

ESSENTIAL CONTRACTIBILITY OF BANACH ALGEBRAS

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In this paper, a new notion, essential contractibility of Banach algebras, is introduced and some of its properties are examined. The main result is to investigate the essential contractibility of (symmetric abstract) Segal algebras.

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1. Introduction

One of the basically cohomological Banach algebras is contractible Banach algebras. A Banach algebra A is said to be contractible when every bounded derivation $D : A \rightarrow X$ is inner for every Banach A -bimodule X . Alternatively, A is contractible if it possesses a diagonal, i.e., an element $u \in A \widehat{\otimes} A$ satisfying $au = ua$ and $\pi_A(u)a = a$ for all $a \in A$. Curtis and Loy showed in [3] that any commutative contractible Banach algebra must be finite dimensional. However, finding an example of an infinite-dimensional contractible Banach algebra remains an open challenge. Another key homological notion in Banach algebras is amenability which was first introduced and studied by Johnson [12]. A stronger notion that amenability, known as essential amenability, was later proposed by Ghahramani and Loy in [8]. They defined a Banach algebra A as essentially amenable if $\mathcal{H}^1(A, X^*) = \{0\}$, where X is a neo-unital Banach A -bimodule, i.e., every continuous derivation from A into the dual of every Banach A -bimodule $X = A \cdot X \cdot A$, is inner. A criterion for the essential amenability of symmetric Segal algebras under certain conditions was established by Ghahramani and Loy [8]. Building on this, Samea extended these ideas to the broader setting involving abstract Segal algebras in [17].

The paper is organized as follows. Section 2 is devoted to introducing the concept of essential contractibility of Banach algebra, along with relevant examples. Furthermore, infinite-dimensional examples of essentially contractible Banach algebras are presented to highlight the significance and potential of this concept for further investigation. In section 3, some hereditary properties of essential contractibility are examined. In section 4, the essential contractibility of Segal Banach algebra is investigated. In addition, several criteria for the essential contractibility of symmetric Segal Banach algebras are provided.

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2. Essential contractibility

Let A be a Banach algebra. For $n \in \mathbb{N}$, define

$$A^n = \left\{ \prod_{i=1}^n a_1 \dots a_n : a_i \in A, 1 \leq i \leq n \right\}.$$

We denote by $[A^n]$ the linear span of A^n . A Banach A -bimodule X is called essential if $X = \overline{[A \cdot X \cdot A]}$ where $A \cdot X \cdot A = \{a \cdot x \cdot b : a, b \in A, x \in X\}$. Additionally, X is said to be neo-unital if $X = A \cdot X \cdot A$. According to the Cohen factorization theorem [10, Theorem 32.22], every essential Banach A -bimodule X is neo-unital whenever A has a bounded approximate identity.

Now, Let A be a Banach algebra and let X be a Banach A -bimodule. A continuous linear map $D : A \rightarrow X$ is called a derivation if it satisfies

$$D(ab) = a \cdot D(b) + D(a) \cdot b,$$

for all $a, b \in A$. For a given $x \in X$, define a continuous linear map $ad_x : A \rightarrow X$ by

$$ad_x(a) = a \cdot x - x \cdot a, \quad (a \in A).$$

This map, known as an inner derivation, satisfies the derivation property.

Definition 2.1. *A Banach algebra A is essentially contractible if for every neo-unital Banach A -bimodule X , every derivation $D : A \rightarrow X$ is inner.*

Let A be a Banach algebra and let $\Phi(A)$ represent the character space of A , consisting of all non-zero multiplicative linear functionals on A . For $\varphi \in \Phi(A)$, a Banach A -bimodule X is called left φ -linked if the left module action is induced by φ ; i.e.,

$$a \cdot x = \varphi(a)x,$$

for all $a \in A$ and $x \in X$. Such an A -bimodule is denoted by ${}_{\varphi}X$. In the same manner, a right φ -linked Banach A -bimodule is defined and it is denoted by X_{φ} . We also use ${}_{\varphi}X_{\varphi}$ to indicate an A -bimodule that is both left and right φ -linked.

Remark 2.1. *Let A and X be a Banach algebras and let $\varphi \in \Phi(A)$ be a fixed character. The Banach algebra ${}_{\varphi}X_{\varphi}$ becomes a neo-unital A -bimodule. Specifically, for some $b \in A$ with $\varphi(b) \neq 0$, we have $x = \frac{b}{\varphi(b)} \cdot x \cdot \frac{b}{\varphi(b)}$, for every $x \in {}_{\varphi}X_{\varphi}$. Any continuous derivation from A into ${}_{\varphi}C_{\varphi}$ is referred to as a point derivation. This implies that every essentially contractible Banach algebra has no non-zero point derivation. This criterion provides us with numerous examples of Banach algebras that are not essentially contractible. As an example, consider the disc algebra $A(\overline{\mathbb{D}})$, defined as*

$$A(\overline{\mathbb{D}}) = \{f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ is analytic}\},$$

where \mathbb{D} denotes the open unit disc in \mathbb{C} . For a point z_0 on the boundary of $\overline{\mathbb{D}}$, and the character $\varphi(f) = f(z_0)$, the linear functional $D : A(\overline{\mathbb{D}}) \rightarrow A(\overline{\mathbb{D}})$ given by $D(f) = f'(z_0)1$ is a point derivation on $A(\overline{\mathbb{D}})$, where $f'(z_0)$ is understood in the sense of the derivative of f at z_0 .

For a Banach algebra A , the projective tensor product of A with itself is denoted by $A \widehat{\otimes} A$ and the diagonal operator from $A \widehat{\otimes} A$ into A is defined by $\pi_A(a \otimes b) = ab$. It is obvious that every contractible Banach algebra is essentially contractible. In the following, it is shown that the essential contractibility with the condition of unitality is the same as the contractibility.

Theorem 2.1. *Every essentially contractible Banach algebra with a bounded central approximate identity is pseudo-contractible.*

Proof. Let (e_α) be the bounded central approximate identity of A . Consider the kernel of the diagonal map π_A , denoted by $\ker \pi_A \subseteq A \widehat{\otimes} A$, and equip it with the usual Banach A -bimodule structure. Since $a \otimes b = \lim_\alpha e_\alpha(a \otimes b)e_\alpha$, for every $a \otimes b \in \ker \pi_A$, it follows from the Cohen factorization theorem that $\ker \pi_A$ is a neo-unital Banach A -bimodule. We set $D_\alpha = ad_{M_\alpha} : A \rightarrow A \widehat{\otimes} A$, where $M_\alpha := e_\alpha \otimes e_\alpha$. Thus $\text{Im}(D) \subseteq \ker \pi_A$. This implies the existence of $N_\alpha \in \ker \pi_A$ such that $ad_{M_\alpha} = ad_{N_\alpha}$. Therefore, $(M_\alpha - N_\alpha)$ forms a central approximate diagonal for A , as it satisfies

$$a \cdot (M_\alpha - N_\alpha) = (M_\alpha - N_\alpha) \cdot a \quad \text{and} \quad a \cdot \pi_A(M_\alpha - N_\alpha) \rightarrow a$$

for all $a \in A$. □

Applying Theorem 2.1 and [9, Theorem 2.4] we deduce the following result.

Corollary 2.1. *Every essentially contractible Banach algebra with unit is contractible.*

A direct consequence of the above corollary is that the discrete group algebra $\ell^1(G)$ is essentially contractible if and only if G is finite.

Remark 2.2. *Consider the semigroup $S = (\mathbb{N}, \max)$. Then, if $\ell^1(S)$ is essentially contractible, then by using Corollary 2.1, $\ell^1(S)$ is contractible. This implies that $\ell^1(S)$ is amenable and consequently the semigroup S has finitely many idempotents [6, Theorem 2], that is a contradiction. Therefore, $\ell^1(S)$ is not essentially contractible.*

In the sequel, some examples that distinguish the essential contractibility from the other concepts of cohomological notions are presented.

Examples

Example 2.1. *An example of (infinite dimensional) Banach algebras that is essentially contractible but not contractible.*

Let A be a nonzero Banach space. By defining the product $a \cdot b = 0$ for all $a, b \in A$, every neo-unital Banach A -bimodule becomes a singular zero set. In fact, $X = A \cdot X \cdot A = A \cdot (A \cdot X \cdot A) \cdot A = \{0\}$. This shows that A is essentially contractible. However, A is not contractible, as it does not have a unit element.

Example 2.2. *An example of a Banach algebra that is essentially contractible but not amenable (contractible).*

Let S be a nonempty set with $|S| \geq 2$.

(a) For a fixed element $s_0 \in S$, define a semigroup structure on S using the multiplication $st = s_0$ for all $s, t \in S$. We show that $\ell^1(S)$ is essentially contractible. Let X be a neo-unital Banach $\ell^1(S)$ -bimodule and $D : \ell^1(S) \rightarrow X$ be a continuous derivation. It is straightforward to see that $f * g = (\sum_{st=s_0} f(s)g(t))\delta_{s_0} = \mathbb{C}\delta_{s_0}$, for all $f, g \in \ell^1(S)$. Since X is neo-unital we get

$$X = \ell^1(S)X\ell^1(S) = \ell^1(S)(\ell^1(S)X\ell^1(S))\ell^1(S) = \mathbb{C}\delta_{s_0}X\mathbb{C}\delta_{s_0} = \delta_{s_0}X\delta_{s_0}.$$

Let $f \in \ell^1(S)$ and $x \in X$. Then for some $y \in X$ we have $x = \delta_{s_0} y \delta_{s_0}$, leading to

$$\begin{aligned} f \cdot x &= f \cdot (\delta_{s_0} y \delta_{s_0}) = (f * \delta_{s_0}) y \delta_{s_0} \\ &= \left(\sum_{s \in S} f(s) \right) x = \delta_{s_0} y (\delta_{s_0} * f) \\ &= (\delta_{s_0} y \delta_{s_0}) \cdot f = x \cdot f. \end{aligned}$$

Specifically, $\delta_{s_0} \cdot x = x \cdot \delta_{s_0} = x$. Additionally, we find

$$D(\delta_{s_0}) = D(\delta_{s_0} * \delta_{s_0}) = \delta_{s_0} D(\delta_{s_0}) + D(\delta_{s_0}) \delta_{s_0} = 2D(\delta_{s_0}),$$

which implies $D(\delta_{s_0}) = 0$. Consequently, $D(f * \delta_{s_0}) = D(f) \delta_{s_0} + f D(\delta_{s_0}) = D(f)$. On the other hand, $D(f * \delta_{s_0}) = (\sum_{s \in S} f(s)) D(\delta_{s_0}) = 0$ and thus $D = 0$. This shows that $\ell^1(S)$ is essentially contractible. However, $\ell^1(S)$ is not amenable, as it lacks a bounded approximate identity.

(b) An element z in S is called left (right) zero if $zs = z$ ($sz = z$) for all $s \in S$. If every member of S is a left (right) zero, then S is called a left (right) zero semigroup. A left zero semigroup that is also a right zero semigroup is called a zero semigroup. If S has a zero 0 and $st = 0$ for all $s, t \in S$, then S is called a null semigroup. Suppose that S is a topological null semigroup. By [1, Theorem 3.4] $M(S)$ does not have a left or a right approximate identity. We show that $M(S)$ is essential contractible. Let X be a neo-unital $M(S)$ -bimodule and $D : M(S) \rightarrow X$ be a continuous derivation. So $D = 0$. In fact, by [1, Lemma 3.3] we have

$$\mu(S)D(\delta_0) = D(\mu(S)\delta_0) = D(\mu * \delta_0) = D(\mu) \cdot \delta_0 + \mu \cdot D(\delta_0) = D(\mu) + \mu(S)D(\delta_0).$$

for all $\mu \in M(S)$.

Example 2.3. *An essentially amenable which is not essentially contractible.*

Let G be a non-compact amenable locally compact group. The group algebra $L^1(G)$ is essentially amenable [8, Corollary 7.1]. These group algebras can not be essentially contractible. In fact, if $L^1(G)$ is essentially contractible, then it is already left φ -essentially contractible for every $\varphi \in \Phi(L^1(G)) \cup \{0\}$. So G must be compact [19, Theorem 2]. Further examples will be introduced progressively in the following sections.

3. Properties of essential contractibility

We will now establish some basic properties of essential contractibility.

Proposition 3.1. *Let A be an essentially contractible Banach algebra and $\Psi : A \rightarrow B$ be a continuous epimorphism. Then B is essentially contractible.*

Proof. Let X be a neo-unital Banach B -bimodule and $D : B \rightarrow X$ be a bounded derivation. The following module actions

$$a \cdot x = \Psi(a)x, \quad x \cdot a = x\Psi(a) \quad (a \in A, x \in X)$$

make X into a neo-unital Banach A -bimodule. Then the map $D \circ \Psi : A \rightarrow X$ is a bounded derivation and since $D \circ \Psi$ is inner, we conclude that D must be inner. \square

Utilizing Theorem 2.1 and Proposition 3.1, we investigate the connection between the finiteness of the underlying group and essentially contractibility of some associated Banach algebras.

Let A be a Banach algebra. The second dual of A , equipped with the first Arens product " \square ", can be considered as a Banach algebra with the properties $a \square F = a \cdot F$ and $F \square a = F \cdot a$ for each $a \in A$ and $F \in A^{**}$ (see [4, Theorem 2.6.15]). Let Y be a closed $L^1(G)$ -submodule of $L^\infty(G)$. If $F \cdot f \in Y$ for all $F \in L^\infty(G)^*$ and $f \in Y$, then Y is called left introverted. So Y^* forms a Banach algebra that its multiplication is induced by the first Arens product. A continuous bounded function $f \in C(G)$ is called almost periodic if the set $\{\ell_s f : s \in G\}$ is relatively norm compact, where $\ell_s f : G \rightarrow \mathbb{C}$ denotes the left translation of f given by $\ell_s f(t) = f(st)$ for all $t \in G$. The space of all almost periodic functions on G is denoted by $AP(G)$, which is an example of such Y . We denote \dot{Y} as a left introverted subspace of $L^\infty(G)$, where G is a maximally almost periodic (MAP) locally compact group and $AP(G) \subset \dot{Y}$ (For more information on MAP groups see [14]). For a locally compact group G , two subbimodules of $L^\infty(G)$, consisting of left uniformly continuous functions and those vanishing at infinity, are denoted by $LUC(G)$ and $L_0^\infty(G)$, respectively.

Proposition 3.2. *Let G be a locally compact group. The following assertions are equivalent.*

- (i) $M(G)$ is essentially contractible.
- (ii) $LUC(G)^*$ is essentially contractible.
- (iii) $L^1(G)^{**}$ is essentially contractible.
- (iv) $L_0^\infty(G)^*$ is essentially contractible.
- (v) \dot{Y}^* is essentially contractible.
- (vi) G is finite.

Proof. If G is finite then $M(G) = LUC(G)^* = L^1(G)^{**} = L_0^\infty(G)^* = \dot{Y}^* = \ell^1(G)$ are contractible and then they are essentially contractible.

(i) If $M(G)$ is essentially contractible, since it is unital, by Theorem 2.1 $M(G)$ is contractible and by [18], G is finite.

(ii) Since the restriction map $\Psi : LUC(G)^* \rightarrow M(G)$ is a continuous epimorphism using Proposition 3.1 and part (i), G is finite.

(iii) By Remark 2.1, $L^1(G)^{**}$ does not have any nonzero continuous point derivation associated to any character in $\Phi(L^1(G)^{**})$. Hence G is finite by applying [5, Theorem 11.17].

(iv) By [13, Theorem 2.11] $M(G)$ is isometrically isomorphic with $E \square L_0^\infty(G)^*$ where E is a norm one right identity of $L_0^\infty(G)$. Furthermore, mapping $F \mapsto E \square F$ is a continuous epimorphism from $L_0^\infty(G)^*$ onto $M(G)$. Again, by applying Proposition 3.1 and part (i), G is finite.

(v) Since the restriction map from \dot{Y}^* into $(AP(G))^*$ is a continuous epimorphism, $(AP(G))^*$ is essentially contractible by Proposition 3.1. On the other hand, we have the isomorphism $M(bG) \cong (AP(G))^*$, where bG is the Bohr compactification of G . Hence, bG must be finite, as established by part (i). Since G is a maximally almost periodic group, the canonical homomorphism from G into bG is injective. Therefore, G must also be finite. \square

Ghahramani and Loy introduced the concept of essential amenability for Banach algebras in [8] and explored various hereditary properties of essential amenable Banach algebras. Building on this work, our goal is to derive some hereditary properties specific to essential contractible Banach algebras.

Let $\{A_i\}_{i \in I}$ be a family of Banach algebras. The product space $\prod_{i \in I} A_i$ contains all mapping $a \mapsto (a(i))_{i \in I}$ from I onto $\bigcup_{i \in I} A_i$. The l_p -direct sum ($1 \leq p < \infty$) and l_∞ -direct sum of $\{A_i\}_{i \in I}$ are defined as following

$$\bigoplus_{i \in I}^p A_i := \left\{ (a(i))_{i \in I} \in \prod_{i \in I} A_i; \|(a(i))_{i \in I}\|_p = \left(\sum_{i \in I} \|a(i)\|^p \right)^{1/p} < \infty \right\},$$

and

$$\bigoplus_{i \in I}^\infty A_i := \left\{ (a(i))_{i \in I} \in \prod_{i \in I} A_i; \|(a(i))_{i \in I}\|_\infty = \max_i \|a(i)\| < \infty \right\}.$$

Moreover, the elements $(a(i))_{i \in I}$ in l_∞ -direct sum of $\{A_i\}_{i \in I}$ for which $\lim_I a(i) = 0$ is denoted by $\bigoplus_{i \in I}^0 A_i$ and is called the c_0 -direct sum of $\{A_i\}_{i \in I}$. The c_0 -direct sum and l_p -direct sum ($1 \leq p \leq \infty$) of the collections of Banach algebras by coordinatewise multiplication, $\|\cdot\|_p$ and $\|\cdot\|_\infty$ form a Banach algebra, respectively.

Let A and B be Banach algebras and $\theta \in \Phi(B)$. The Cartesian multiplication $A \times B$ by the following multiplication

$$(a, b)(a', b') = (aa' + \theta(b)a' + \theta(b')a, bb'),$$

and the norm

$$\|(a, b)\| = \|a\|_A + \|b\|_B,$$

for all $a, a' \in A$ and $b, b' \in B$, becomes a Banach algebra. Such product is called the θ -Lau product of A and B and is denoted by $A \times_\theta B$. In the case of $B = \mathbb{C}$ and $\theta = id_{\mathbb{C}}$, then $A \times_\theta \mathbb{C}$ is denoted by A^\sharp and is called the unitization of A .

Proposition 3.3. *Let $\{A_i\}_{i \in I}$ be a family of Banach algebras and A, B be Banach algebras.*

- (i) *If A is essentially contractible and I is a closed two-sided ideal of A , then A/I is essentially contractible.*
- (ii) *Let I be a closed two-sided ideal of A with $A^n \subseteq I$ for some $n \geq 2$. If I is essentially contractible and A/I is contractible then A is essentially contractible.*
- (iii) *If $\bigoplus_{i \in I}^p A_i$ ($1 \leq p \leq \infty$) or $\bigoplus_{i \in I}^0 A_i$ is essentially contractible, then so is A_i for each $i \in I$.*
- (iv) *If $A \times_\theta B$ is essentially contractible, then A and B are essentially contractible.*
- (v) *If A^\sharp is essentially contractible then so is A .*

Proof. (i) It is a direct consequence of Proposition 3.1.

(ii) Let $D : A \rightarrow X$ be a bounded derivation and X be a neo-unital Banach A -bimodule. Since $A^n \subseteq I$ so

$$X = A \cdot X \cdot A = A \cdot (A \cdot X \cdot A) \cdot A = A^2 \cdot X \cdot A^2 = \dots = A^n \cdot X \cdot A^n \subseteq I \cdot X \cdot I \subseteq X.$$

This implies that X is a neo-unital Banach I -bimodule. Since $D|_I : I \rightarrow X$ is a bounded derivation thus there exists $x \in X$ such that $D|_I = ad_x$. Define $\tilde{D} = D - ad_x$. Then $\tilde{D} : A \rightarrow X$ is a bounded derivations such that $\tilde{D}|_I = 0$, and so drops to a map, denoted by \bar{D} , from A/I into X . i.e., $\bar{D}(a + I) = \tilde{D}(a)$. Set

$$Y = \{x \in X : ax = xa = 0 \text{ for all } a \in I\}.$$

Then Y is a closed submodule of X . Also Y is a Banach A/I -bimodule by the usual module actions

$$(a + I) \cdot y = ay \quad \text{and} \quad y \cdot (a + I) = ya, \quad (a \in A, y \in Y).$$

Since $\tilde{D}|_I = 0$, we obtain

$$a \cdot \tilde{D}(b) = \tilde{D}(ab) - \tilde{D}(a) \cdot b = 0, \quad \tilde{D}(b) \cdot a = \tilde{D}(ba) - b \cdot \tilde{D}(a) = 0, \quad (a \in I, b \in A).$$

This means that $\overline{D}(A/I) \subseteq Y$. Therefore, there exists $y \in Y$ such that $\overline{D} = ad_y$ and so $D = ad_{x+y}$

(iii) For every $i \in I$, A_i is a homomorphic image of $\bigoplus_{i \in I}^p A_i$ or $\bigoplus_{i \in I}^0 A_i$, so it is essentially contractible using Proposition 3.1.

(iv) Let X be a neo-unital Banach A -bimodule and $D : A \rightarrow X$ be a continuous derivation. Then X is a neo-unital Banach $(A \times_\theta B)$ -bimodule by the following module actions

$$(a, b) \cdot x = a \cdot x + \theta(b)x, \quad x \cdot (a, b) = x \cdot a + \theta(b)x, \quad (a \in A, b \in B, x \in X).$$

We define $\widehat{D} : A \times_\theta B \rightarrow X$ by $\widehat{D} = D \circ p_A$, where $p_A : A \times_\theta B \rightarrow A$ is the projection map on the first variable. Continuity of D and p_A implies that \widehat{D} is continuous. On the other hand, we have

$$\begin{aligned} \widehat{D}((a, b)(a', b')) &= D \circ p_A((aa' + \theta(b)a' + \theta(b')a, bb')) \\ &= D(aa' + \theta(b)a' + \theta(b')a) \\ &= aD(a') + D(a)a' + \theta(b)D(a') + \theta(b')D(a) \\ &= (a, b) \cdot D(a') + D(a) \cdot (a', b') \\ &= (a, b) \cdot \widehat{D}((a, b)) + \widehat{D}((a, b)) \cdot (a', b'). \end{aligned}$$

So \widehat{D} is a derivation and then is inner. Therefore D is inner and we conclude that A is essentially contractible. Similarly, B is essentially contractible.

(v) Note that $A \times_\theta \mathbb{C} = A^\sharp$. So by part (iv) the proof is complete. \square

It is important to note that the converse of Proposition 3.3 (iii) does not hold. For instance, the direct sum $\bigoplus_{n \in \mathbb{N}} \mathbb{C} = \ell^\infty$ is not essentially contractible.

Recall that an element δ in Banach algebra A is called a central idempotent if $\delta^2 = \delta \cdot \delta = \delta$ and $\delta \cdot a = a \cdot \delta$ for all $a \in A$.

Corollary 3.1. *Let A be an essentially contractible Banach algebra with a central idempotent δ . Then A has a contractible closed two-sided ideal.*

Proof. Let A be essentially contractible and δ be a central idempotent. Define the set $I = \{a \in A : a\delta = 0\}$, which is a closed two-sided ideal in A . By applying Proposition 3.3 (i), the quotient algebra A/I is essentially contractible. Since $A/I \cong A\delta$, it follows that $A\delta$ is also essentially contractible. Furthermore, by Theorem 2.1, $A\delta$ is contractible, as it is unital with unit element δ . \square

4. Essential contractibility of Segal Banach algebras.

Segal algebras are the certain classes of Banach algebra associated with a locally compact group G that was introduced by Reiter [15]. For a locally compact group G , a Segal algebra $S^1(G)$ is a left ideal of $L^1(G)$ that satisfies the following conditions:

- (i) $\overline{S^1(G)}^{\|\cdot\|_1} = L^1(G)$;
- (ii) $(S^1(G), \|\cdot\|_{S^1})$ is a Banach space and $\|\cdot\|_1 \leq \|\cdot\|_{S^1}$;

(iii) $S^1(G)$ is left translation invariant, and $\|L_x f\|_{S^1} = \|f\|_{S^1}$ for all $x \in G$ and every $f \in S^1(G)$, where $L_x f$ is defined by $L_x f(y) = f(x^{-1}y)$ for all $y \in G$;

(iv) For every $x \in G$ and $f \in S^1(G)$, the map $x \mapsto L_x f$ is continuous in $\|\cdot\|_{S^1}$ -norm.

A Segal algebra $S^1(G)$ is called symmetric if it also has the following conditions:

(v) $S^1(G)$ is right translation invariant, and $\|R_x f\|_{S^1} = \|f\|_{S^1}$ for all $x \in G$ and every $f \in S^1(G)$, where $R_x f$ is defined by $R_x f(y) = f(xy)$ for all $y \in G$;

(vi) For every $x \in G$ and $f \in S^1(G)$, the map $x \mapsto R_x f$ is continuous in $\|\cdot\|_{S^1}$ -norm.

It is well-known that every Segal algebra has a left approximate identity that is bounded in $\|\cdot\|_1$ see [15].

An abstract Segal algebra is a general concept of Segal algebra. A dense left ideal B of a Banach algebra A is called an abstract Segal algebra when it admits a norm $\|\cdot\|_B$ for which $(B, \|\cdot\|_B)$ is a Banach algebra and

(i) there exists $M > 0$ such that $\|b\|_A \leq M\|b\|_B$ for each $b \in B$.

(ii) there exists $C > 0$ such that $\|ab\|_B \leq C\|a\|_A\|b\|_B$ for each $a \in A, b \in B$.

Definition 4.1. Let I be a two-sided ideal of a Banach algebra A . We say that I has property (\mathcal{P}) if I admits a norm $\|\cdot\|_I$ which $(I, \|\cdot\|_I)$ is a Banach algebra and

$$\|ab\|_I, \|ba\|_I \leq \|a\|_A\|b\|_I, \quad (a \in A, b \in I).$$

An abstract Segal algebra B is called symmetric if it also has property (\mathcal{P}) .

The famous theorem of the real analysis, the continuous extension theorem, can also be generalized to bounded derivations in the case where the range is a Banach space. In fact, the same technique works here.

Proposition 4.1. Let A be a Banach algebra, X be a Banach A -bimodule, \mathcal{D} be a dense subalgebra of A and $T : \mathcal{D} \rightarrow X$ be a bounded derivation. Then there exists a bounded operator $\tilde{T} : A \rightarrow X$ which is a derivation and $\tilde{T}|_{\mathcal{D}} = T$. This extension is unique. Furthermore, if T is inner, then \tilde{T} is inner.

Proof. Take $a \in A$ and choose $\{a_n\} \subset \mathcal{D}$ such that $a_n \rightarrow a$. Then $\{a_n\}$ is a Cauchy sequence in A and so $\{T(a_n)\}$ is a Cauchy sequence in B . Since B is complete so $\{T(a_n)\}$ is convergent. We set $\tilde{T}(a) := \lim_{n \rightarrow \infty} T(a_n)$. If $a_n \rightarrow a$ and $b_n \rightarrow a$, using continuity of T , we conclude that $\lim_{n \rightarrow \infty} T(a_n - b_n) = 0$. It means that \tilde{T} is well-defined. It is easy to see that \tilde{T} is a linear derivation. Also, we have

$$\|\tilde{T}(a)\| = \|\lim_{n \rightarrow \infty} T(a_n)\| = \lim_{n \rightarrow \infty} \|T(a_n)\| \leq \lim_{n \rightarrow \infty} \|T\| \|a_n\| = \|T\| \|a\|.$$

This means that \tilde{T} is bounded. Finally, if \tilde{T}_1 and \tilde{T}_2 are two continuous extensions of T , then for $a \in A$ we have

$$\tilde{T}_1(a) = \lim_{n \rightarrow \infty} \tilde{T}_1(a_n) = \lim_{n \rightarrow \infty} T(a_n) = \lim_{n \rightarrow \infty} \tilde{T}_2(a_n) = \tilde{T}_2(a).$$

Furthermore, if T is inner, the uniqueness of extension and continuity of action imply that \tilde{T} is inner. □

The extension obtained from Proposition 4.1 gives us criterions for essential contractibility of Segal algebras and dense substructures.

Proposition 4.2. *Let A be a Banach algebra, and I be a dense subalgebra of A . If there exists an integer $n \geq 2$ such that $A^n \subseteq I$ and I is essentially contractible, then A is essentially contractible.*

Proof. Let X be a neo-unital Banach A -bimodule, and let $D : A \rightarrow X$ be a continuous derivation. It follows that $X = A \cdot (A \cdot X \cdot A) \cdot A$, and by induction, we obtain the inclusion $X = A^n \cdot X \cdot A^n \subseteq I \cdot X \cdot I \subseteq X$. This means that X is a neo-unital Banach I -bimodule. Consequently, there exists an element $x \in X$ such that $D|_I = ad_x$. Since D is the unique extension of $D|_I$ we conclude that D is inner by Proposition 4.1. \square

Corollary 4.1. *Let B be an essentially contractible abstract Segal algebra with respect to A . If $A^n \subseteq B$ for some integer $n \geq 2$ then A is essentially contractible.*

Now, we are ready to present the main result that was proved amenability version of that in [8].

Theorem 4.1. *Suppose that A is a contractible Banach algebra and I is a non-closed two-sided ideal in A with the property (\mathcal{P}) . If I has an approximate identity, which is an approximate identity for A , then I is essentially contractible.*

Proof. Let X be a neo-unital Banach I -bimodule and let $D : I \rightarrow X$ be a continuous derivation. Since I has property (\mathcal{P}) , X can be regarded as a Banach A -bimodule, as demonstrated in the proof of [8, Theorem 7.1]. Hence, using Proposition 4.1, there exists a unique extension $\tilde{D} : A \rightarrow X$. Since A is contractible, \tilde{D} is an inner derivation and consequently D is also inner. \square

Applying Proposition 4.2 and Theorem 4.1 we obtain the following result.

Corollary 4.2. *Suppose that A is a Banach algebra. Let I be a non-closed two-sided ideal of A with property (\mathcal{P}) and it has an approximate identity which is an approximate identity for A . Then A is essentially contractible if and only if I is essentially contractible*

In the following, we discuss the essential contractibility of Segal algebras associated with groups and hypergroups. For further details on hypergroups, we refer to the work of Jewett [11].

Corollary 4.3. *The following Banach algebras are essentially contractible:*

- (i) *A symmetric Segal algebra $S^1(G)$, for a finite group G ;*
- (ii) *A Segal algebra $S^1(H)$, for a commutative finite hypergroup H ;*
- (iii) *$\overline{[B^2]}^{\|\cdot\|_B}$, for a symmetric abstract Segal algebra B with respect to a contractible Banach algebra.*

Proof. (i) By Theorem 4.1 and [16, Corollary 4.1.3].

(ii) By Theorem 4.1 and [2, Proposition 3.2].

(iii) By Theorem 4.1 and [17, Theorem 3.5]. \square

H. Samea achieved several important results regarding abstract symmetric Segal algebras, as discussed in [17], which led to the following subsequent findings.

Lemma 4.1. *Let A be a contractible Banach algebra and B be a symmetric abstract Segal algebra with respect to A . If $[B^2]$ is dense in B , then B is essentially contractible.*

Proof. Since A is contractible, it is unital, and by [17, Theorem 3.5], A has an approximate identity in $[\overline{B^2}]^{\|\cdot\|_B}$. Additionally, by [17, Theorem 3.5] symmetry of $[\overline{B^2}]^{\|\cdot\|_B}$ implies that $[\overline{B^2}]^{\|\cdot\|_B}$ has the property (\mathcal{P}) . Therefore, applying Theorem 4.1, we conclude that $B = [\overline{B^2}]^{\|\cdot\|_B}$ is essentially contractible. \square

Lemma 4.2. *Let A be a contractible Banach algebra and I be a dense two-sided ideal in A with the property (\mathcal{P}) . If $[I^2]$ is dense in I , then I is essentially contractible.*

Proof. Since A is contractible, it is unital. Therefore, by [17, Lemma 4.5] I is a symmetric abstract Segal algebra. So by Lemma 4.1, I is essentially contractible. \square

Corollary 4.4. *Every symmetric abstract Segal algebra with respect to a contractible Banach algebra with an approximate identity is essentially contractible.*

At the end, the aim is to provide an example of an (infinite dimensional) essentially contractible Banach algebra using matrix algebras. Let A be a Banach algebra and $n \in \mathbb{N}$. The vector space of all $n \times n$ -matrices (a_{ij}) with entries from A , denoted by $M_n(A)$, is a Banach algebra when equipped with the norm

$$\|(a_{ij})\|_{M_n(A)} = \sum_{i,j=1}^n \|a_{i,j}\|_A.$$

Lemma 4.3. *Let A be a Banach algebra, B be a symmetric abstract Segal algebra with respect to A . Then $M_n(B)$ is a symmetric abstract Segal algebra with respect to $M_n(A)$.*

Proof. It is obvious that $M_n(B)$ is a dense left ideal in $M_n(A)$. First, let $a = (a_{i,j})$ and $b = (b_{i,j})$ be in $M_n(A)$ and $M_n(B)$, respectively. Then

$$\|(b_{ij})\|_A = \sum_{i,j=1}^n \|b_{ij}\|_A \leq M \sum_{i,j=1}^n \|b_{ij}\|_B = M \|(b_{ij})\|_B$$

Finally, let $(a_{ij})(b_{ij}) = (c_{ik})$, where $c_{ik} = \sum_j a_{ij} b_{jk}$. Using the submultiplicativity property of B we have

$$\|c_{ik}\|_B = \left\| \sum_{j=1}^n a_{ij} b_{jk} \right\|_B \leq \sum_{j=1}^n \|a_{ij}\|_A \|b_{jk}\|_B.$$

Summing over all i, k , we obtain

$$\|(c_{ik})\|_{M_n(B)} = \sum_{i,k=1}^n \|c_{ik}\|_B \leq \sum_{i,k=1}^n \sum_{j=1}^n \|a_{ij}\|_A \|b_{jk}\|_B.$$

Reorganizing the sums gives:

$$\|(c_{ik})\|_{M_n(B)} \leq \left(\sum_{i,j=1}^n \|a_{ij}\|_A \right) \left(\sum_{j,k=1}^n \|b_{jk}\|_B \right) = \|(a_{ij})\|_{M_n(A)} \|(b_{ij})\|_{M_n(B)}$$

In a similar way we obtain

$$\|(b_{ij})(a_{ij})\|_{M_n(B)} \leq \|(a_{ij})\|_{M_n(A)} \|(b_{ij})\|_{M_n(B)}$$

\square

In the sequel, we are prepared to present the example that was promised.

Example 4.1. Let A be a contractible Banach algebra, B be a symmetric abstract Segal algebra with respect to A and $n \in \mathbb{N}$. By Lemma 4.3, $M_n(B)$ is a symmetric abstract Segal algebra with respect to $M_n(A)$. Using [17, Lemma 3.5] $\overline{[M_n(B)^2]}^{\|\cdot\|_{M_n(B)}}$ is a symmetric abstract Segal algebra with respect to $M_n(A)$, which has an approximate identity (e_α) for $M_n(A)$. Moreover, by [7, Theorem 3.2] $M_n(A)$ is contractible, Corollary 4.4 implies that $\overline{[M_n(B)^2]}^{\|\cdot\|_{M_n(B)}}$ is essentially contractible. Note that $\overline{[M_n(B)^2]}^{\|\cdot\|_{M_n(B)}}$ is not contractible; else it should have a unit element, namely E . Hence we have $e_\alpha = e_\alpha E \rightarrow E$. Therefore, $e_\alpha I \rightarrow I$ and $e_\alpha I \rightarrow EI = E$, implies that $E = I$, where I is the identity matrix. This would imply $M_n(A) = \overline{[M_n(B)^2]}^{\|\cdot\|_{M_n(B)}} \subseteq M_n(B)$, which is a contradiction.

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6. Conclusions

In this paper we introduce a new notion, namely essential contractibility, for Banach algebras. We present some examples to illustrate that this notion differs from the older notions, such as contractibility, amenability and essential amenability. We investigate certain hereditary properties of essential contractible Banach algebras. At the end, we study essential contractibility of abstract Segal algebras.

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