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ANALYTICAL TIME-DOMAIN SOLUTIONS FOR HYSTERETICALLY DAMPED SDOF OSCILLATOR: CLOSED-FORM EXPRESSIONS AND NON-CAUSALITY ANALYSIS

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In this paper, a theoretical analysis of the hysteretically damped SDOF oscillator is presented. Closed-form time domain solutions are obtained analytically. Although the hysteretic model is in agreement with experimental observations since it solves the problem of the frequency dependence of the dissipated energy during a vibration cycle, its transcription into the time domain remains a significant challenge. In this work, impulse and free vibration responses are evaluated using inverse Fourier and inverse Laplace transforms respectively to solve the governing differential equations. The existence of a free vibration solution for the hysteretic model is established. Based on impulse and free vibration responses, the complete harmonic response is then extracted by separating the real and the imaginary parts of the Duhamel integral expression. The parameters affecting the observed non-causality are identified and discussed.

Keywords: Hysteretic damping, viscous damping, Causality, energy dissipation.

1. Introduction

In vibration theory, viscous and hysteretic models are the most used to describe damping of dynamic movement. The flaw in the well-known viscous model is the frequency dependence of the dissipated energy. Cyclic deformation tests have shown that the internal friction doesn't obey to liquid viscosity law, but depends on deformation amplitude and independent of strain velocity [4]. According to Lazan [18, 28], energy dissipated by cycle is practically independent of frequency [14]. The hysteretic model was initially used in flutter analysis [30].

To characterize the hysteretic model, several papers have dealt with the time domain impulse response. In ref. [13], an impulse response of hysteretically damped SDOF oscillator as Fourier integral was presented. The solution contain a small precursor response that make the model physically unrealisable. Same author, in [3], reported that hysteretic model violates the causality principle. In Ref. [9], using an elaborate method, the authors obtained the time domain impulse response which include impulse response

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precursor and confirmed the non-causality of the model. This impulse response precursor was investigated in [7, 2].

On the side of free vibration, Soroka [10] used complex stiffness model and obtained natural damped frequencies and logarithmic decays for comparing with viscous model. Bishop [11], questioned Soroka's assumption and reported that the hysteretic damping should be extended to cover non-harmonic motion. Several publications [6, 12, 1] have reached similar conclusions than Bishop and pointed out that the hysteretic model is designed only for harmonic motions. In [15], the authors extends the applicability of the hysteretic damping to treat inharmonic loadings but it was questioned by [17]. In another publication [16], Chen et al. present an integro-differential equation of motion which, according to the author, can be extended to arbitrary loads . Ribeiro *et al.* [20, 21, 22] dealt with hysteretic model in free vibration, they presented a model that combines hysteretic and viscous models with complex initial conditions in order to overcome the incoherence of the hysteretic model in free vibration. Maia [8] concluded that the free vibration should be considered only in the perspective of an equivalent viscously damped model.

Facing all the imperfections of the hysteretic model, the present paper shows a mathematical methodology allowing to obtain a concise and practical time-domain solutions for the hysteretically damped SDOF oscillator submitted to impulse force and harmonic loading as well as under free vibration behavior. In each case, the corresponding response is plotted and the contribution of observed non-causality is discussed.

2. Mathematical model

Consider an SDOF system, where m, k, c, represent mass, stiffness and damping coefficient, respectively. Damping can be characterized with loss factor η obtained by $\eta = W/2\pi V$, where W is the dissipated energy and V is the peak potential energy stored in the spring [3], η can be written by:

$$\eta = c|\omega|/k \tag{1}$$

A damping mechanism $c(\omega)$ allowing to measure the hysteretic damping at different frequencies was introduced [3], and takes its origin from Eq.(1):

$$c(\omega) = k\eta(\omega)/|\omega| \tag{2}$$

Considering $c(\omega)$, the equation of motion governing the oscillator is as follows [3]:

$$m\ddot{u}(t) + c(\omega)\dot{u}(t) + ku(t) = F(t) \tag{3}$$

In frequency domain, it leads to:

$$(-m\omega^2 + k\{1 + i\eta(\omega)\text{sgn}(\omega)\})X(\omega) = F(\omega) \tag{4}$$

Converting the equation back to the time domain yields the equation of motion in complex stiffness form:

$$m\ddot{u}(t) + k\left[1 + i\eta\,(\omega)\,\text{sgn}(\omega)\right]u(t) = F(t) \tag{5}$$

In the following sections, Eq.(5) is solved for impulsive and harmonic excitation, and also in free vibration. $\omega_0 = \sqrt{k/m}$ is the natural frequency of the system. We define $\omega_z = \omega_0 \sqrt{1 + i\eta \operatorname{sgn}(\omega)}$ as a complex frequency, then Eq.(5) becomes:

$$\ddot{u}(t) + \omega_z^2 u(t) = \frac{F(t)}{m} \tag{6}$$

3. **Evaluation of impulse response**

Considering an impulse force modeled by Dirac delta function $\delta(t)$ in Eq.(6):

$$\ddot{u} + \omega_z^2 u(t) = \frac{\delta(t)}{m} \tag{7}$$

Transfer function $H(i\omega)$ is obtained with Fourier transform:

$$H(i\omega) = \frac{1}{m\left(-\omega^2 + \omega_z^2\right)} \tag{8}$$

The time domain response $u(t)$ is given by inverse Fourier transform:

$$u(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(i\omega) \mathrm{e}^{i\omega t} \, \mathrm{d}\omega = \frac{1}{2\pi m} \int_{-\infty}^{+\infty} \left(\frac{\mathrm{e}^{i\omega t}}{-\omega^2 + \omega_z^2} \right) \mathrm{d}\omega \tag{9}$$

According to the sign of ω, w_z splits into: $\omega_{z1} = \omega_0\sqrt{1+i\eta}$, when $\omega \geq 0$ and $\omega_{z2} = \omega_0\sqrt{1-i\eta}$, when $\omega < 0$. When $\omega \geq 0$, Eq.(9), becomes:

$$u(t) = \frac{1}{2\pi m} \left(\int_{-\infty}^{0} \frac{\mathrm{e}^{i\omega t}}{-\omega^2 + \omega_{z2}^2} \, \mathrm{d}\omega + \int_{0}^{+\infty} \frac{\mathrm{e}^{i\omega t}}{-\omega^2 + \omega_{z1}^2} \, \mathrm{d}\omega \right) \qquad \omega \in \mathbb{R} \tag{10}$$

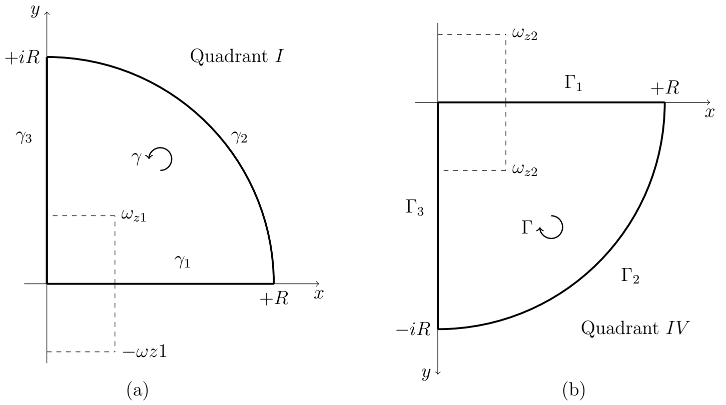
$$u(t) = \frac{1}{2\pi m} \left(\int_{0}^{+\infty} \frac{\mathrm{e}^{-i\omega t} \, \mathrm{d}\omega}{-\omega^2 + \omega_{z2}^2} + \int_{0}^{+\infty} \frac{\mathrm{e}^{i\omega t} \, \mathrm{d}\omega}{-\omega^2 + \omega_{z1}^2} \right) = \frac{1}{2\pi m} \left[I_2(t) + I_1(t) \right] \tag{11}$$

$I_1(t)$ and $I_2(t)$ are evaluated using the residue theorem according to Eq.(12) and Eq.(13). To this purpose, we analytically extend the two integrals to the complex plane by introducing the complex variable $z = x + iy$, which yields the complex-valued functions $I_{z1}(t)$ and $I_{z2}(t)$ of the functions $f(z)$ and $g(z)$ along the contours (γ) and (Γ), respectively, as shown in (Fig.1). The explicit contour integration and residues calculus are presented in *Appendix A*. The final result is given by Eq.(14):

$$I_{z1}(t) = \oint_{\gamma} f(z) \, \mathrm{d}z = \oint_{\gamma} \left(\frac{\mathrm{e}^{izt}}{-z^2 + \omega_{z1}^2} \right) \mathrm{d}z = 2\pi i \sum_{k=1}^{m} \operatorname{Res}\left[f\left(z_k\right) \right] \tag{12}$$

$$I_{z2}(t) = \oint_{\Gamma} g(z) \, \mathrm{d}z = \oint_{\Gamma} \left(\frac{\mathrm{e}^{-izt}}{-z^2 + \omega_{z2}^2} \right) \mathrm{d}z = -2\pi i \sum_{k=1}^{m} \operatorname{Res}\left[g\left(z_k\right) \right] \tag{13}$$

$$u(t) = \frac{i}{2m} \left(\frac{\mathrm{e}^{-i\omega_{z2} t}}{\omega_{z2}} - \frac{\mathrm{e}^{i\omega_{z1} t}}{\omega_{z1}} \right) - \frac{i}{2\pi m} \left(\int_{+\infty}^{0} \frac{\mathrm{e}^{-yt} \, \mathrm{d}y}{y^2 + \omega_{z1}^2} + \int_{0}^{+\infty} \frac{\mathrm{e}^{-yt} \, \mathrm{d}y}{y^2 + \omega_{z2}^2} \right) \tag{14}$$



(a) (b)

Fig. 1. Integration domain for $I_{z1}(t)$ and $I_{z2}(t)$

The second term of Eq.(14) is composed of two residual integrals which converge to the same solution as that one proposed by [9]. Eq.(14) leads to:

$$u(t) = \frac{i}{2m}\left(\frac{\mathrm{e}^{-i\omega_{z2}t}}{\omega_{z2}} - \frac{\mathrm{e}^{i\omega_{z1}t}}{\omega_{z1}}\right) + \frac{\eta}{\pi}\int_0^{+\infty} \frac{\mathrm{e}^{-y\omega_0 t}}{(1+y^2)^2+\eta^2}\,\mathrm{d}y \tag{15}$$

Several authors [8, 7] addressed Eq.(15). According to [9] residual integral's contribution in Eq.(15) is insignificant. After verifying this statement and considering this study is restricted when $t \geq 0$, Eq.(15) can be reduced to:

$$u(t) = \frac{i}{2m}\left(\frac{\mathrm{e}^{-i\omega_{z2}t}}{\omega_{z2}} - \frac{\mathrm{e}^{i\omega_{z1}t}}{\omega_{z1}}\right) \tag{16}$$

To simplify Eq. (16), we use Euler's formula and then write ω_{z1} and ω_{z2} in algebraic form: $\omega_{z1} = \omega_{1r} + i\omega_{1i}$ and $\omega_{z2} = \omega_{2r} + i\omega_{2i}$. The real and imaginary parts of ω_{z1} and ω_{z2} are written as follows. Eq.(16) leads to Eq.(19) :

$$\omega_{1r} = \omega_{2r} = \left[\omega_0 \sqrt[4]{1+\eta^2}\cos\left(\frac{1}{2}\arctan(\eta)\right)\right] \tag{17}$$

$$\omega_{1i} = -\omega_{2i} = \left[\omega_0 \sqrt[4]{1+\eta^2}\sin\left(\frac{1}{2}\arctan(\eta)\right)\right] \tag{18}$$

$$u(t) = \frac{-i}{2m}\left(\frac{\mathrm{e}^{i(\omega_{1r}+i\omega_{1i})t}}{\omega_{1r}+i\omega_{1i}} - \frac{\mathrm{e}^{-i(\omega_{2r}+i\omega_{2i})t}}{\omega_{2r}+i\omega_{2i}}\right) = \frac{-i}{2m}\left(\frac{\mathrm{e}^{i\omega_{1r}t}\mathrm{e}^{-\omega_{1i}t}}{\omega_{1r}+i\omega_{1i}} - \frac{\mathrm{e}^{-i\omega_{2r}t}\mathrm{e}^{\omega_{2i}t}}{\omega_{2r}+i\omega_{2i}}\right) \tag{19}$$

With Euler's formula, Eq.(19) simplifies to:

$$u(t) = \frac{-i}{2m}\left(\frac{[\cos\omega_{1r}t + i\sin\omega_{1r}t]\times\mathrm{e}^{-\omega_{1i}t}}{\omega_{1r}+i\omega_{1i}} - \frac{[\cos\omega_{2r}t - i\sin\omega_{2r}t]\times\mathrm{e}^{\omega_{2i}t}}{\omega_{2r}+i\omega_{2i}}\right) \tag{20}$$

As obtained by [9], Eq.(20) becomes:

$$u(t) = \frac{-\mathrm{e}^{-\omega_{1i}t}}{2m}\left(\frac{-\sin\omega_{1r}t + i\cos\omega_{1r}t}{\omega_{1r}+i\omega_{1i}} - \frac{\sin\omega_{2r}t + i\cos\omega_{2r}t}{\omega_{2r}+i\omega_{2i}}\right) \tag{21}$$

Extracting the real part of Eq.(21) yields the impulse response $u_{hy}(t)$:

$$u_{hy}(t) = \Re(u(t)) = \frac{e^{-\omega_{1i}t}}{m\left(\omega_{1r}^2 + \omega_{1i}^2\right)}\left(\omega_{1i}\cos\omega_{1r}t - \omega_{1r}\sin\omega_{1r}t\right) \tag{22}$$

Substituting ω_{1r}, ω_{1i}, ω_{2r} and ω_{2i} from Eq.(17) and Eq.(18) into Eq.(22) yields:

$$u_{hy}(t) = \frac{e^{-\omega_0\sqrt[4]{1+\eta^2}\sin\left(\frac{1}{2}\arctan(\eta)\right)t}}{m\omega_0\sqrt[4]{1+\eta^2}}\sin\left[\omega_0\sqrt[4]{1+\eta^2}\cos\left[\frac{1}{2}\arctan(\eta)\right]t - \frac{1}{2}\arctan(\eta)\right] \tag{23}$$

Using Taylor series expansion for Eq.(17) and Eq.(18) when $\eta < 0.6$, we obtain:

$$\omega_{1r} = \omega_{2r} = \omega_0\sqrt[4]{1+\eta^2}\cos\left(\frac{1}{2}\arctan(\eta)\right) \simeq 1 + \frac{\eta^2}{8} \simeq \omega_0\sqrt{1+\frac{\eta^2}{4}}$$

$$\omega_{1i} = -\omega_{2i} = \omega_0\sqrt[4]{1+\eta^2}\sin\left(\frac{1}{2}\arctan(\eta)\right) \simeq \frac{\eta}{2}\omega_0$$

$$\sqrt[4]{1+\eta^2} \simeq 1 + \frac{\eta^2}{4}$$

Substituting the above approximations into Eq.(23), we obtain:

$$u_{hy}(t) = \frac{1}{m\omega_0\left(1+\frac{1}{2}\eta^2\right)}\sin\left[\omega_0\left(\sqrt{1+\frac{1}{2}\eta^2}\right)t - \frac{1}{2}\arctan(\eta)\right]e^{-\frac{1}{2}\eta\omega_0 t} \tag{24}$$

Writing Eq.(24) as a function of the damping rate ξ, such as $\eta = 2\xi$, then:

$$u_{hy}(t) = \frac{1}{m\omega_0\left(1+\xi^2\right)}\sin\left[\omega_0\sqrt{1+\xi^2}t - \frac{1}{2}\arctan(2\xi)\right]e^{-\xi\omega_0 t} \tag{25}$$

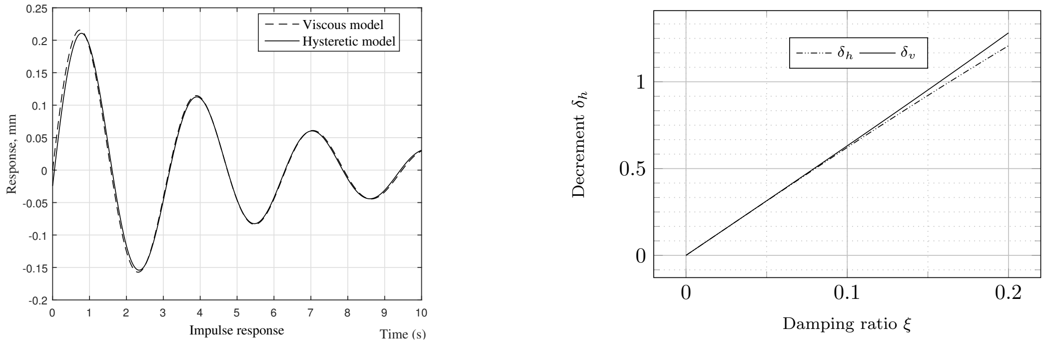
$$u_{hy}(t) = \rho_h\sin(\omega_h t - \phi)\,e^{-\xi\omega_0 t} \tag{26}$$

In Eq.(25), we introduce: $\omega_h = \omega_0\sqrt{1+\xi^2}$, as hysteretic damped frequency, $\rho_h = \dfrac{1}{m\omega_0\left(1+\xi^2\right)}$, as maximal amplitude and $\phi = \dfrac{1}{2}\arctan(2\xi)$, such as non-causal phase angle. By plotting Eq.(26) with the classical impulse response of viscous model, we obtain (Fig.2a). The hysteretic model exhibits a non-causal behaviour. At $t = 0$, we see that the hysteretic model response is ahead of that of the viscous one. It should be noted that, the ignored residual integral of Eq.(15) slightly amplifies the non-causality. We can state that, Eq.(26) faithfully reflects the expected response. The logarithmic decrements δ_h and δ_v of the two impulse responses are given by Eq.(27), shown in (Fig.2b), indicates that the correlation with the viscous model increases when ξ decreases. When $\xi \leq 10\%$ as in civil engineering structures, both logarithmic decrements provide same values.

$$\delta_h = \ln\left(\frac{u_{h(n)}}{u_{h(n+1)}}\right) = \frac{2\pi\xi}{1+\xi^2} \quad ; \quad \delta_v = \ln\left(\frac{u_{v(n)}}{u_{v(n+1)}}\right) = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \tag{27}$$

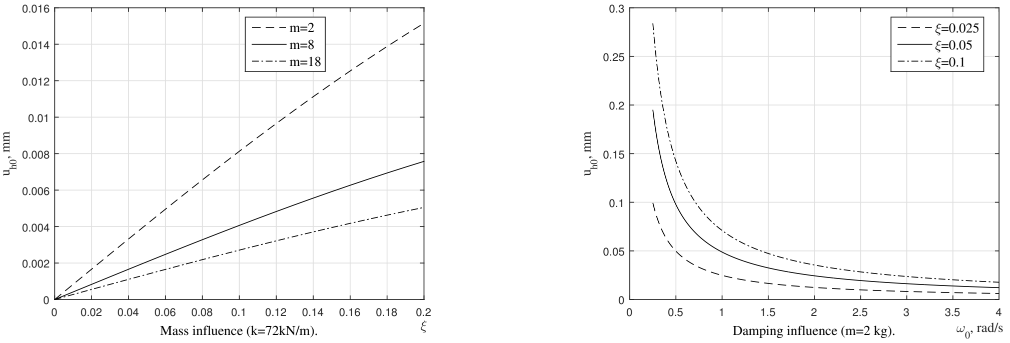
We now examine the parameters governing the non-causal displacement having already been studied in [31]. At $t = 0$, $u_{hy}(t)$ given in Eq.(25) reduces to u_{h0}:

$$u_{h0} = \frac{1}{m\omega_0\left(1+\xi^2\right)}\sin\left(\frac{1}{2}\arctan(2\xi)\right) \tag{28}$$



(a) (b)

Fig. 2. Impulse response and logarithmic decrement



(a) (b)

Fig. 3. Non-causal displacement u_{h0}

This term in Eq.(28), taken in absolute value, represents the non-causal displacement. u_{h0} is inversely proportional to both mass and natural frequency. The effect of m and ω_0 are shown in (Fig.3a) and (Fig.3b), respectively. It clearly appears that non-causality is more apparent in lightweight, flexible and highly damped structures.

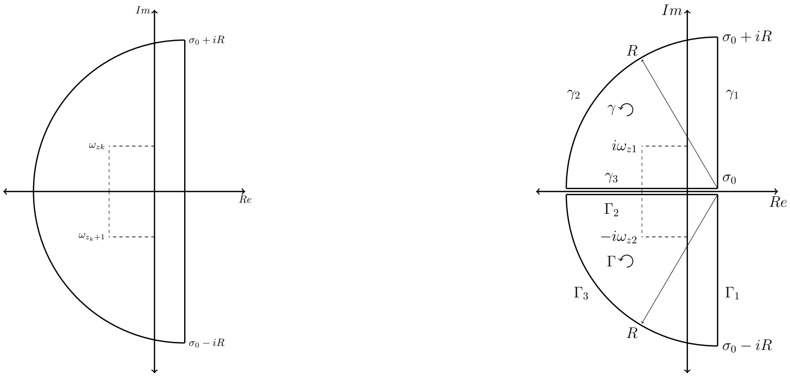
4. **Evaluation of free vibration response**

This response is obtained when $F(t) = 0$ in Eq.(6) (I.C are:u_0 and $\dot{u}_0$). The motion equation is solved with Laplace transforms. So, transfer function $X(s)$ is:

$$TL\left[\ddot{u}(t) + \omega_z u(t)\right] = TL\left[\frac{F(t)}{m}\right] \tag{29}$$

$$\left(s^2 X(s) - s.u_0 - \dot{u}_0\right) + \omega_z^2 X(s) = \frac{F(s)}{m} \tag{30}$$

$$X(s) = \frac{F(s)}{m\left(s^2 + \omega_z^2\right)} + \frac{\dot{u}_0 + su_0}{s^2 + \omega_z^2} \tag{31}$$



(a) Typical Bromwich contour (b) Integration domain for I_a and I_b

Fig. 4. Integration domain for inverse Laplace transform

In free vibration $F(s) = 0$. Inverse Laplace transform $u_f(t)$ is given by:

$$u_f(t) = TL^{-1}\left(\frac{\dot{u}_0}{s^2+\omega_z^2} + \frac{u_0 s}{s^2+\omega_z^2}\right) \tag{32}$$

The mathematical equivalence of Eq.(32) is:

$$u_f(t) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} X(s)\, e^{st} ds = \frac{1}{2\pi i}\lim_{\omega\to\infty}\int_{\sigma_0-i\omega}^{\sigma_0+i\omega}\left(\frac{\dot{u}_0 e^{st}}{s^2+\omega_z^2} + \frac{u_0 s e^{st}}{s^2+\omega_z^2}\right) ds \tag{33}$$

$$u_f(t) = \frac{1}{2\pi i}\left(\dot{u}_0 I_a(t) + u_0 I_b(t)\right) \tag{34}$$

With,

$$I_a(t) = \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{e^{st}\, ds}{s^2+\omega_z^2} = \lim_{\omega\to\infty}\int_{\sigma_0-i\omega}^{\sigma_0+i\omega} \frac{e^{st}\, ds}{s^2+\omega_z^2} \tag{35}$$

$$I_b(t) = \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{s e^{st}\, ds}{s^2+\omega_z^2} = \lim_{\omega\to\infty}\int_{\sigma_0-i\omega}^{\sigma_0+i\omega} \frac{s e^{st}\, ds}{s^2+\omega_z^2} \tag{36}$$

Note that, $\omega_{z1} = \omega_0\sqrt{1+i\eta}$ when $\omega \geq 0$ and $\omega_{z2} = \omega_0\sqrt{1-i\eta}$ when $\omega < 0$. So:

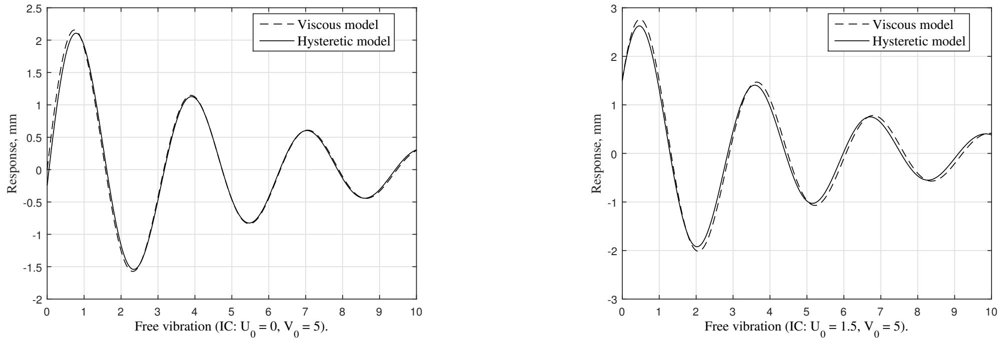
$$I_a = \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{e^{st}\, ds}{s^2+\omega_z^2} = \lim_{\omega\to\infty}\left(\int_{\sigma_0-i\omega}^{\sigma_0-i0} \frac{e^{st}\, ds}{s^2+\omega_{z2}^2} + \int_{\sigma_0+i0}^{\sigma_0+i\omega} \frac{e^{st}\, ds}{s^2+\omega_{z1}^2}\right) = I_{a2}(t) + I_{a1}(t) \tag{37}$$

and

$$I_b = \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{s e^{st}\, ds}{s^2+\omega_z^2} = \lim_{\omega\to\infty}\left(\int_{\sigma_0-i\omega}^{\sigma_0-i0} \frac{e^{st}\, ds}{s^2+\omega_{z2}^2} + \int_{\sigma_0+i0}^{\sigma_0+i\omega} \frac{s e^{st}\, ds}{s^2+\omega_{z1}^2}\right) = I_{b2}(t) + I_{b1}(t) \tag{38}$$

Residue theorem along (γ) and (Γ), sown in (Fig.4), is used to calculate $I_a(t)$ and $I_b(t)$ in Eq.(37) and Eq.(38). The full procedure of contour integration and residue calculus is carried out in *Appendix B*. Eq.(34) becomes:

$$u_f(t) = \frac{i\dot{u}_0}{2}\left(\frac{e^{-\omega_{z2}t}}{\omega_{z2}} - \frac{e^{i\omega_{z1}t}}{\omega_{z1}}\right) + \frac{u_0}{2}\left(e^{-i\omega_{z2}t} + e^{i\omega_{z1}t}\right) \tag{39}$$



(a) (b)

Fig. 5. Free vibration for hysteretic and viscous model, $\xi = 0.05$

The real part of first term in Eq.(39) is obtained in *Section* 2, the second follows after simplification. Thus, the free vibration response $u_{hf}(t)$ leads to:

$$u_{hf}(t) = \left(\frac{\dot{u}_0}{\omega_0\,(1+\xi^2)} \sin\left(\omega_0 \sqrt{1+\xi^2} t - \frac{1}{2} \arctan(2\xi) \right) \right.$$

$$\left. + u_0 \cos\left(\omega_0 \sqrt{1+\xi^2 t} \right) \right) e^{-\,\xi\omega_0 t} \quad (40)$$

As introduced in impulse response, $\omega_h = \omega_0 \sqrt{1+\xi^2}$, $\varphi = \frac{1}{2} \arctan(\eta)$, the free vibration response is given by:

$$u_{hf}(t) = \left(u_0 \cos(\omega_h t) + \frac{\dot{u}_0}{\omega_0\,(1+\xi^2)} \sin(\omega_h t - \varphi\,) \right) e^{-\,\xi\omega_0 t} \quad (41)$$

The response given in above equation is plotted in (Fig.5) with the viscous model one. The figure shows that the two responses remain very similar, regardless of the initial conditions. Non-causality is clearly apparent in the hysteretic model response. The peaks of the hysteretic response are always smaller than those of the viscous response. We also note that if $\phi = 0$, there will be a better correlation with the viscous model. From a theoretical point of view, and in light of the response illustrated in (Fig.5), the hysteretic damping model does accept a free vibration solution, a conclusion that contrasts with the views of several authors who reject this possibility. It's important to note that while the impulse response of the hysteretic model is based on approximations, the free vibration response is accurate and free from such simplications.

5. **Evaluation of the harmonic response**

The harmonic response is evaluated with Duhamel integral using Eq.(26) when $F(t) = p_0 \mathrm{e}^{\imath\bar{\omega}t}$. Assuming that the system is at rest, the response is:

$$u_s = \int_0^t \rho_h \sin(\omega_h \tau - \phi)\, \mathrm{e}^{-\xi\omega_0\tau} p_0 \mathrm{e}^{\imath\bar{\omega}(t-\tau)}\, \mathrm{d}\tau \quad (42)$$

$$u_s = \rho_h p_0 \int_0^t \left(\frac{\mathrm{e}^{i(\omega_h \tau - \phi)} - \mathrm{e}^{-i(\omega_h \tau - \phi)}}{2i} \right) \mathrm{e}^{-\xi \omega_0 \tau} \mathrm{e}^{i\overline{\omega}(t-\tau)} \, \mathrm{d}\tau \tag{43}$$

$$u_s = \frac{\rho_h p_0}{2i} \left[\mathrm{e}^{i(\overline{\omega} t - \phi)} \int_0^t \mathrm{e}^{-\xi \omega_0 \tau + i(\omega_h - \overline{\omega})\tau} \, \mathrm{d}\tau - \mathrm{e}^{i(\overline{\omega} t + \phi)} \int_0^t \mathrm{e}^{-\xi \omega_0 \tau - i(\omega_h + \overline{\omega})\tau} \, \mathrm{d}\tau \right] \tag{44}$$

To simplify the equations, we note: $\alpha = \xi \omega_0$, $A = \omega_h - \overline{\omega}$, $B = \omega_h + \overline{\omega}$, then:

$$u_s = \frac{\rho_h p_0}{2i} \left[\mathrm{e}^{i(\overline{\omega} t - \phi)} \int_0^t \mathrm{e}^{-\alpha \tau + iA\tau} \, \mathrm{d}\tau - \mathrm{e}^{i(\overline{\omega} t + \phi)} \int_0^t \mathrm{e}^{-\alpha \tau - iB\tau} \, \mathrm{d}\tau \right] \tag{45}$$

$$u_s = \frac{\rho_h p_0}{2i} \left(\mathrm{e}^{i(\overline{\omega} t - \phi)} \left[\frac{\mathrm{e}^{(-\alpha + iA)t}}{-\alpha + iA} \right]_0^t + \mathrm{e}^{i(\overline{\omega} t + \phi)} \left[\frac{\mathrm{e}^{-(\alpha + iB)t} - 1}{\alpha + iB} \right]_0^t \right) \tag{46}$$

$$\begin{aligned} u_s = \frac{\rho_h p_0}{2i} \Bigg[& \mathrm{e}^{i(\overline{\omega} t - \phi)} \left(\frac{\mathrm{e}^{-\alpha t} \times \mathrm{e}^{iAt}}{-\alpha + iA} \right) + \mathrm{e}^{i(\overline{\omega} t + \phi)} \left(\frac{\mathrm{e}^{-\alpha t} \times \mathrm{e}^{iBt}}{\alpha + iB} \right) \\ & - \left(\frac{\mathrm{e}^{i(\overline{\omega} t - \phi)}}{-\alpha + iA} \right) - \left(\frac{\mathrm{e}^{i(\overline{\omega} t + \phi)}}{\alpha + iB} \right) \Bigg] \end{aligned} \tag{47}$$

Rationalizing each denominators, Eq.(47):

$$\begin{aligned} u_s = \frac{\rho_h p_0}{2i} \Bigg[& \mathrm{e}^{i(\overline{\omega} t - \phi)} (-\alpha - iA) \left(\frac{\mathrm{e}^{-\alpha t} \times \mathrm{e}^{iAt}}{\alpha^2 + A^2} \right) + \mathrm{e}^{i(\overline{\omega} t + \phi)} (\alpha - iB) \left(\frac{\mathrm{e}^{-\alpha t} \times \mathrm{e}^{iBt}}{\alpha^2 + B^2} \right) \\ & - (-\alpha - iA) \left(\frac{\mathrm{e}^{i(\overline{\omega} t - \phi)}}{\alpha^2 + A^2} \right) - (\alpha - iB) \left(\frac{\mathrm{e}^{i(\overline{\omega} t + \phi)}}{\alpha^2 + B^2} \right) \Bigg] \end{aligned} \tag{48}$$

Using Euler's formula for the expression bellow:

$$\mathrm{e}^{i(\overline{\omega} t - \phi)} = \cos(\overline{\omega} t - \phi) + i \sin(\overline{\omega} t - \phi) \qquad \mathrm{e}^{i(\overline{\omega} t + \phi)} = \cos(\overline{\omega} t + \phi) + i \sin(\overline{\omega} t + \phi)$$

$$\mathrm{e}^{iAt} = \cos(At) + i \sin(At) \qquad \mathrm{e}^{iBt} = \cos(Bt) + i \sin(Bt)$$

The imaginary part $u_{hs}(t)$ of Eq.(48) gives the response to $F(t) = \sin(\overline{\omega} t)$:

$$\begin{aligned} u_{hs}(t) = \frac{\rho_h p_0}{2} \Bigg[& \left(\frac{\alpha \cos(\omega_h t - \phi) - A \sin(\omega_h t - \phi)}{\alpha^2 + A^2} - \frac{\alpha \cos(\omega_h t - \phi) - B \sin(\omega_h t - \phi)}{\alpha^2 + B^2} \right) \mathrm{e}^{-\alpha t} \\ & - \frac{\alpha \cos(\overline{\omega} t - \phi) - A \sin(\overline{\omega} t - \phi)}{\alpha^2 + A^2} + \frac{\alpha \cos(\overline{\omega} t + \phi) + B \sin(\overline{\omega} t + \phi)}{\alpha^2 + B^2} \Bigg] \end{aligned} \tag{49}$$

To get the full response, including the transient part, we add Eq.(41) to Eq.(49):

$$\begin{aligned} u_{hs}(t) = & \left(u_0 \cos(\omega_h t) + \frac{\dot{u}_0}{\omega_0 (1 + \xi^2)} \sin(\omega_h t - \varphi) \right) e^{-\xi \omega_0 t} \\ & + \frac{\rho_h p_0}{2} \Bigg[\left(\frac{\alpha \cos(\omega_h t - \phi) - A \sin(\omega_h t - \phi)}{\alpha^2 + A^2} - \frac{\alpha \cos(\omega_h t - \phi) - B \sin(\omega_h t - \phi)}{\alpha^2 + B^2} \right) \mathrm{e}^{-\alpha t} \\ & - \frac{\alpha \cos(\overline{\omega} t - \phi) - A \sin(\overline{\omega} t - \phi)}{\alpha^2 + A^2} + \frac{\alpha \cos(\overline{\omega} t + \phi) + B \sin(\overline{\omega} t + \phi)}{\alpha^2 + B^2} \Bigg] \end{aligned} \tag{50}$$

For reference, the complete harmonic response of the viscous model is given by:

$$\begin{aligned} u_v(t) = &\left(u_0\cos\left(\omega_d t\right) + \frac{\dot{u}_0 + \omega_0 \xi u_0}{\omega_d}\sin\left(\omega_d t\right)\right) e^{-\ \xi\omega_0 t} \\ &+ \frac{p_0}{k}\frac{2\xi\beta}{(1-\beta^2)^2+(2\xi\beta)^2}\left(\left(1-\beta^2\right)\sin\overline{\omega}t - 2\xi\beta\cos\overline{\omega}t\right) \\ &+ \frac{p_0}{k}\frac{e^{-\ \xi\omega_0 t}}{(1-\beta^2)^2+(2\xi\beta)^2}\left[2\xi\beta\cos\left(\omega_d t\right) + \frac{\omega_0}{\omega_d}\left(2\beta\xi^2 - \beta\left(1-\beta^2\right)\right)\sin\left(\omega_d t\right)\right] \end{aligned} \tag{51}$$

At resonance, the above equation becomes:

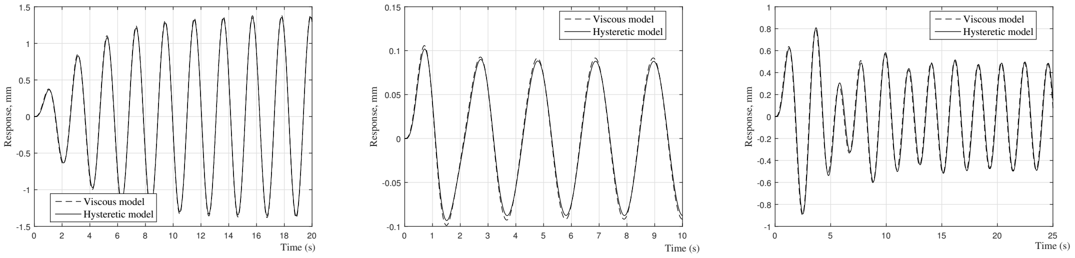
$$\begin{aligned} u_v(t) = &\left(u_0\cos\left(\omega_d t\right) + \frac{\dot{u}_0 + \omega_0 \xi u_0}{\omega_d}\sin\left(\omega_d t\right)\right) \\ &+ \frac{1}{2\xi}\frac{p_0}{k}\left(\left(\frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_d t) + \cos(\omega_d t)\right)\mathrm{e}^{-\xi\omega_0 t} - \cos\left(\omega_0 t\right)\right) \end{aligned} \tag{52}$$

(Fig.6) is plotted form Eq.(50), Eq.(51) and Eq.(52). It shows the harmonic response of the hysteretic and viscous models for a system initially at rest. Whether the system is in resonance, underdamped or overdamped, the harmonic response is always causal, as it observed in (Fig.7), near $t = 0$. Moreover, we observe a better correlation with the viscous model when the non-causal phase angle $\phi = 0$. (Fig.8) confirms these observations near $t = 0$. We therefore conclude that the contribution of the non-causal phase angle to the harmonic response is insignificant. Consequently, to make Eq.(50) more compact, we ignore ϕ, with no perceptible effect on the harmonic response. This yields Eq.(53), which is the final form of the harmonic response when $F(t) = \sin(\overline{\omega}t)$. We note that, if we consider initial conditions u_0 and $\dot{u}_0$, the harmonic response exhibit same causal characteristics and correlation with the viscous model. We also note that the transient part of the response is less correlated, to viscous model response, than the permanent part.

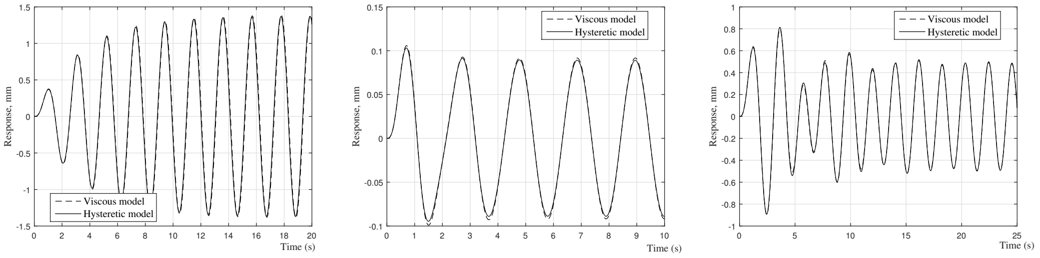
$$\begin{aligned} u_{hs}(t) = &\left(u_0\cos\left(\omega_h t\right) + \frac{\dot{u}_0}{\omega_0\left(1+\xi^2\right)}\sin\left(\omega_h t\right)\right) e^{-\ \xi\omega_0 t} \\ &+ \frac{\rho_h p_0}{2}\left[\left(\frac{\alpha\cos(\omega_h t) - A\sin(\omega_h t)}{\alpha^2+A^2} - \frac{\alpha\cos(\omega_h t) - B\sin(\omega_h t)}{\alpha^2+B^2}\right)\mathrm{e}^{-\alpha t}\right. \\ &\left. - \frac{\alpha\cos(\overline{\omega}t) - A\sin(\overline{\omega}t)}{\alpha^2+A^2} + \frac{\alpha\cos(\overline{\omega}t) + B\sin(\overline{\omega}t)}{\alpha^2+B^2}\right] \end{aligned} \tag{53}$$

6. **Conclusion**

This work focused on the time-domain transcription of the hysteretically damped model. An analytical approach that provided exact solutions in the time domain for a hysteretically damped SDOF system, was presented addressing the three common cases: impulse loading, harmonic excitation, and free vibration.

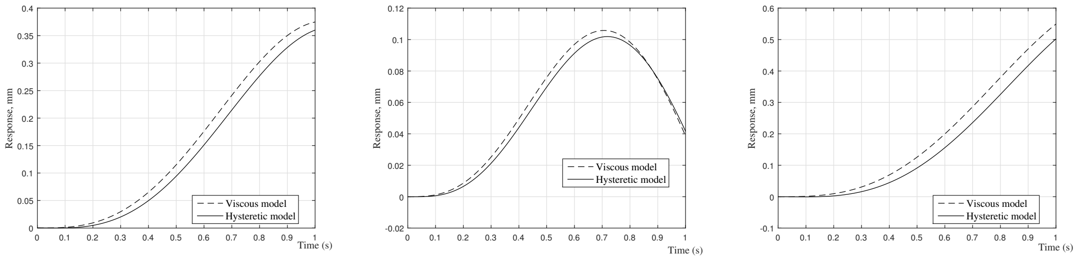


(a) $\beta = 1, \phi = \frac{1}{2}\arctan 2\xi$ (b) $\beta \leq 1, \phi = \frac{1}{2}\arctan 2\xi$ (c) $\beta \geq 1, \phi = \frac{1}{2}\arctan 2\xi$



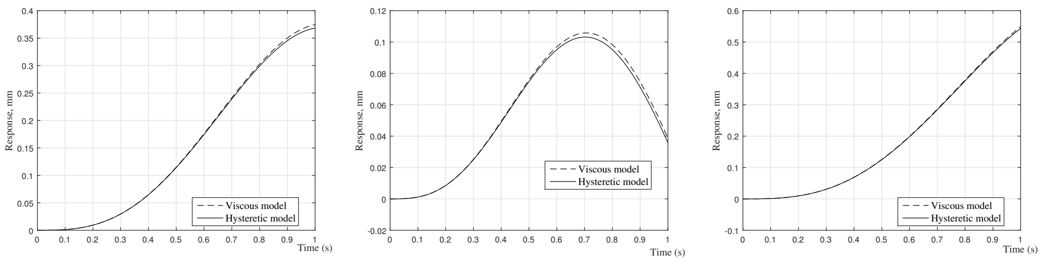
(d) $\beta = 1, \phi = 0$ (e) $\beta \leq 1, \phi = 0$ (f) $\beta \geq 1, \phi = 0$

Fig. 6. Harmonic response, $u_0 = 0, \dot{u}_0 = 0$



(a) $\beta = 1$ (b) $\beta \leq 1$ (c) $\beta \geq 1$

Fig. 7. Harmonic response, $u_0 = 0, \dot{u}_0 = 0, \phi = \frac{1}{2}\arctan(2\xi)$



(a) $\beta = 1$ (b) $\beta \leq 1$ (c) $\beta \geq 1$

Fig. 8. Harmonic response, $u_0 = 0, \dot{u}_0 = 0, \phi = 0$

The impulse and free vibration responses are non-causal. We retained that non-causality in impulse response is more apparent in cases of flexible, highly damped and lightweight structures.

The existence of a free vibration solution for the hysteretic model is demonstrated unlike the generalized statement in the literature. Although, the free vibration for the hysteretic model is questioned, the obtained response agrees with the viscous model as long as the damping ratio remains low. It follows that the hysteretic model accepts a free vibration solution. The obtained response is exact and don't contain a residual integral such as the impulse response.When the non-causal phase angle ϕ is zero, we noted a better correlation with the viscous model response.

The complete harmonic response for a sinusoidal force is obtained. It allows accurate prediction of the oscillator's transient and steady-state responses. By omitting the non-causal phase angle ϕ from the solution allows for better correlation with the viscous model.

Appendix A. **Integrals for impulse response**

$I_{z1}(t)$ given by Eq.(12) is evaluated by parametrizing z over the contour (γ):

$$I_{z1}(t)=\oint_{\gamma} f(z)dz=\int_0^{+\infty}\frac{\mathrm{e}^{ixt}\,\mathrm{d}x}{-x^2+\omega_{z1}^2}+\int_{\gamma 2}\frac{\mathrm{e}^{izt}\,\mathrm{d}z}{-z^2+\omega_{z1}^2}+i\int_{+\infty}^{0}\frac{\mathrm{e}^{-yt}\,\mathrm{d}y}{y^2+\omega_{z1}^2} \tag{54}$$

Since the following two properties are satisfied:
(i):γ_2 is the circular arc in the upper half-plane,
(ii): $\lim\limits_{z\to+\infty} h_1(z)=\lim\limits_{R\to+\infty}\left(\dfrac{1}{-z^2+\omega_{z1}^2}\right)=0.$

Thus, *Jordan's Lemma* [27, 29, 25], states that: $\lim_{R\longrightarrow+\infty}\int_{\gamma 2}\left(\frac{\mathrm{e}^{izt}}{-z^2+\omega_{z1}^2}\right)\,\mathrm{d}z=0$

$f(z)$ has two poles at the roots of the denominator. Only ω_{z1}, located within (γ) contributes. Residue $f(z)$ at is ω_{z1} is given by:

$\mathrm{Res}\,(f,\omega_{z1})=\lim\limits_{R\to\omega_{z1}}(z-\omega_{z1})\left(\dfrac{\mathrm{e}^{izt}}{-z^2+\omega_{z1}^2}\right)\Rightarrow \mathrm{Res}\,(f,\omega_{z1})=-\dfrac{\mathrm{e}^{i\omega_{z1}t}}{2\omega_{z1}}$

From Eq.(54), $I_1(t)$ yields to:

$$I_1(t)=\int_0^{+\infty}\left(\frac{\mathrm{e}^{i\omega t}}{-\omega^2+\omega_{z1}^2}\right)\mathrm{d}\omega=-\pi i\frac{\mathrm{e}^{i\omega_{z1}t}}{\omega_{z1}}+i\int_{+\infty}^{0}\frac{\mathrm{e}^{-yt}}{y^2+\omega_{z1}^2}\,\mathrm{d}y \tag{55}$$

$I_{z2}(t)$ given by Eq.(13) is evaluated by parametrizing z over the contour (Γ):

$$I_{z2}(t)=\oint_{\Gamma}\frac{\mathrm{e}^{-izt}\,\mathrm{d}z}{-z^2+\omega_{z2}^2}=\int_0^{+\infty}\frac{\mathrm{e}^{-ixt}\,\mathrm{d}x}{-x^2+\omega_{z2}^2}+\int_{\Gamma 2}\frac{\mathrm{e}^{-izt}\,\mathrm{d}z}{-z^2+\omega_{z2}^2}+i\int_{-\infty}^{0}\frac{\mathrm{e}^{yt}\,\mathrm{d}y}{y^2+\omega_{z2}^2} \tag{56}$$

$I_1(t)$ procedure applies to $I_2(t)$ in lower half-plane, from Eq.(54), $I_2(t)$ leads to:

$$I_2(t)=\int_0^{+\infty}\left(\frac{\mathrm{e}^{-i\omega t}}{-\omega^2+\omega_{z2}^2}\right)\mathrm{d}\omega=\pi i\frac{\mathrm{e}^{-i\omega_{z2}t}}{\omega_{z2}}+i\int_0^{+\infty}\frac{\mathrm{e}^{-yt}}{y^2+\omega_{z2}^2}\,\mathrm{d}y \tag{57}$$

Substituting $I_1(t)$ and $I_2(t)$ in Eq.(11), inverse Fourier transform of $H(i\omega)$ yields to:

$$u(t) = \frac{i}{2m}\left(\frac{\mathrm{e}^{-i\omega_{z2}t}}{\omega_{z2}} - \frac{\mathrm{e}^{i\omega_{z1}t}}{\omega_{z1}}\right) - \frac{i}{2\pi m}\left(\int_{+\infty}^{0} \frac{\mathrm{e}^{-yt}\ \mathrm{d}y}{y^2+\omega_{z1}^2} + \int_{0}^{+\infty} \frac{\mathrm{e}^{-yt}\ \mathrm{d}y}{y^2+\omega_{z2}^2}\right) \tag{58}$$

Appendix B. Integrals for free vibration

(1) **Evaluation of $I_a(t)$:** From Eq.(37), $I_{a1}(t)$ is writing as follows:

$$I_{a1}(t) = \lim_{\omega\to\infty} \int_{\sigma_0+i0}^{\sigma_0+i\omega} f_1(s)ds = \lim_{\omega\to\infty} \int_{\sigma_0+i0}^{\sigma_0+i\omega} \frac{e^{st}\ ds}{s^2+\omega_{z1}^2} \tag{59}$$

Considering contour integral of $f_1(s)$ along γ as shown in (Fig.4), we obtain:

$$\oint_{\gamma} f_1(s)\ \mathrm{d}s = \int_{\gamma_1} f_1(s)ds + \int_{\gamma_2} f_1(s)ds + \int_{\gamma_3} f_1(s)ds = 2\pi i \sum_{k=1}^{m} \mathrm{Res}\left[f_1(\omega_{zk})\right] \tag{60}$$

Along γ, s is parametrizing as below:
-Along γ_1 : $\int_{\gamma_1} f_1(s)\ \mathrm{d}s = I_{a1}(t)$
-Along γ_2: $y = 0, \theta = -\pi, s = x\mathrm{e}^{-i\pi} = -x$, so $ds = -\ dx$,
-Along γ_3 : $s = R\mathrm{e}^{i\theta}$, $ds = iR\mathrm{e}^{i\theta}\ d\theta$
Eq.(60) leads to:

$$I_{a1}(t) + \int_{\pi/2}^{\pi} \frac{e^{R\mathrm{e}^{i\theta}t}}{R^2\mathrm{e}^{i2\theta}+\omega_{z1}^2} iR\mathrm{e}^{i\theta}\ d\theta - \lim_{R\to+\infty} \int_{-R}^{\sigma_0} \frac{e^{-xt}\ dx}{x^2+\omega_{z1}^2} = 2\pi i Res\left[f_1(\omega_{z1})\right] \tag{61}$$

The second term in Eq.(61) will vanish by Jordan's Lemma. Since $f_1(s)$ has two poles at $\pm i\omega_{z1}$, only $s = i\omega_{z1}$ inside γ contributes, thus we obtain:
$Res\left[f_1(\omega_{z1})\right] = \lim_{s\to i\omega_{z1}} (s - i\omega_{z1}) \dfrac{e^{st}}{s^2+\omega_{z1}^2} = \dfrac{\mathrm{e}^{i\omega_{z1}t}}{2i\omega_{z1}}$
Eq.(61) leads to:

$$I_{a1}(t) - \lim_{R\to+\infty} \int_{-R}^{\sigma_0} \frac{e^{-xt}\ dx}{x^2+\omega_{z1}^2} = 2\pi i\left(\frac{\mathrm{e}^{i\omega_{z1}t}}{2i\omega_{z1}}\right) = \frac{\pi\mathrm{e}^{i\omega_{z1}t}}{\omega_{z1}} \tag{62}$$

$I_{a2}(t)$ is evaluated identically to $I_{a1}(t)$.From Eq.(37), $I_{a2}(t)$ is as follows:

$$I_{a2}(t) = \lim_{\omega\to\infty} \int_{\sigma_0-i\omega}^{\sigma_0+i0} f_2(s)ds = \lim_{\omega\to\infty} \int_{\sigma_0-i\omega}^{\sigma_0+i0} \frac{e^{st}\ ds}{s^2+\omega_{z2}^2} \tag{63}$$

Considering contour integral of $f_2(s)$ along (Γ) as shown in (Fig.4), we obtain:

$$\oint_{\Gamma} f_2(s)\ \mathrm{d}s = \int_{\Gamma_1} f_2(s)ds + \int_{\Gamma_2} f_2(s)ds + \int_{\Gamma_3} f_2(s)ds = 2\pi i \sum_{k=1}^{m} \mathrm{Res}\left[f_2(\omega_{zk})\right] \tag{64}$$

Eq.(64) leads to:

$$I_{a2}(t) + \lim_{R\to+\infty} \int_{-R}^{\sigma_0} \frac{e^{-xt}\ dx}{x^2+\omega_{z2}^2} = 2\pi i\left(\frac{\mathrm{e}^{-i\omega_{z2}t}}{-2i\omega_{z2}}\right) = -\frac{\pi\mathrm{e}^{-i\omega_{z2}t}}{\omega_{z2}} \tag{65}$$

(2) **Evaluation of $I_b(t)$:** From Eq.(38), $I_{b1}(t)$ is writing as follows::

$$I_{b1}(t) = \lim_{\omega \to \infty} \int_{\sigma_0+i0}^{\sigma_0+i\omega} g_1(s)ds = \lim_{\omega \to \infty} \int_{\sigma_0+i0}^{\sigma_0+i\omega} \frac{se^{st}\ ds}{s^2+\omega_{z1}^2} \tag{66}$$

Considering contour integral of $(g_1(s))$ along (Γ) as shown in (Fig.4), we obtain:

$$\oint_{\gamma} g_1(s)\ \mathrm{d}s = \int_{\gamma_1} g_1(s)ds + \int_{\gamma_2} g_1(s)ds + \int_{\gamma_3} g_1(s)ds = 2\pi i \sum_{k=1}^{m} \mathrm{Res}\left[g_1\left(\omega_{zk}\right)\right] \tag{67}$$

Using the same parametrization of s as for $I_{a1}(t)$, Eq.(67)reduces to:

$$I_{b1}(t) + \lim_{R \to +\infty} \int_{\pi/2}^{\pi} \frac{R\mathrm{e}^{i\theta} e^{R\mathrm{e}^{i\theta} t}}{R^2\mathrm{e}^{i2\theta}+\omega_{z1}^2} iR\mathrm{e}^{i\theta}\ d\theta + \lim_{R \to +\infty} \int_{-R}^{\sigma_0} \frac{xe^{-xt}\ dx}{x^2+\omega_{z1}^2} = 2\pi i Res\left[g_1\left(\omega_{z1}\right)\right] \tag{68}$$

The second term in Eq.(69) will vanish by Jordan's Lemma. Since $g_1(s)$ has two poles at $\pm i\omega_{z1}$, only $s = i\omega_{z1}$ inside (γ) contributes, thus we obtain:
$Res\left[g_1\left(\omega_{z1}\right)\right] = \lim_{s \to i\omega_{z1}} \left(s - i\omega_{z1}\right) \dfrac{se^{st}}{s^2+\omega_{z1}^2} = \dfrac{\mathrm{e}^{i\omega_{z1}t}}{2}$
Eq.(69) leads to:

$$I_{b1}(t) + \lim_{R \to +\infty} \int_{-R}^{\sigma_0} \frac{-xe^{-xt}\ dx}{x^2+\omega_{z1}^2} = 2\pi i \left(\frac{\mathrm{e}^{i\omega_{z1}t}}{2}\right) = \pi i \mathrm{e}^{i\omega_{z1}t} \tag{69}$$

$I_{b2}(t)$ is evaluated same as $I_{b1}(t)$. From Eq.(38), $I_{b2}(t)$ is as following:

$$I_{b2}(t) = \lim_{\omega \to \infty} \int_{\sigma_0+i0}^{\sigma_0+i\omega} g_2(s)ds = \lim_{\omega \to \infty} \int_{\sigma_0+i0}^{\sigma_0+i\omega} \frac{se^{st}\ ds}{s^2+\omega_{z2}^2} \tag{70}$$

Considering contour integral of $g_2(s)$ along (Γ) as shown in (Fig.4), we obtain:

$$\oint_{\Gamma} g_2(s)\ \mathrm{d}s = \int_{\Gamma_1} g_2(s)ds + \int_{\Gamma_2} g_2(s)ds + \int_{\Gamma_3} g_2(s)ds = 2\pi i \sum_{k=1}^{m} \mathrm{Res}\left[g_2\left(\omega_{zk}\right)\right] \tag{71}$$

After performing the same procedure as for $I_{b1}(t)$, we obtain:

$$I_{b2}(t) + \lim_{R \to +\infty} \int_{\sigma_0}^{-R} \frac{-xe^{-xt}\ dx}{x^2+\omega_{z2}^2} = 2\pi i \left(\frac{\mathrm{e}^{-ia_1t}}{2}\right) = \pi i \mathrm{e}^{-i\omega_{z2}t} \tag{72}$$

Combining Eq.(62) and Eq.(65), cancels the second terms. The same holds for Eq.(69) and Eq.(72). Finally, $I_a(t)$ and I_b yields to:

$$I_a(t) = I_{a1}(t) + I_{a2}(t) = \frac{\pi\mathrm{e}^{i\omega_{z1}t}}{\omega_{z1}} - \frac{\pi\mathrm{e}^{-i\omega_{z2}t}}{\omega_{z2}} \tag{73}$$

$$I_b(t) = I_{b1}(t) + I_{b2}(t) = \pi i \mathrm{e}^{i\omega_{z1}t} + \pi i \mathrm{e}^{-i\omega_{z2}t} \tag{74}$$

According to Eq.(34), the time domain response yields to:

$$u_f(t) = \frac{i\dot{u}_0}{2}\left(\frac{e^{-i\omega_{z2}t}}{\omega_{z2}} - \frac{e^{i\omega_{z1}t}}{\omega_{z1}}\right) + u_0\left(\frac{e^{-i\omega_{z2}t}}{2} + \frac{e^{i\omega_{z1}t}}{2}\right) \tag{75}$$

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