

ON THE ARITHMETIC PROPERTIES OF PARTITIONS INTO PARTS SIMULTANEOUSLY 4-REGULAR AND 9-DISTINCT

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In 2017, Keith presented a survey of several restricted partition functions. Following this, the authors initiated an arithmetic and combinatorial study of partitions that are both ℓ -regular (parts indivisible by ℓ) and t -distinct (parts appearing fewer than t times). Denoting the number of such partitions of n by $RD^{(\ell,t)}(n)$, Ramanujan-type congruences were obtained for various pairs (ℓ, t) . In this paper we establish infinite families of Ramanujan-type congruences modulo 4, 6 and 12 for $RD^{(4,9)}(n)$ together with a collection of congruences modulo 24.

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1. Introduction

A *partition* of a nonnegative integer n is a finite sequence of positive integers which sum up to n . We denote a partition λ by $(\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \dots, \lambda_k^{\alpha_k})$ if $\lambda_1 > \lambda_2 > \dots > \lambda_k$ are the distinct positive integers (called parts) appearing in λ . Each part λ_i appears exactly α_i times for $1 \leq i \leq k$ and $\alpha_i \geq 1$. Here and throughout we shall use the following standard q -series notation.

$$(a; q)_n := \begin{cases} \prod_{i=0}^{n-1} (1 - aq^i) & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

Moreover, $(a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n$ and f_k is defined by

$$f_k := (q^k; q^k)_\infty = \prod_{n \geq 1} (1 - q^{nk}).$$

For a positive integer $\ell > 1$, a partition is called ℓ -regular if none of the parts are divisible by ℓ . Let $b_\ell(n)$ denote the number of ℓ -regular partitions of

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n . The generating function for $b_\ell(n)$ satisfies the identity

$$\sum_{n \geq 0} b_\ell(n)q^n = \frac{f_\ell}{f_1}.$$

The arithmetic of $b_\ell(n)$ has been studied by many authors. For example, see [3, 5, 8, 10, 20, 21, 22].

For $t > 1$, a partition λ is said to be an ℓ -distinct partition if $\alpha_i < t$ for all $1 \leq i \leq k$. One can show by elementary techniques that the number of ℓ -distinct partitions of n is equal to $b_\ell(n)$. For more details on ℓ -distinct partitions, we refer the reader to [6, 7].

Denote by $\text{RD}^{(\ell,t)}(n)$ the number of partitions of n that are simultaneously ℓ -regular and t -distinct. For example, $\text{RD}^{(4,9)}(n) = 9$ since the partitions of 6 into parts simultaneously 4-regular and 9-distinct are

$$(6), (5, 1), (3^2), (3, 2, 1), (3, 1^3), (2^3), (2^2, 1^2), (2, 1^4), (1^6).$$

This class of partitions was first introduced by Keith in [11] and has since been extensively studied by various authors. For example, see [1, 2, 12, 14, 15, 17, 18]. The generating function for $\text{RD}^{(\ell,t)}(n)$ satisfies the identity

$$\sum_{n \geq 0} \text{RD}^{(\ell,t)}(n)q^n = \frac{f_t f_\ell}{f_1 f_{t\ell}}. \quad (1)$$

In [13] the authors explored these partitions combinatorially and arithmetically, and established the following Ramanujan-type congruences.

Theorem 1.1 (Nadji and Ahmia, 2025). *For all $n \geq 0$, we have*

$$\text{RD}^{(4,9)}(4n + 3) \equiv 0 \pmod{3},$$

$$\text{RD}^{(4,9)}(6n + 2) \equiv 0 \pmod{2},$$

$$\text{RD}^{(4,9)}(6n + 3) \equiv 0 \pmod{3},$$

$$\text{RD}^{(4,9)}(6n + 4) \equiv 0 \pmod{4},$$

$$\text{RD}^{(4,9)}(6n + 5) \equiv 0 \pmod{6},$$

$$\text{RD}^{(4,9)}(6n + 7) \equiv 0 \pmod{12},$$

$$\text{RD}^{(4,9)}(8n + 5) \equiv 0 \pmod{6}.$$

In this paper we establish infinite families of Ramanujan-type congruences for $\text{RD}^{(4,9)}(n)$ modulo 4, 6, and 12, as well as several congruences modulo 24. The rest of the paper is organized as follows. We collect in Section 2 the essential preliminaries needed for the proofs in the subsequent sections. In Section 3, we establish a family of congruences modulo 4. Sections 4 and 5 are devoted to two infinite families of congruences modulo 6 and 12, respectively. Finally, in Section 6, we present several Ramanujan-type congruences modulo 24.

2. Preliminary results

In this section we list a few dissection formulas which are useful in proving our main results. *Ramanujan's general theta function* $f(a, b)$ [4, p. 34, (18.1)] is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}.$$

Two special cases are

$$\psi(q) := f(q; q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} = \frac{f_2^2}{f_1},$$

and

$$f(-q) := f(-q^2; -q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q; q)_{\infty} = f_1.$$

Lemma 2.1. *The following 2-dissections hold.*

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q \frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \quad (2)$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4}. \quad (3)$$

Xia and Yao [23] proved Equation (2) and Hirschhorn et al. [9] proved Equation (3).

The following result appears in the work of Ramanujan [16, p. 212].

Lemma 2.2. *We have the 5-dissection*

$$f_1 = f_{25}(a - q - q^2/a), \quad (4)$$

where

$$a := (q^{10}, q^{15}; q^{25})_{\infty} / (q^5, q^{20}; q^{25})_{\infty}.$$

The next two results are due to Cui and Gu [5, Thm. 2.1 and Thm. 2.2].

Lemma 2.3. *For any odd prime p , we have*

$$\psi(q) = \sum_{k=0}^{(p-3)/2} q^{k(k+1)/2} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}). \quad (5)$$

Furthermore, $\frac{m^2+m}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}$ for $0 \leq m \leq (p-3)/2$.

Lemma 2.4. *For any prime $p \geq 5$, we have*

$$f_1 = \sum_{\substack{k=(1-p)/2 \\ k \neq \pm \frac{p-1}{6}}}^{(p-1)/2} (-1)^k q^{k(3k+1)/2} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}, \quad (6)$$

where

$$\frac{\pm p - 1}{6} = \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod{6}, \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod{6}. \end{cases}$$

Lemma 2.5. For all primes p and all $k, m \geq 1$, we have

$$f_{pm}^{p^k-1} \equiv f_m^{p^k} \pmod{p^k}. \quad (7)$$

The following generating function dissections were proved by the authors in [13].

Theorem 2.1 (Nadji and Ahmia [13], 2025). We have

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(2n)q^n = \frac{f_6^7 f_9^7}{f_3^9 f_{18}^5} + 2q \frac{f_6^6 f_9^4}{f_3^8 f_{18}^2} + 4q^2 \frac{f_6^5 f_9 f_{18}}{f_3^7}, \quad (8)$$

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n+2)q^n = 2 \frac{f_2^6 f_3^4}{f_1^8 f_6^2}, \quad (9)$$

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n+4)q^n = 4 \frac{f_2^5 f_3 f_6}{f_1^7}, \quad (10)$$

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n+5)q^n = 6 \frac{f_2^2 f_3^2 f_6^2}{f_1^6}. \quad (11)$$

Let p be any odd prime and δ be any integer relatively prime to p . The Legendre symbol $\left(\frac{\delta}{p}\right)$ is defined by

$$\left(\frac{\delta}{p}\right) = \begin{cases} 1, & \text{if } \delta \text{ is a quadratic residue of } p, \\ -1, & \text{if } \delta \text{ is a quadratic non-residue of } p. \end{cases}$$

3. Congruences for $\text{RD}^{(4,9)}(n)$ modulo 4

Theorem 3.1. Let $p \equiv 3 \pmod{4}$ be a prime, $1 \leq i \leq p-1$ and $\alpha \geq 0$. Then for all $n \geq 0$ we have

$$\text{RD}^{(4,9)}\left(12p^{2\alpha+1}(pn+i) + p^{2\alpha+2} + 1\right) \equiv 0 \pmod{4}.$$

Proof. In view of (7) and (8), with $p = 2$ and $k = 1$, we have

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(2n)q^n = \frac{f_6^7 f_9^7}{f_3^9 f_{18}^5} + 2q \frac{f_6^6 f_9^4}{f_3^8 f_{18}^2} + 4q^2 \frac{f_6^5 f_9 f_{18}}{f_3^7} \equiv \frac{f_6^7 f_9^7}{f_3^9 f_{18}^5} + 2q f_6^2 \pmod{4}.$$

Extracting the terms where the exponent of q is congruent to 1 modulo 3 from both sides of the above congruence yields

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n+2)q^n \equiv 2f_6^2 \pmod{4},$$

and since the right hand side of this congruence is a series in q^2 , it follows that

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(12n+2)q^n \equiv 2f_1^2 \pmod{4}. \quad (12)$$

Define $a(n)$ for $n \geq 0$ by

$$\sum_{n \geq 0} a(n)q^n = f_1^2. \quad (13)$$

Combining Equations (12) and (13), we deduce that

$$\text{RD}^{(4,9)}(12n+2) \equiv 2a(n) \pmod{4}. \quad (14)$$

Let $-(p-1)/2 \leq k, m \leq (p-1)/2$. Then the congruence

$$\frac{3k^2+k}{2} + \frac{3m^2+m}{2} \equiv \frac{p^2-1}{12} \pmod{p}$$

is equivalent to

$$(6k+1)^2 + (6m+1)^2 \equiv 0 \pmod{p}. \quad (15)$$

Since $p \equiv 3 \pmod{4}$, we have $(\frac{-1}{p}) = -1$, which implies that (15) holds if and only if $k = m = (\pm p - 1)/6$. Substituting (6) into (13) and then extracting the terms in which the power of q is congruent to $(p^2 - 1)/12$ modulo p , and then divide by $q^{\frac{p^2-1}{12}}$ yields

$$\sum_{n \geq 0} a\left(pn + \frac{p^2-1}{12}\right) q^{pn} = f_{p^2}^2.$$

Therefore,

$$\sum_{n \geq 0} a\left(p^2n + \frac{p^2-1}{12}\right) q^n = f_1^2,$$

and for all $n \geq 0$,

$$a\left(p^2n + pi + \frac{p^2-1}{12}\right) = 0, \quad (16)$$

where $1 \leq i \leq p-1$. By induction on (16) we can prove that for $n \geq 0$ and $\alpha \geq 0$,

$$a\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{12}\right) = a(n). \quad (17)$$

Replacing n by $p^2n + pi + \frac{p^2-1}{12}$ ($1 \leq i \leq p-1$) in Equation (17) and using (16), we find that for all nonnegative integers n and α ,

$$a\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{12}\right) = 0.$$

Finally, replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{12}$ in Equation (14) ($1 \leq i \leq p-1$) we obtain the desired result. \square

4. Congruences for $\text{RD}^{(4,9)}(n)$ modulo 6

Theorem 4.1. *For all integers $\alpha \geq 0$ and $n \geq 0$,*

$$\text{RD}^{(4,9)}(6 \cdot 5^{2\alpha+2}n + 6 \cdot 5^{2\alpha+1}i + 5^{2\alpha+2} + 1) \equiv 0 \pmod{6}, \quad (18)$$

where $1 \leq i \leq 4$.

Proof. By (7) and (9), with $p = 3$ and $k = 1$, we arrive at

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n + 2)q^n = 2 \frac{f_2^6 f_3^4}{f_1^8 f_6^2} \equiv 2f_1^4 \pmod{6}. \quad (19)$$

Employing (4) into (19), and then extracting the terms of the form q^{5n+4} from both sides of the resulting equation, we get

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(30n + 26)q^n \equiv 2f_5^4 \pmod{6}, \quad (20)$$

which implies that

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(150n + 26)q^n \equiv 2f_1^4 \pmod{6}. \quad (21)$$

From (19) and (21), we conclude that

$$\text{RD}^{(4,9)}(6n + 2) \equiv \text{RD}^{(4,9)}(150n + 26) \pmod{6},$$

for all $n \geq 0$, from which it follows by induction that for all $\alpha \geq 0$,

$$\text{RD}^{(4,9)}(6n + 2) \equiv \text{RD}^{(4,9)}(6 \cdot 5^{2\alpha+2}n + 5^{2\alpha+2} + 1) \pmod{6}. \quad (22)$$

Finally, (20) also yields that

$$\text{RD}^{(4,9)}(150n + 30i + 26) \equiv 0 \pmod{6} \quad (23)$$

for $1 \leq i \leq 4$, and (18) follows immediately from (22) and (23). \square

5. Congruences for $\text{RD}^{(4,9)}(n)$ modulo 12

Theorem 5.1. *For any prime $p \equiv 5 \pmod{6}$, $\alpha \geq 0$, and $n \geq 0$, we have*

$$\text{RD}^{(4,9)}(6p^{2\alpha+1}(pn + i) + 3p^{2\alpha+2} + 1) \equiv 0 \pmod{12},$$

for all $1 \leq i \leq p - 1$.

Proof. In view of (7) and (10), with $p = 3$ and $k = 1$, we find that

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n + 4)q^n = 4 \frac{f_2^5 f_3 f_6}{f_1^7} \equiv 4 \frac{f_2^2 f_6^2}{f_1 f_3} = 4\psi(q)\psi(q^3) \pmod{12}. \quad (24)$$

Define $b(n)$ for $n \geq 0$ by

$$\sum_{n \geq 0} b(n)q^n = \psi(q)\psi(q^3). \quad (25)$$

From Equations (24) and (25) we have

$$\text{RD}^{(4,9)}(6n + 4) \equiv 4b(n) \pmod{12}. \tag{26}$$

Now, consider the congruence equation

$$\frac{k^2 + k}{2} + 3 \cdot \frac{m^2 + m}{2} \equiv \frac{4p^2 - 4}{8} \pmod{p}.$$

which is equivalent to

$$(2k + 1)^2 + 3 \cdot (2m + 1)^2 \equiv 0 \pmod{p}, \tag{27}$$

where $0 \leq k, m \leq (p - 1)/2$ and p is a prime number such that $\left(\frac{-3}{p}\right) = -1$. Since $\left(\frac{-3}{p}\right) = -1$ for $p \equiv 5 \pmod{6}$, the congruence relation (27) holds if and only if $k = m = (p - 1)/2$. Substituting (5) into (25) and extract the terms in which the powers of q are congruent to $(p^2 - 1)/2$ modulo p , and then divide by $q^{\frac{p^2-1}{2}}$, we find that

$$\sum_{n \geq 0} b\left(pn + \frac{p^2 - 1}{2}\right) q^{pn} = \psi(q^{p^2})\psi(q^{3p^2}),$$

which implies that

$$\sum_{n \geq 0} b\left(p^2n + \frac{p^2 - 1}{2}\right) q^n = \psi(q)\psi(q^3),$$

and for $n \geq 0$,

$$b\left(p^2n + pi + \frac{p^2 - 1}{2}\right) = 0, \tag{28}$$

where $1 \leq i \leq p - 1$. By induction on (28) we obtain that for all $n, \alpha \geq 0$,

$$b\left(p^{2\alpha}n + \frac{p^{2\alpha} - 1}{2}\right) = b(n). \tag{29}$$

Replacing n by $p^2n + pi + \frac{p^2-1}{2}$ ($1 \leq i \leq p - 1$) in (29) and using (28), we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$b\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2} - 1}{2}\right) = 0.$$

Replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{2}$ in Equation (26), we obtain the desired result. \square

6. Congruences for $\text{RD}^{(4,9)}(n)$ modulo 24

Theorem 6.1. For all $n \geq 0$,

$$\text{RD}^{(4,9)}(24n + 23) \equiv 0 \pmod{24}, \tag{30}$$

$$\text{RD}^{(4,9)}(48n + 29) \equiv 0 \pmod{24}, \tag{31}$$

$$\text{RD}^{(4,9)}(96n + 89) \equiv 0 \pmod{24}. \tag{32}$$

Proof. Applying (7) on (11), with $p = k = 2$, we find that

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n + 5)q^n = 6 \frac{f_2^2 f_3^2 f_6^2}{f_1^6} \equiv 6 \frac{f_3^2 f_6^2}{f_1^2} \pmod{24}.$$

Substituting (2) into the above equation, we get

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n + 5)q^n \equiv 6 \frac{f_4^4 f_6^3 f_{12}^2}{f_2^5 f_8 f_{24}} + 12q \frac{f_4 f_6^4 f_8 f_{24}}{f_2^4 f_{12}} \pmod{24}. \quad (33)$$

If we extract the even and the odd powers from both sides of (33), we find that

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(12n + 5)q^n \equiv 6 \frac{f_2^4 f_3^3 f_6^2}{f_1^5 f_4 f_{12}} \equiv 6 \frac{f_2^2 f_3^3 f_6^2}{f_1 f_4 f_{12}} \pmod{24}, \quad (34)$$

and

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(12n + 11)q^n \equiv 12 \frac{f_2 f_3^4 f_4 f_{12}}{f_1^4 f_6} \equiv 12 \frac{f_4 f_6 f_{12}}{f_2} \pmod{24}. \quad (35)$$

Employing (3) in (34), we arrive at

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(12n + 5)q^n \equiv 6 \frac{f_4^2 f_6^4}{f_{12}^2} + 6q \frac{f_2^2 f_6^2 f_{12}^2}{f_4^2} \pmod{24}. \quad (36)$$

Extracting the terms involving the odd and the even powers from both sides of (36), we obtain

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(24n + 5)q^n \equiv 6 \frac{f_2^2 f_3^4}{f_6^2} \equiv 6f_2^2 \pmod{24}, \quad (37)$$

and

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(24n + 17)q^n \equiv 6 \frac{f_1^2 f_3^2 f_6^2}{f_2^2} \equiv 6 \frac{f_3^2 f_6^2}{f_1^2} \pmod{24}. \quad (38)$$

Congruence (30) follows from (35) by collecting the terms involving q^{2n+1} from both sides, and congruence (31) follows from (37) by equating the odd powers on both sides.

If we substitute (2) into (38), and then extract the odd powers from both sides of the resulting equation, we arrive at

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(48n + 41)q^n \equiv 12 \frac{f_4 f_6 f_{12}}{f_2} \pmod{24}. \quad (39)$$

Congruence (32) follows immediately from (39). \square

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REFERENCES

- [1] K. C. Ajeyakashi, H. S. Sumanth Bharadwaj, and S. Chandankumar, Extending recent congruence results on t -Schur overpartitions. *Ann Univ Ferrara* **71** (2025), Article 60.
- [2] A. M. Alanazi, A. O. Munagi, and M. P. Saikia, Some Properties of Overpartitions into Nonmultiples of Two Integers. arXiv:2412.18938.
- [3] O. Beckwith and C. Bessenrodt, Multiplicative properties of the number of k -regular partitions. *Ann. Comb.* **20** (2016), 231–250.
- [4] B. C. Berndt, *Ramanujan's Notebooks Part III*. Springer-Verlag, New York, 1991.
- [5] S. Cui and N. Gu, Arithmetic properties of ℓ -regular partitions. *Adv. Appl. Math.* **51** (2013), 507–523.
- [6] A. Fink, R. Guy and M. Krusemeyer, Partitions with parts occurring at most thrice. *Contrib. Discrete Math.* **3** (2008).
- [7] P. Hagsis, Partitions with a restriction on the multiplicity of the summands. *Trans. Am. Math. Soc.* **155** (1971), 375–384.
- [8] M. D. Hirschhorn and J. A. Sellers, Elementary proofs of parity results for 5-regular partitions. *Bull. Aust. Math. Soc.* **81** (2010), 58–63.
- [9] M. D. Hirschhorn, F. Garvan, and J. Borwein, Cubic analogs of the Jacobian cubic theta function $\Theta(z; q)$. *Canad. J. Math.* **45** (1993), 673–694.
- [10] W. J. Keith, Congruences for 9-regular partitions modulo 3. *Ramanujan J.* **35** (2014), 157–164.
- [11] W. J. Keith, Partitions into Parts Simultaneously Regular, Distinct, And/or Flat, Nathanson, M. (eds) Combinatorial and Additive Number Theory II. CANT 2015-2016, Springer Proceedings in Mathematics and Statistics, **2020** (2017), Springer, Cham.
- [12] W. J. Keith, A bijection for partitions simultaneously s -regular and t -distinct. *Integers* **23** (2023), Article 9.
- [13] M. L. Nadji and M. Ahmia, Partitions into parts simultaneously regular and distinct. *J. Ramanujan Math. Soc.* **40**(2) (2025), 125–150
- [14] M. L. Nadji, M. Ahmia, and J. L. Ramírez, Arithmetic properties of biregular overpartitions. *Ramanujan J.* **67** (2025), Article 13.
- [15] M. L. Nadji, M. P. Saikia, and J. A. Sellers, Arithmetic properties of t -Schur overpartitions. *Quaest. Math.*, in press.
- [16] S. Ramanujan, Collected Papers. Cambridge Univ. Press, Cambridge, 1927.
- [17] P. Buragohain and N. Saikia, Some new congruences for simultaneously s -regular and t -distinct partition function. *Russ Math* **68** (2024), 14–17.
- [18] P. Buragohain and N. Saikia, Congruences for simultaneously s -regular, t -regular and s -distinct partition function. *Vietnam J. Math.* **53** (2025), 407–426.
- [19] The On-Line Encyclopedia of Integer Sequences. <https://oeis.org>, 2025.
- [20] D. Penniston, Arithmetic of ℓ -regular partition functions. *Int. J. Number Theory* **4** (2008), 295–302.
- [21] J. J. Webb, Arithmetic of the 13-regular partition function modulo 3. *Ramanujan J.* **25** (2011), 49–56.

- [22] E. X. W. Xia and O. X. M. Yao, Parity results for 9-regular partitions. *Ramanujan J.* **34** (2014), 109–117.
- [23] E. X. W. Xia and O. X. M. Yao. Analogues of Ramanujan’s partition identities. *Ramanujan J.* **31** (2013), 373–396.