then  $X^*X^{**} \subseteq \mathcal{A}^*$  if and only if  $Z_{\mathcal{A}^{**}}(X^{**}) = X^{**}$ .

**Remark 2.1.** In the preceding corollary, if we take  $X = \mathcal{A}$ , we obtain Lemma 3.1 (i) of [6] and in Theorem 2.1, if we take  $X = \mathcal{A}$ , we obtain Theorem 2.6.17 of [4].

### 3. Module Arens regularity

Let  $X, Y$  and  $Z$  be normed spaces and  $\mathfrak{A}$ -modules. Let  $f : X \times Y \rightarrow Z$  be a continuous map such that

$$f(\alpha \cdot x, y) = \alpha \cdot f(x, y), \quad f(x, y \cdot \alpha) = f(x, y) \cdot \alpha \tag{1}$$

for all  $\alpha \in \mathfrak{A}, x \in X, y \in Y$  and

$$f(x_1 \pm x_2, y) = f(x_1, y) \pm f(x_2, y), \quad f(x, y_1 \pm y_2) = f(x, y_1) \pm f(x, y_2) \tag{2}$$

for all  $x_1, x_2, x \in X, y_1, y_2, y \in Y$ . In this case we say that  $f$  is an  $\mathfrak{A}$ -bimodule map. Note that the map  $f$  is not necessary bilinear. However, it can be bilinear if  $X, Y$  and  $Z$  are left essential  $\mathfrak{A}$ -modules (see the proof of Theorem 3.2).

We consider the set

$$J_Z = \overline{\text{span}}\{f(x \cdot \alpha, y) - f(x, \alpha \cdot y) : x \in X, y \in Y, \alpha \in \mathfrak{A}\}.$$

The relations (1) and (2) show that the set  $J_Z$  is a closed subspace and  $\mathfrak{A}$ -submodule of  $Z$ , and so  $Z/J_Z$  is an  $\mathfrak{A}$ -module. Now, define  $J_X = \{x \in X : f(x, y) \in J_Z (\forall y \in Y)\}$  and  $J_Y = \{y \in Y : f(x, y) \in J_Z (\forall x \in X)\}$  then since  $f(0, y) = f(x, 0) = 0$  for each  $x, y$  both  $J_X$  and  $J_Y$  contain 0 and clearly are subspaces and  $\mathfrak{A}$ -submodules of  $X$  and  $Y$ , respectively.

Let  $X$  be a normed space, let  $Y \subseteq X$  be subspace of  $X$ . The annihilators  $Y^\perp$  of  $Y$  is defined by

$$Y^\perp = \{f \in X' : \langle f, x \rangle = 0 \quad (x \in Y)\}.$$

Then  $Y^\perp$  is closed subspace of  $X'$ . Henceforth, the weak topology generated by the family  $\mathcal{F}$  of  $w^*$ -continuous functionals on a dual space  $X$  denoted by  $\sigma(X, \mathcal{F})$ .

For the  $\mathfrak{A}$ -bimodule map  $f : X \times Y \rightarrow Z$ , its adjoint is defined by

$$f^* : Z^* \times X \rightarrow Y^*, \quad \langle f^*(z', x), y \rangle = \langle z', f(x, y) \rangle \quad (z' \in Z^*, x \in X, y \in Y).$$

satisfies in the relations (1) and (2). For each  $x \in X$ , the mapping  $Z^* \rightarrow Y^* : z' \mapsto f^*(z', x)$  is  $\sigma(Z^*, J_Z)$  to  $\sigma(Y^*, J_Y)$  continuous. This process may be repeated to define  $f^{**} = (f^*)^* : Y^{**} \times Z^* \rightarrow X^*$ , and then  $f^{***} = (f^{**})^* : X^{**} \times Y^{**} \rightarrow Z^{**}$ . The map  $f^{***}$  is the unique extension of  $f$  such that  $X^{**} \rightarrow Z^{**}; x'' \mapsto f^{***}(x'', y'')$  is  $\sigma(X^{**}, J_X^\perp)$  to  $\sigma(Z^{**}, J_Z^\perp)$  continuous for all  $y'' \in Y^{**}$  and  $Y^{**} \rightarrow Z^{**}; y'' \mapsto f^{***}(x, y'')$  is  $\sigma(Y^{**}, J_Y^\perp)$  to  $\sigma(Z^{**}, J_Z^\perp)$  continuous for all  $x \in X$  (see [8] for the classical case). Note that the transpose  $f^t$  of  $f$  is a continuous map which satisfies the relations (1) and (2).

The canonical images of  $x \in X$  in  $X^{**}$  will be denoted by  $\hat{x}$ . Let  $x'' \in X^{**}$  and  $y'' \in Y^{**}$ . Then there exist nets  $(x_j) \subset X$  and  $(y_k) \subset Y$  with  $\hat{x}_j \xrightarrow{J_X^\perp} x''$  and  $\hat{y}_k \xrightarrow{J_Y^\perp} y''$ . We have

$$f^{***}(x'', y'') = \lim_j \lim_k f(\widehat{x_j, y_k}),$$

$$f^{t***t}(x'', y'') = \lim_k \lim_j f(\widehat{x_j, y_k}).$$

The *first module topological center* of  $f$  (as an  $\mathfrak{A}$ -bimodule map) is

$$\mathfrak{Z}_{\mathfrak{A}}^{(1)}(f) = \{x'' \in X^{**} : y'' \rightarrow f^{***}(x'', y'')$$

is  $\sigma(Y^{**}, J_Y^\perp)$  to  $\sigma(Z^{**}, J_Z^\perp)$ -continuous}.

The *second module topological center* of  $f$  (as an  $\mathfrak{A}$ -bimodule map) is

$$\mathfrak{Z}_{\mathfrak{A}}^{(2)}(f) = \{y'' \in Y^{**} : x'' \rightarrow f^{t***t}(x'', y'')$$

is  $\sigma(X^{**}, J_X^\perp)$  to  $\sigma(Z^{**}, J_Z^\perp)$ -continuous}.

Let  $f : X \times Y \rightarrow Z$  be an  $\mathfrak{A}$ -bimodule map. Then the map  $\tilde{f} : X/J_X \times Y/J_Y \rightarrow Z/J_Z$  defined by  $\tilde{f}(x + J_X, y + J_Y) = f(x, y) + J_Z$  is well defined and is an  $\mathfrak{A}$ -bimodule map.

Consider the map  $R_{z'} : Y \rightarrow X^*; y \mapsto f^{t*}(z', y)$ , where  $z' \in Z^*$ . Then  $R_{z'}$  is left  $\mathfrak{A}$ -module homomorphism if and only if  $z' \in J_Z^\perp$ . In other words,

$$\begin{aligned} R_{z'} \text{ is left } \mathfrak{A}\text{-module homomorphism} &\iff R_{z'}(\alpha \cdot y) = \alpha \cdot R_{z'}(y) \\ &\iff \langle f^{t*}(z', \alpha \cdot y), x \rangle = \langle \alpha \cdot f^{t*}(z', y), x \rangle \\ &\iff \langle z', f^t(\alpha \cdot y, x) \rangle = \langle z', f^t(y, x \cdot \alpha) \rangle \\ &\iff \langle z', f(x, \alpha \cdot y) \rangle = \langle z', f(x \cdot \alpha, y) \rangle \\ &\iff \langle z', f(x, \alpha \cdot y) - f(x \cdot \alpha, y) \rangle = 0 \\ &\iff z' \in J_Z^\perp. \end{aligned}$$

for all  $\alpha \in \mathfrak{A}$ ,  $y \in Y$  and  $x \in X$ . The above observation leads us to the following definition:

**Definition 3.1.** *The map  $f$  is called module Arens regular (as an  $\mathfrak{A}$ -module) if  $\mathfrak{A}$ -module homomorphisms  $R_{z'}$  are weakly compact for any  $z' \in J_Z^\perp$ .*

**Theorem 3.1.** *The following statements are equivalent.*

- (i)  $f$  is module Arens regular (as an  $\mathfrak{A}$ -module);
- (ii) The map  $L_{z'} : X \rightarrow Y^*$ ;  $x \mapsto f^*(z', x)$  is weakly compact for any  $z' \in J_Z^\perp$ .
- (iii) (Iterated limit condition) For bounded sequences  $(x_j) \subset X$ ,  $(y_k) \subset Y$  and  $z' \in J_Z^\perp$  we have

$$\lim_j \lim_k \langle z', f(x_j, y_k) \rangle = \lim_k \lim_j \langle z', f(x_j, y_k) \rangle$$

whenever both the iterated limits exist;

- (iv)  $(f^{***} - f^{t***t})(x'', y'') \in J_Z^{\perp\perp}$  for all  $x'' \in X^{**}$  and  $y'' \in Y^{**}$ ;
- (v) For each  $x''$  the mapping  $Y'' \rightarrow Z''$ ;  $y'' \mapsto f^{***}(x'', y'')$  is  $\sigma(Y^{**}, J_Y^\perp)$ - $\sigma(Z'', J_Z^\perp)$ -continuous;
- (vi) For each  $y''$  the mapping  $X^{**} \rightarrow Z''$ ;  $x'' \mapsto f^{t***t}(x'', y'')$  is  $\sigma(X^{**}, J_Z^\perp)$ - $\sigma(Z^{**}, J_Z^\perp)$ -continuous;
- (vii)  $\mathfrak{Z}_{\mathfrak{A}}^{(1)}(f) = X^{**}$ ;
- (viii)  $\mathfrak{Z}_{\mathfrak{A}}^{(2)}(f) = Y^{**}$ .

*Proof.* First we show that  $R_{z'}^*(x'') = f^{t**}(x'', z')$  for all  $z' \in J_Z^\perp$ . For each  $x'' \in X^{**}$  and  $y \in Y$  we have

$$\langle R_{z'}^*(x''), y \rangle = \langle x'', R_{z'}(y) \rangle = \langle x'', f^{t*}(z', y) \rangle = \langle f^{t**}(x'', z'), y \rangle,$$

and so

$$R_{z'}^*(x'') = f^{t**}(x'', z'). \quad (3)$$

We also get

$$\begin{aligned} R_{z'}^* \text{ is module Arens regular} &\iff R_{z'}^* \text{ is weak}^* \text{-weak compact for each } z' \in J_Z^\perp \\ &\iff R_{z'}^*(w^* - \lim_j x_j^{**}) = w - \lim_j R_{z'}^*(x_j'') \end{aligned}$$

when  $(x_j'') \subset X^{**}$ . Now, it follows from (3) and the above equivalent conditions that

$$f^{t**}(w^* - \lim_j x_j'', z') = w - \lim_j f^{t**}(x_j'', z'). \quad (4)$$

On the other hand, for each  $y'' \in Y^{**}$  and  $z' \in J_Z^\perp$  we have

$$\begin{aligned} \langle f^{t***t}(w^* - \lim_j x_j'', y''), z' \rangle &= \langle y'', f^{t**}(w^* - \lim_j x_j'', z') \rangle \\ &\stackrel{(4)}{=} \langle y'', w - \lim_j f^{t**}(x_j'', z') \rangle \\ &= \lim_j \langle y'', f^{t**}(x_j'', z') \rangle \\ &= \lim_j \langle f^{t**}(x_j'', y''), z' \rangle \\ &= \langle \sigma(Z^{**}, J_Z^\perp) - \lim_j f^{t***t}(x_j'', y''), z' \rangle. \end{aligned}$$

The above argument shows that (i),(vi) and (viii) are equivalent. Also by the Grothendieck Criterion for compactness and  $\sigma(X^{**}, J_Y^\perp)$ - $\sigma(Z^{**}, J_Z^\perp)$ -continuity of the mapping  $y'' \mapsto f^{***}(x'', y'')$  and  $x'' \mapsto f^{t***t}(x'', y'')$  we observe that (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv). The equivalence (ii) $\iff$ (v) $\iff$ (vii) is similar to the equivalence (i) $\iff$ (vi) $\iff$ (viii).  $\square$

Recall that a left Banach  $\mathfrak{A}$ -module  $X$  is called *left essential* if the linear span of  $\mathfrak{A} \cdot X = \{a \cdot x : a \in \mathfrak{A}, x \in X\}$  is dense in  $X$ . Right essential  $\mathfrak{A}$ -modules and (two-sided) essential  $\mathfrak{A}$ -bimodules are defined similarly.

**Theorem 3.2.** *Let  $X, Y$  and  $Z$  be left essential  $\mathfrak{A}$ -bimodules. Then the map  $f$  is module Arens regular (as an  $\mathfrak{A}$ -module) if and only if  $\tilde{f}$  is Arens regular.*

*Proof.* First we show that  $\tilde{f}$  is a continuous bilinear map. Indeed, if we assume that  $X$  is a left essential  $\mathfrak{A}$ -module and  $x \in X$ , then there is a sequence  $(x_n) \subseteq \mathfrak{A} \cdot X$  such that  $\lim_n x_n = x$ . Assume that  $x_n = \sum_{m=1}^{K_n} \alpha_{n,m} \cdot x_{n,m}$  for some finite sequences  $(\alpha_{n,m})_{m=1}^{K_n} \subseteq \mathfrak{A}$  and  $(x_{n,m})_{m=1}^{K_n} \subseteq X$ . Then for  $\lambda \in \mathbb{C}$ ,

$$\begin{aligned} f(\lambda x_n, y) &= f(\lambda \sum_{m=1}^{K_n} \alpha_{n,m} \cdot x_{n,m}, y) = \sum_{m=1}^{K_n} f((\lambda \alpha_{n,m}) \cdot x_{n,m}, y) \\ &= \sum_{m=1}^{K_n} (\lambda \alpha_{n,m}) \cdot f(x_{n,m}, y) = \sum_{m=1}^{K_n} \lambda f(\alpha_{n,m} \cdot x_{n,m}, y) = \lambda f(x_n, y), \end{aligned}$$

and so, by the continuity of  $f$ ,  $f(\lambda x, y) = \lambda f(x, y)$  for all  $x \in X$  and  $y \in Y$ .

Similarly,  $f(x, \lambda y) = \lambda f(x, y)$  for all  $x \in X$  and  $y \in Y$ . Thus  $\tilde{f}$  is a continuous bilinear map. Now, for each  $x \in X, y \in Y$  and  $\varphi \in (Z/J_Z)^*$  we have

$$\begin{aligned} \langle \tilde{f}^*(\varphi, x + J), y + J \rangle &= \langle \varphi, \tilde{f}(x + J, y + J) \rangle \\ &= \langle \varphi, f(x, y) + J_Z^\perp \rangle \\ &= \langle \varphi, f(x, y) \rangle \quad (\varphi \in (Z/J_Z)^* \cong J_Z^\perp). \end{aligned}$$

Hence

$$\langle \tilde{f}^*(\varphi, x + J_X), y + J \rangle = \langle f^*(\varphi, x), y \rangle \quad (x \in X, y \in Y, \varphi \in (Z/J_Z)^*). \quad (5)$$

For  $y'' + J_Y^{\perp\perp} \in Y^{**}/J^{\perp\perp} \cong (Y/J_Y)^{**}$  take a bounded net  $(y_j + J_Y) \subset Y/J_Y$  such that  $\widehat{y_j + J_Y} \xrightarrow{w^*} y'' + J_Y^{\perp\perp}$ . Then

$$\begin{aligned} \langle \tilde{f}^{**}(y'' + J_Y^{\perp\perp}, \varphi), x + J_X \rangle &= \langle y'' + J_Y^{\perp\perp}, \tilde{f}^*(\varphi, x + J_X) \rangle \\ &= \lim_j \langle \tilde{f}^*(\varphi, x + J_X), y_j + J_Y \rangle \\ &\stackrel{(5)}{=} \lim_j \langle f^*(\varphi, x), y_j \rangle. \end{aligned}$$

Thus

$$\langle \tilde{f}^{**}(y'' + J_Y^{\perp\perp}, \varphi), x + J_X \rangle = \lim_j \langle \varphi, f(x, y_j) \rangle. \quad (6)$$

Also for  $x'' + J_X^{\perp\perp} \in X^{**}/J^{\perp\perp} \cong (X/J_X)^{**}$  take a bounded net  $(x_k + J_X) \subset X/J_X$  such that  $\widehat{x_k + J_X} \xrightarrow{w^*} x'' + J_X^{\perp\perp}$ . Then

$$\begin{aligned} \langle \tilde{f}^{***}(x'' + J_X^{\perp\perp}, y'' + J_Y^{\perp\perp}), \varphi \rangle &= \langle x'' + J_X^{\perp\perp}, \tilde{f}^{**}(y'' + J_Y^{\perp\perp}, \varphi) \rangle \\ &= \lim_k \langle \tilde{f}^{**}(y'' + J_Y^{\perp\perp}, \varphi), x_k + J_X \rangle \\ &\stackrel{(6)}{=} \lim_k \lim_j \langle \varphi, f(x_k, y_j) \rangle, \end{aligned}$$

and so

$$\langle \tilde{f}^{***}(x'' + J_X^{\perp\perp}, y'' + J_Y^{\perp\perp}), \varphi \rangle = \langle w^* - \lim_k w^* - \lim_j f(x_k, y_j), \varphi \rangle.$$

Similarly,

$$\langle \tilde{f}^{t***t}(x'' + J_X^{\perp\perp}, y'' + J_Y^{\perp\perp}), \varphi \rangle = \langle w^* - \lim_j w^* - \lim_k f(x_k, y_j), \varphi \rangle.$$



Therefore  $\tilde{f}$  is Arens regular if and only if  $\tilde{f}^{***} = \tilde{f}^{t***t}$  if and only if

$$\langle w^* - \lim_k w^* - \lim_j f(x_k, y_j), \varphi \rangle = \langle w^* - \lim_j w^* - \lim_k f(x_k, y_j), \varphi \rangle \quad (\forall \varphi \in J_Z^\perp)$$

if and only if  $f$  is module Arens regular. □

Suppose that  $\mathcal{J}$  is the closed ideal of  $\mathcal{A}$  generated by elements of the form  $(a \cdot \alpha)b - a(\alpha \cdot b)$ , for all  $a \in \mathcal{A}$  and  $\alpha \in \mathfrak{A}$ . When  $X = Y = Z = \mathcal{A}$  and  $f = \pi$ , then  $J_Z = \mathcal{J}$  and  $J_X = \{x \in \mathcal{A} : xy \in \mathcal{J}(y \in \mathcal{A})\}$  and  $\mathcal{J} \subseteq J_X$  (as  $\mathcal{J}$  is an ideal) and if  $\mathcal{A}$  has an approximate identity (not necessarily bounded) then clearly  $J_X \subseteq \mathcal{J}$ . Similarly for  $J_Y$ . In this case  $J_X = J_Y = J_Z = \mathcal{J}$  and equalities (1) will be the following compatible actions  $\mathfrak{A}$  over  $\mathcal{A}$ :

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \quad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}). \quad (7)$$

Also  $\tilde{f}$  is well-defined in general (as it is bilinear, it is enough to show that it maps  $(0, 0)$  to  $0$ , but this is clear by definition of  $J_X$  and  $J_Y$ ). Henceforth, we assume that  $\tilde{\pi} : \mathcal{A}/\mathcal{J} \times \mathcal{A}/\mathcal{J} \rightarrow \mathcal{A}/\mathcal{J}; (a + \mathcal{J}, b + \mathcal{J}) \mapsto \pi(a, b) + \mathcal{J}$ . Obviously,  $\tilde{\pi}$  is well-defined and since  $\pi$  is bilinear, so is  $\tilde{\pi}$ .

One should recall that the ideal  $\mathcal{J}$  in [9] and this paper is defined to be the closed ideal of  $\mathcal{A}$  generated by elements of the form  $(a \cdot \alpha)b - a(\alpha \cdot b)$ , for  $a \in \mathcal{A}$  and  $\alpha \in \mathfrak{A}$ , whereas in [10] it was defined as the closed ideal of  $\mathcal{A}$  generated by elements of the form  $\alpha \cdot (ab) - (ab) \cdot \alpha$ , for  $a \in \mathcal{A}$  and  $\alpha \in \mathfrak{A}$ . It has been observed in [9] that  $\mathcal{J}$  is indeed the closed subspace generated by elements of the form  $(a \cdot \alpha)b - a(\alpha \cdot b)$  for  $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$ .

The *first module topological center* of  $\mathcal{A}^{**}$  (as an  $\mathfrak{A}$ -module) is

$$\mathcal{Z}_{\mathfrak{A}}^{(1)}(\mathcal{A}^{**}) = \{b'' \in \mathcal{A}^{**} : a'' \rightarrow b'' \square a'' \text{ is } \sigma(\mathcal{A}^{**}, \mathcal{J}^\perp)\text{-continuous}\}.$$

Also, like in the classic case, we can define the *second module topological center* of  $\mathcal{A}^{**}$  by

$$\mathcal{Z}_{\mathfrak{A}}^{(2)}(\mathcal{A}^{**}) = \{b'' \in \mathcal{A}^{**} : a'' \rightarrow a'' \diamond b'' \text{ is } \sigma(\mathcal{A}^{**}, \mathcal{J}^\perp)\text{-continuous}\}.$$

Obviously,  $\mathcal{Z}_{\mathfrak{A}}^{(1)}(\mathcal{A}^{**})$  and  $\mathcal{Z}_{\mathfrak{A}}^{(2)}(\mathcal{A}^{**})$  are  $\sigma(\mathcal{A}^{**}, \mathcal{J}^\perp)$ -closed subalgebras of  $(\mathcal{A}^{**}, \square)$  containing  $\mathcal{A}$ . It is shown in [10] that

$$\mathcal{Z}_{\mathfrak{A}}^{(1)}(\mathcal{A}^{**}) = \{a'' \in \mathcal{A}^{**} : a'' \square b'' - a'' \diamond b'' \in \mathcal{J}^{\perp\perp} \quad (\forall b'' \in \mathcal{A}^{**})\}.$$

Similarly, we can show that

$$\mathcal{Z}_{\mathfrak{A}}^{(2)}(\mathcal{A}^{**}) = \{b'' \in \mathcal{A}^{**} : a'' \square b'' - a'' \diamond b'' \in \mathcal{J}^{\perp\perp} \quad (\forall a'' \in \mathcal{A}^{**})\}.$$

The above argument for  $\mathcal{J}$  shows that Proposition 2.2 of [10] remains valid, and so we have the following result.

**Theorem 3.3.** *Let  $\mathcal{A}$  be a Banach algebra and a Banach  $\mathfrak{A}$ -bimodule with compatible actions. The following are equivalent.*

- (i)  $\mathcal{A}$  is module Arens regular;
- (ii)  $\mathcal{A}/\mathcal{J}$  is Arens regular;
- (iii)  $\tilde{\pi}$  is Arens regular;
- (iv)  $\pi$  is module Arens regular.
- (v)  $a'' \square b'' - a'' \diamond b'' \in \mathcal{J}^{\perp\perp}, \quad a'', b'' \in \mathcal{A}^{**}$ .
- (vi)  $\mathcal{Z}_{\mathfrak{A}}^{(1)}(\mathcal{A}^{**}) = \mathcal{A}^{**}$ ;

$$(vii) \mathcal{Z}_{\mathfrak{A}}^{(2)}(\mathcal{A}^{**}) = \mathcal{A}^{**}.$$

*Proof.* It is shown in [1, Theorem 2.4] (i) and (ii) are equivalent. It is routine to check that (ii) and (iii) are equivalent. The equivalence of (iii) and (iv) follows from Theorem 3.2 by taking  $X = Y = Z = \mathcal{A}$ . Since  $\mathcal{J}$  is the closed subspace of  $\mathcal{A}$  [9, Lemma 3.1], it follows immediately from [10, Proposition 2.2] that (i)  $\iff$  (v)  $\iff$  (vi)  $\iff$  (vii). Now, it follows from the proof of Theorem 3.2 that (iii) and (iv) are equivalent.  $\square$

Let  $\mathcal{A}$  be a Banach  $\mathfrak{A}$ -module with compatible actions (7). For  $Y \leq \mathcal{A}^{(n)}$  and a non-negative integer number  $n$ , define  $Y^{(n\perp)}$  by induction:  $Y^{(0\perp)} = Y \leq \mathcal{A}$ ,  $Y^{(1\perp)} = Y^\perp \leq \mathcal{A}'$ , and  $Y^{(n\perp)} = (Y^{((n-2)\perp)})^{\perp\perp} \leq (\mathcal{A}^{(n-2)})'' = \mathcal{A}^{(n)}$ . It is well-known that  $(\mathcal{A}/\mathcal{J})^{(2n)} = \mathcal{A}^{(2n)}/\mathcal{J}^{(2n\perp)}$  and  $(\mathcal{A}/\mathcal{J})^{(2n-1)} = \mathcal{J}^{((2n-1)\perp)}$ . It is easy to see that the concepts of module Arens regularity and Arens regularity for  $\mathcal{A}^{(n)}$  coincide when  $\mathcal{A}$  is a commutative Banach  $\mathfrak{A}$ -module, but for the non-commutative case we have the following result.

**Theorem 3.4.** *Let  $n$  be a even natural number. If  $\mathcal{A}^{(n)}$  is module Arens regular, then  $\mathcal{A}^{(n)}/\mathcal{J}^{(n\perp)}$  is Arens regular.*

*Proof.* First note that if  $\lambda \in \mathcal{J}^\perp$ , the compatible actions (7) show that  $\alpha \cdot \lambda \in \mathcal{J}^\perp$  and  $\lambda \cdot \alpha \in \mathcal{J}^\perp$  for all  $\alpha \in \mathfrak{A}$ . let  $\mathcal{N}$  be the closed ideal of  $\mathcal{A}^{**}$  generated by  $(a'' \cdot \alpha) \square b'' - a'' \square (\alpha \cdot b'')$ , for  $a'', b'' \in \mathcal{A}^{**}$  and  $\alpha \in \mathfrak{A}$ . Then clearly  $\mathcal{J} \subseteq \mathcal{N}$ . Take two bounded nets  $(a_j), (b_k) \subset \mathcal{A}$  with  $\hat{a}_j \xrightarrow{\mathcal{J}^\perp} a''$  and  $\hat{b}_k \xrightarrow{\mathcal{J}^\perp} b''$ , then  $\hat{a}_j \cdot \alpha \xrightarrow{\mathcal{J}^\perp} a'' \cdot \alpha$  and  $\hat{\alpha} \cdot b_k \xrightarrow{\mathcal{J}^\perp} \alpha \cdot b''$  and so

$$\langle (a'' \cdot \alpha) \square b'' - a'' \square (\alpha \cdot b''), \lambda \rangle = \lim_j \lim_k \langle \lambda, (a_j \cdot \alpha) b_k - a_j (\alpha \cdot b_k) \rangle = 0.,$$

for all  $\lambda \in \mathcal{J}^\perp$ . Therefore  $\mathcal{N} \subseteq \mathcal{J}^{\perp\perp}$ . It follows from Theorem 3.3 that  $\mathcal{A}^{**}$  is module Arens regular if and only if  $\mathcal{A}^{**}/\mathcal{N}^{\perp\perp}$  is Arens regular. On the other hand, for any  $a''''', b'''' \in \mathcal{A}^{****}$ , if  $a'''' \square b'''' - a'''' \diamond b'''' \in \mathcal{N}^{\perp\perp}$  then  $a'''' \square b'''' - a'''' \diamond b'''' \in \mathcal{J}^{(4\perp)}$ . This shows that the image of  $a'''' \square b''''$  and  $a'''' \diamond b''''$  in  $(\mathcal{A}^{****}/\mathcal{J}^{(4\perp)}, \square)$  and  $(\mathcal{A}^{****}/\mathcal{J}^{(4\perp)}, \diamond)$  are equal. Thus module Arens regularity of  $\mathcal{A}^{**}$  implies Arens regularity of  $\mathcal{A}^{**}/\mathcal{J}^{\perp\perp}$ . Now let  $\mathcal{N}_n$  be the corresponding closed ideal of  $\mathcal{A}^{(n)}$  generated by  $(a^{(n)} \cdot \alpha) \square b^{(n)} - a^{(n)} \square (\alpha \cdot b^{(n)})$ , for  $a^{(n)}, b^{(n)} \in \mathcal{A}^{(n)}$  and  $\alpha \in \mathfrak{A}$ . Similar to the above argument we can show that  $\mathcal{N}_n \subseteq \mathcal{J}^{(n\perp)}$  and module Arens regularity of  $\mathcal{A}^{(n)}$  implies Arens regularity of  $\mathcal{A}^{(n)}/\mathcal{J}^{(n\perp)}$ .  $\square$

#### 4. Module Arens regularity of semigroup algebras

In this section we find conditions on a (discrete) inverse semigroup  $S$  such that the map

$$\omega : \ell^1(S) \times \ell^1(S) \longrightarrow \ell^1(S); (\delta_s, \delta_t) \mapsto \delta_{st}$$

is module Arens regular (as  $\ell^1(E)$ -module). Throughout this section  $S$  is an inverse semigroup with the set of idempotents  $E$ , where the order of  $E$  is defined by

$$e \leq d \iff ed = e \quad (e, d \in E).$$

Since  $E$  is a commutative subsemigroup of  $S$  [11, Theorem V.1.2], actually a semilattice,  $\ell^1(E)$  could be regarded as a commutative subalgebra of  $\ell^1(S)$ , and

thereby  $\ell^1(S)$  is a Banach algebra and a Banach  $\ell^1(E)$ -module with compatible actions [12]. Here we let  $\ell^1(E)$  act on  $\ell^1(S)$  by multiplication from right and trivially from left, that is

$$\delta_e \cdot \delta_s = \delta_s, \quad \delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e \quad (s \in S, e \in E).$$

In this case, the ideal  $\mathcal{J}$  (see section 3) is the closed linear span of  $\{\delta_{set} - \delta_{st} : s, t \in S, e \in E\}$ . We consider an equivalence relation on  $S$  as follows:

$$s \approx t \iff \delta_s - \delta_t \in \mathcal{J} \quad (s, t \in S).$$

For an inverse semigroup  $S$ , the quotient  $S/\approx$  is a discrete group (see [10] and [13]). Indeed,  $S/\approx$  is homomorphic to the maximal group homomorphic image  $G_S$  [14] of  $S$  [15]. In particular,  $S$  is amenable if and only if  $G_S$  is amenable [14, 16]. As in [1, Theorem 3.3], we may observe that  $\ell^1(S)/\mathcal{J} \cong \ell^1(G_S)$ . With the notations of the previous section,  $\ell^1(S)/\mathcal{J}$  is a commutative  $\ell^1(E)$ -bimodule with the following actions:

$$\delta_e \cdot (\delta_s + \mathcal{J}) = \delta_s + \mathcal{J}, \quad (\delta_s + \mathcal{J}) \cdot \delta_e = \delta_{se} + \mathcal{J} \quad (s \in S, e \in E).$$

**Theorem 4.1.** *Let  $S$  an inverse semigroup with the set of idempotents  $E$ . Then*

$$Z_{\ell^1(E)}^{(1)}(\ell^1(S)^{**})/\mathcal{J}^{\perp\perp} = Z_{\ell^1(E)}^{(2)}(\ell^1(S)^{**})/\mathcal{J}^{\perp\perp} = \ell^1(G_S).$$

*Proof.* It is shown in [1, Theorem 2.4] that for any  $a'', b'' \in \ell^1(S)^{**}$ ,  $a'' \square b'' - a'' \diamond b'' \in \mathcal{J}^{\perp\perp}$  if and only if the images of  $a'' \square b''$  and  $a'' \diamond b''$  in  $(\ell^1(S)^{**}/\mathcal{J}^{\perp\perp}, \square)$  and  $(\ell^1(S)^{**}/\mathcal{J}^{\perp\perp}, \diamond)$  are equal. For the first module topological center, the equality is proved in [13, Theorem 2.6]. For the second module topological center, we have

$$\begin{aligned} & Z_{\ell^1(E)}^{(2)}(\ell^1(S)^{**})/\mathcal{J}^{\perp\perp} \\ &= \{b'' \in \ell^1(S)^{**} : a'' \square b'' - a'' \diamond b'' \in \mathcal{J}^{\perp\perp}, \forall a'' \in \ell^1(S)^{**}\}/\mathcal{J}^{\perp\perp} \\ &= \{b'' + \mathcal{J}^{\perp\perp} \in \ell^1(S)^{**}/\mathcal{J}^{\perp\perp} : (a'' \square b'') + \mathcal{J}^{\perp\perp} = (a'' \diamond b'') + \mathcal{J}^{\perp\perp}, \forall a'' \in \ell^1(S)^{**}\} \\ &= Z^{(2)}(\ell^1(S)^{**}/\mathcal{J}^{\perp\perp}) = Z^{(2)}((\ell^1(S)/\mathcal{J})^{**}) = Z^{(2)}(\ell^1(G_S)^{**}) = \ell^1(G_S), \end{aligned}$$

where  $Z_t^{(2)}(\ell^1(G_S)^{**})$  is the second module topological center of  $\ell^1(G_S)^{**}$  which is computed in [17]. □

**Remark 4.1.** *It is well known that  $G_S$  is finite if and only if  $\ell^1(G_S)^{**}$  is Arens regular and so the finiteness of  $G_S$  is equivalent to the following equalities:*

$$Z_{\ell^1(E)}^{(1)}(\ell^1(S)^{**})/\mathcal{J}^{\perp\perp} = Z_{\ell^1(E)}^{(2)}(\ell^1(S)^{**})/\mathcal{J}^{\perp\perp} = \ell^1(G_S)^{**}.$$

**Theorem 4.2.** *Let  $S$  be an inverse semigroup with the set of idempotents  $E$ . Then the following are equivalent.*

- (i)  $G_S$  is finite;
- (ii)  $\ell^1(S)$  is  $\ell^1(E)$ -module Arens regular;
- (iii)  $\ell^1(G_S)$  is Arens regular;
- (iv)  $\omega$  is module Arens regular;
- (v)  $\tilde{\omega}$  is Arens regular.

*Proof.* The equivalence of (i) and (ii) is the consequence of [15] and [1, Theorem 3.3]. The equivalence of other parts follows from Theorem 3.3 with  $\mathcal{A} = \ell^1(S)$  and  $\mathfrak{A} = \ell^1(E)$ . □

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