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D-SHADOWING PROPERTY OF G-SPACES AND THEIR ORBIT SPACES

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We introduce the definitions of $\overline{d}$-shadowing, asymptotic average shadowing properties and study various of transitive, chain transitive properties for the map f on G-spaces. We show that $\overline{d}$-shadowing of f is iteration-invariant, and f and its corresponding induced map $\widehat{f}$ are equivalent in terms of the above dynamical properties under certain additional conditions.

Keywords: G-spaces, $\overline{d}$-shadowing, chain transitivity, asymptotic average shadowing properties.

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1. Introduction

Shadowing properties originally stem from differential dynamical systems. When studying dynamical systems, exact solutions can often be elusive due to various influencing factors, leading researchers to approximate system orbits using numerical methods. The errors that arise during this process result in "pseudo-orbits." Therefore, investigating the shadowing properties of dynamical systems aids in understanding the impact of errors generated by numerical computations on the approximation of system orbits, thereby enabling better handling and correction of "pseudo-orbits" and enhancing the accuracy of understanding and predicting the behavior of dynamical systems. Recently, various shadowing properties are being defined and studied for group actions, set-valued maps, iterated funtion system and so on. The present paper focuses on G-space.

In the realm of topological dynamical systems, the concept of G-spaces serves as a framework for exploring the dynamic impact of group actions on topological spaces. Delving into the realm of G-spaces enables researchers to delve deeper into how dynamical systems behave and interact under the influence of group actions, offering a more profound understanding of the interplay between topology and dynamics across various contexts. Furthermore, the study of G-spaces can lead to the discovery of novel phenomena, unveil connections between diverse mathematical domains, and establish a theoretical basis for applications in physics, engineering, biology, and other fields. In recent times, an increasing number of scholars have turned their attention to investigating G-spaces. Ali and Ekta introduced and examined the concept of chain transitivity for mappings on G-spaces in their work referenced as [2]. Phinao and Khundrakpam explored the G-mixing, G-sensitive, and G-shadowing characteristics of f in [9]. Raad and Iftichar delved into the G-asymptotic average shadowing sequence alongside G-chain transitivity in their publication mentioned as [11]. For additional sources, kindly refer to [4, 7, 8]. The aim of the present paper is to define $\overline{d}$-shadowing, asymptotic average shadowing properties and study various of transitive, chain transitive properties for maps on G-spaces. The specific layout of the present paper is as follows. In Section 2, we introduce some preliminaries and definitions. In Section 3,

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we study the implications between f and $\widehat{f}$ in terms of $\overline{d}$-shadowing and asymptotic average shadowing, and we show $\overline{d}$-shadowing of f is iteration-invariant. In Section 4, we study the implications between f and $\widehat{f}$ in terms of transitive, weakly mixing, mixing, totally transitive and totally chain transitive.

2. **Preliminaries and Basic Concepts**

Let X be a compact metric space with metric d, $f : X \to X$ be a self-continuous map and $\mathbf{N} = \{1, 2, \cdots\}$, $\mathbf{Z}^+ = \{0, 1, 2, 3, \cdots\}$. Consider G as a topological group and $\varphi : G \times X \to X$ as a continuous mapping [1]. We can say that (X, G, φ) or simply X forms a metric G-space if the given conditions are satisfied:

(1) $\varphi(e, x) = x$, $\forall x \in X$, in which e is the identity of G.
(2) $\varphi(g_1, \varphi(g_2, x)) = \varphi(g_1 g_2, x)$, $\forall x \in X, \forall g_1, g_2 \in G$.

Then X is said to be compact metric G-space. To facilitate notation, the expression $\varphi(g, x)$ is commonly shortened to gx.

For $x \in X$, the $G - orbit$ of x, denoted by $G_f(x)$, is given as the set $\{gf^k(x)|g \in G, k \geq 0\}$. The set X/G of all G-orbits in X with the quotient topology induced by the quotient map $\pi : X \to X/G$ defined by $\pi(x) = G(x)$, is called the *orbit space* of X and the map ϕ is called the *orbit map*. Note that π is open and continuous. We say f is a *pseudoequivariant map*, if $f(G(x)) = G(f(x))$ for any $x \in X$. A pseudoequivariant map f can naturally induces a continuous map $\widehat{f} : X/G \to X/G$, given by $\widehat{f}(G(x)) = G(f(x))$, and the metric d can induce a metric $\widehat{d}$ on X/G.

Firstly, we review the definitions of $\overline{d}$-shadowing and asymptotic average shadowing properties of $f : X \to X$ and extend them to G-space.

Definition 2.1. *The sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *is a* δ-ergodic pseudo orbit, *if*

$$\lim_{n\to\infty} \frac{1}{n}|\{i \in \mathbf{Z}^+ : 0 \leq i < n, d(f(x_i), x_{i+1}) < \delta\}| = 1,$$

in which $|A|$ *is the cardinal number of* A. *We say that* f *has* $\overline{d}$-shadowing property, *if for any* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for any* δ*-ergodic pseudo orbit* $\{x_i\}_{i=0}^{\infty}$, *there exists* $z \in X$ *satisfying*

$$\limsup_{n\to\infty} \frac{1}{n}|\{i \in \mathbf{Z}^+ : 0 \leq i < n, d(f^i(z), x_i) < \epsilon\}| > \frac{1}{2}.$$

Definition 2.2. *The sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *is a* (G, δ)-ergodic pseudo orbit, *if there exists* $\{g_i\}_{i=0}^{\infty} \subset G$

$$\lim_{n\to\infty} \frac{1}{n}|\{i \in \mathbf{Z}^+ : 0 \leq i < n, d(g_i f(x_i), x_{i+1}) < \delta\}| = 1.$$

We say that f *has* $G - \overline{d}$-shadowing property, *if for any* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for any* (G, δ)*-ergodic pseudo orbit* $\{x_i\}_{i=0}^{\infty}$, *there exist* $z \in X$ *and* $\{g_i\}_{i=0}^{\infty} \subset G$ *satisfying*

$$\limsup_{n\to\infty} \frac{1}{n}|\{i \in \mathbf{Z}^+ : 0 \leq i < n, d(f^i(z), g_i x_i) < \epsilon\}| > \frac{1}{2}.$$

Definition 2.3. [5] *The sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *is said to be a* asymptotic average pseudo orbit, *if* $\lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n-1} d(f(x_i), x_{i+1}) = 0$. *We say that* f *has* asymptotic average shadowing property *(AASP, for short), if for any asymptotic average pseudo orbit* $\{x_i\}_{i=0}^{\infty}$, *there exists* $z \in X$ *such that*

$$\lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n-1} d(f^i(z), x_i) = 0.$$

Definition 2.4. [6] *The sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *is said to be an* G-asymptotic average pseudo orbit*, if there exists* $\{t_i\}_{i=0}^{\infty} \subset G$ *such that*

$$\lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n-1} d(t_i f(x_i), x_{i+1}) = 0.$$

We say that f *has* G-asymptotic average shadowing property *(G-AASP, for short), if for any G-asymptotic average pseudo orbit* $\{x_i\}_{i=0}^{\infty}$*, there exist* $z \in X$ *and* $\{g_i\}_{i=0}^{\infty} \subset G$ *such that* $\lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n-1} d(f^i(z), g_i x_i) = 0$.

Now, we review the definitions of some kinds of transitivity and chain transitivity of $f : X \to X$ and extend them to G-space. For $\delta > 0$ and $x, y \in X$, a δ-chain of f from x to y of length $n \in \mathbb{N}$ is a finite sequence $x_0 = x, x_1, \cdots, x_n = y$ satisfying for any $0 \le i \le n-1$, $d(f(x_i), x_{i+1}) < \delta$. Let $N(U, V) = \{n \in \mathbb{N} : f^n(U) \bigcap V \neq \emptyset\}$. we say that f is

(1) *(topologically) transitive*, if for any nonempty open sets $U, V \subset X$, $N(U, V) \neq \emptyset$.
(2) *(topologically) weakly mixing*, if for any nonempty open sets $U, V \subset X$, $N(U, V)$ contains arbitrarily long runs of positive integers, that is, there is a strictly increasing subsequence $\{n_i\}$ of $\mathbb{N}$ such that $\bigcup_{i=1}^{\infty}\{n_i, n_i + 1, \cdots, n_i + i\} \subset N(U, V)$, (see [3] for details).
(3) *(topologically) mixing*, if for any nonempty open sets $U, V \subset X$, there exists $N \in \mathbb{N}$ such that for any positive integer $n \ge N$, $f^n(U) \cap V \neq \emptyset$.
(4) *totally transitive*, if for any $n \in \mathbb{N}$, f^n is transitive.
(5) *chain transitive*, if for any $x, y \in X$ and any $\epsilon > 0$, there is an ϵ-chain from x to y.
(6) *totally chain transitive*, if for any $n \in \mathbb{N}$, f^n is chain transitive.
(7) *chain mixing*, if for any $x, y \in X$ and any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for any positive integer $n \ge N$, there is an ϵ-chain from x to y with length n.

It is easy to see that transitivity ⇒ chain transitivity, and mixing ⇒ chain mixing.

Definition 2.5. *Let* (X, d) *be a metric G-space.* f *is said to be*

(1) G-transitive[10]*, if for any non-empty open sets* $U, V \subset X$*, there exist* $n \in \mathbf{Z}^+$ *and* $g \in G$ *such that* $gf^n(U) \bigcap V \neq \emptyset$.
(2) G-weakly mixing*, if for any nonempty open sets* $U, V \subset X$*, there is a strictly increasing subsequence* $\{n_i\}$ *of* $\mathbb{N}$ *and* $\{g_{n_i}\} \subset G$ *such that* $g_{n_i} f^{n_i}(U) \bigcap V \neq \emptyset, \forall i \ge 0$*, (see* [3] *for details).*
(3) G-mixing*, if for any nonempty open sets* $U, V \subset X$*, there exists* $N \in \mathbb{N}$ *such that for any positive integer* $n \ge N$*, there exists* $g_n \in G$ *such that* $g_n f^n(U) \cap V \neq \emptyset$.
(4) G-totally transitive*, if for any* $n \in \mathbb{N}$*,* f^n *is G-transitive.*
(5) G-chain transitive*, if for any* $x, y \in X$ *and any* $\epsilon > 0$*, there is an* (G, ϵ)*-chain from* x *to* y.
(6) G-totally chain transitive*, if for any* $n \in \mathbb{N}$*,* f^n *is G-chain transitive.*
(7) G-chain mixing*, if for any* $x, y \in X$ *and any* $\epsilon > 0$*, there is* $N \in \mathbb{N}$ *such that for any positive integer* $n \ge N$*, there is an* (G, ϵ)*-chain from* x *to* y *with length* n.

3. **Shadowing properties**

Ekta Shah [10] studied that if $f : X \to X$ is a pseudo-equivariant map and $\pi : X \to X/G$ is a covering map, then f has G-shadowing if and only if the induced map $\widehat{f} : X/G \to X/G$ has shadowing. Raad Safah Abood and Iftichar M. T. proved that under the same assumptions, the G-average shadowing of f is equivalent to the average shadowing of $\widehat{f}$. Now we study the implications between f and $\widehat{f}$ in terms of $\overline{d}$-shadowing and AASP.

Theorem 3.1. *Let* X *be a compact metric G-space and* $f : X \to X$ *be a pseudo-equivariant map. Suppose that* $\pi : X \to X/G$ *is a covering map. Then* $f : X \to X$ *has* $G - \overline{d}$*-shadowing property if and only if* $\widehat{f} : X/G \to X/G$ *has* $\overline{d}$*-shadowing property.*

Proof. Sufficiency $\Leftarrow$: Suppose that $\widehat{f}: X/G \to X/G$ has $\overline{d}$-shadowing property. Let $\epsilon > 0$. Since π is a covering map and X is compact, there exists $\delta > 0$ such that for any $\pi(x) \in X/G$, $\pi^{-1}(U_\delta(\pi(x))) = \bigcup_{\alpha\in\Lambda} U_\alpha$, where each U_α in X, $\alpha \in \Lambda$, $\alpha \neq \beta \Rightarrow U_\alpha \bigcap U_\beta = \emptyset$ and $\pi|_{U_\alpha} : U_\alpha \to U_\alpha(\pi(x))$ is a homeomorphism. For ϵ-neighborhood $U_\epsilon(x)$ of x, consider U_α which contains x if $diamU_\alpha < \epsilon$, we have $\pi^{-1}|_{U_\alpha}(U_\alpha(\pi(x))) \subset U_\alpha \subset U_\epsilon(x)$. If $diamU_\alpha \geq \epsilon$, then choose $U'_\alpha \subset U_\alpha$ such that $diamU'_\alpha < \epsilon$ and $x \in U'_\alpha$, we have $\pi^{-1}|_{U'_\alpha}(U_\alpha(\pi(x))) = U'_\alpha \subset U_\epsilon(x)$. Since $\widehat{f}$ has $\overline{d}$-shadowing property, there is an $\eta > 0$ such that every η-ergodic pseudo orbit of $\widehat{f}$ is $\delta - \overline{d}$-shadowed by a point of X/G. Since π is continuous and X is compact, there is $\gamma > 0$ such that $d(x, y) < \gamma \Rightarrow \widehat{d}(\pi(x), \pi(y)) < \eta$.

Next, to complete the sufficiency of the proof, it is sufficient to prove that any (G, γ)-ergodic pseudo orbit of f is $\epsilon - \overline{d}$ shadowed by some point of X.

Let $\{x_i\}_{i=0}^\infty \subset X$ be a (G, γ)-ergodic pseudo orbit of f. Then there is $\{g_i\}_{i=0}^\infty \subset G$ such that $\lim_{n\to\infty} \frac{1}{n}|i \geq 0 : 0 \leq i < n, d(g_i f(x_i), x_{i+1}) < \gamma| = 1$. Since for any $i \in \mathbf{Z}^+$, $d(g_i f(x_i), x_{i+1}) < \gamma \Rightarrow \widehat{d}(\pi(f_(x_i)), \pi(x_{i+1})) < \eta \Rightarrow \widehat{d}(G(f(x_i)), G(x_{i+1})) < \eta$ $\Rightarrow \widehat{d}(\widehat{f}(G(x_i)), G(x_{i+1})) < \eta$,

$$\lim_{n\to\infty} \frac{1}{n}|i \geq 0 : 0 \leq i < n, \widehat{d}(\widehat{f}(G(x_i)), G(x_{i+1})) < \eta| = 1.$$

That is $\{G(x_i)\}_{i=0}^\infty$ is a η-ergodic pseudo orbit of $\widehat{f}$. Since $\widehat{f}$ has $\overline{d}$-shadowing, there exists $G(x) \in X/G$ such that

$$\limsup_{n\to\infty} \frac{1}{n}|\{i \in \mathbf{Z}^+ : 0 \leq i < n, \widehat{d}(\widehat{f}^i(G(x)), G(x_i)) < \delta\}| > \frac{1}{2}.$$

For any $i \in \mathbf{Z}^+$, since $\widehat{d}(G(f^i(x)), G(x_i)) = \widehat{d}(\widehat{f}^i(G(x)), G(x_i)) < \delta$ and $\pi(f^i(x)) \in U_\delta(\pi(x_i)) \Rightarrow f^i(x) \in \pi^{-1}(U_\delta(\pi(x_i))) \subset U_\epsilon(x_i)$ implies $d(f^i(x), x_i) > \epsilon$,

$$\limsup_{n\to\infty} \frac{1}{n}|\{i \in \mathbf{Z}^+ : 0 \leq i < n, d(f^i(x), x_i) < \epsilon\}| > \frac{1}{2}.$$

So, f has $G - \overline{d}$-shadowing.

Necessity $\Rightarrow$: Suppose that f has $G - \overline{d}$-shadowing. Let $\epsilon > 0$. Since π is continuous and X is compact, there exists $r > 0$ such that for any $x, y \in X$, $d(x, y) < r$ implies $d_1(\pi(x), \pi(y)) < \epsilon$. Since f has $G - \overline{d}$-shadowing, there is an $\eta > 0$ such that any (G, η)-ergodic pseudo orbit of f can be $\overline{d} - r$-shadowed by some point of X. Since π is a covering map and X is compact, there exists $\delta > 0$ such that for any $x \in X$, there exists α_x such that

$$(\pi|_{U_{\alpha_x}})^{-1}(U_\alpha(\pi(x))) \subset U_\eta(x).$$

Next, to complete the necessity of the proof, it is sufficient to prove that any δ-ergodic pseudo orbit of $\widehat{f}$ is $\epsilon - \overline{d}$-shadowed by some point of X/G.

Let $\{G(x_i)\}_{i=0}^\infty$ be a (G, δ)-ergodic pseudo orbit of $\widehat{f}$. Then

$$\lim_{n\to\infty} \frac{1}{n}|\{i \in \mathbf{Z}^+ : 0 \leq i < n, \widehat{d}(\widehat{f}(G(x_i)), G(x_{i+1})) < \delta\}| = 1.$$

Note that for any $i \in \mathbf{Z}^+$, $\widehat{d}(\widehat{f}(G(x_i)), G(x_{i+1})) < \delta$ implies that there exists $\alpha_{x_{i+1}}$ such that $x_{i+1} \in (\pi|_{U_{\alpha_{x_{i+1}}}})^{-1}(U_\alpha(\pi(f(x_i)))) \subset U_\eta(f(x_i))$. That is for any $n \in \mathbf{N}$,

$$|\{0 \leq i < n : d(f(x_i), x_{i+1}) < \eta\}| \geq |\{0 \leq i < n : \widehat{d}(\widehat{f}(G(x_i)), G(x_{i+1})) < \delta\}|.$$

Then

$$\lim_{n\to\infty} \frac{1}{n}|\{i \in \mathbf{Z}^+ : 0 \leq i < n, d(f(x_i), x_{i+1}) < \eta\}| = 1.$$

Thus $\{x_i\}_{i=0}^{\infty}$ is a η-ergodic pseudo orbit of f. Also, it is a (G,η)-ergodic pseudo orbit of f. Since f has $G-\overline{d}$-shadowing, there exist $x \in X$ and $\{g_i\}_{i=0}^{\infty} \in G$ such that

$$\limsup_{n\to\infty} \frac{1}{n}|i \in \mathbf{Z}^+ : 0 \le i < n, d(f^i(x), g_i x_i) < \gamma| > \frac{1}{2}.$$

Note that for any $i \in \mathbf{Z}^+$, $d(f^i(x), g_i x_i) < \gamma$, then

$$\widehat{d}(\widehat{f}^i(G(x)), G(x_i)) = \widehat{d}(G(f^i(x), G(x_i))) = \widehat{d}(\pi(f^i(x)), \pi(g_i x_i)) < \epsilon.$$

Thus,

$$\limsup_{n\to\infty} \frac{1}{n}|i \in \mathbf{Z}^+ : 0 \le i < n, \widehat{d}(\widehat{f}^i(G(x)), G(x_i)) < \epsilon| > \frac{1}{2}.$$

So, $\widehat{f}$ has $\overline{d}$-shadowing property.

□

To establish the equivalence between the G-AASP of f and AASP of $\widehat{f}$, we first introduce the following two definitions.

Definition 3.1. *$f : X \to Y$ is strong continuous, if for any $\epsilon > 0$ and any $x \in X$, there exists $\delta > 0$ such that $d(x,y) < r\delta$ implies $d(f(x), f(y)) < r\epsilon$, $\forall x, y \in X, \forall r > 0$.*

Definition 3.2. *A strong continuous map $f : X \to Y$ is called a strong covering map, if for each $y \in Y$, there exists $\delta > 0$ such that for any $r > 0$*

$$f^{-1}(B(y, r\delta)) = U_{\alpha\in\Lambda} B(x_\alpha, r\alpha),$$

where $x_\alpha \in X$, $\alpha, \beta \in \Lambda$, $\alpha \ne \beta$ implies $B(x_\alpha, r\alpha) \bigcap B(x_\beta, r\beta) = \emptyset$ and

$$f \mid_{B(x_\alpha, r\alpha)}: B(x_\alpha, r\alpha) \to B(y, r\delta)$$

is a homeomorphism.

Clearly, if f is strong continuous then it is continuous. If f is a strong covering map then it is a covering map.

Theorem 3.2. *Let (X,d) be a compact metric G-space, $f : X \to X$ be pseudoequi variant and $\pi : X \to X/G$ be a strong covering map. Then $f : X \to X$ has G-AASP if and only if $\widehat{f} : X/G \to X/G$ has AASP.*

Proof. $\Leftarrow$:Suppose that $\widehat{f} : X/G \to X/G$ has AASP. Let $\epsilon > 0$. Since X is compact and π is a strong covering map, there exists $\delta > 0$ such that for any $r > 0$, $d(x,y) < r\delta$ implies $\widehat{d}(\pi(x), \pi(y)) < r\epsilon$, $\forall x, y \in X$. And there exists $\eta > 0$ such that for any $r > 0$ and any $\pi(x) \in X/G$, $\pi^{-1}(B(\pi(x), r\eta)) = U_{\alpha\in\Lambda} B(x_\alpha, r\alpha)$, where $x_\alpha \in X$, $\alpha, \beta \in \Lambda$, $\alpha \ne \beta$ implies $B(x_\alpha, r\alpha) \bigcap B(x_\beta, r\beta) = \emptyset$ and

$$f \mid_{B(x_\alpha, r\alpha)}: B(x_\alpha, r\alpha) \to B(y, r\eta)$$

is a homeomorphism. For $B(x, r\epsilon)$, consider $B(x_\alpha, r\alpha)$ which contains x if $\alpha < \epsilon$, then

$$\pi^{-1} \mid_{B(x_\alpha, r\alpha)} (B(y, r\eta)) \subset B(x_\alpha, r\alpha) \subset B(x, r\epsilon).$$

If $\alpha \ge \epsilon$, then choose $\alpha' < \alpha$ such that $\alpha' < \epsilon$ and $x \in B(x_\alpha, r\alpha')$. Then

$$\pi^{-1} \mid_{B(x_\alpha, r\alpha')} (B(y, r\eta)) \subset B(x_\alpha, r\alpha') \subset B(x, r\epsilon).$$

Let $\{x_i\}_{i=0}^{\infty}$ be a G-asymptotic average pseudo orbit. Then there exists $\{t_i\}_{i=0}^{\infty}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \Sigma_{i=0}^{n-1} d(t_i f(x_i), x_{i+1}) = 0.$$

Thus for $\delta > 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{n}\Sigma_{i=0}^{n-1} d(t_i f(x_i), x_{i+1}) < \delta$, $\forall n \geq N$. Let for each $i \geq 0$, $d(t_i f(x_i), x_{i+1}) < r_i\delta$, then

$$\widehat{d}(\pi(t_i f(x_i)), \pi(x_{i+1})) = \widehat{d}(G(f(x_i)), G(x_{i+1})) = \widehat{d}(\widehat{f}(G(x_i)), G(x_{i+1})) < r_i\epsilon,$$

and when $n > N$, $\Sigma_{i=0}^{n-1} \frac{r_i}{n} < 1$. Thus $\frac{1}{n}\Sigma_{i=0}^{n-1}\widehat{d}(\widehat{f}(G(x_i)), G(x_{i+1})) < \epsilon$. Then

$$\lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n-1}\widehat{d}(\widehat{f}(G(x_i)), G(x_{i+1})) = 0.$$

That is, $\{G(x_i)\}_{i=0}^{\infty}$ is an asymptotic average pseudo orbit of $\widehat{f}$.

Since $\widehat{f}$ has AASP, there exists $G(z) \in X/G$ such that

$$\lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n-1}\widehat{d}(\widehat{f}^i(G(z)), G(x_i)) = 0.$$

Then for $\eta > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$$\frac{1}{n}\Sigma_{i=0}^{n-1}\widehat{d}(\widehat{f}^i(G(z)), G(x_i)) < \eta.$$

Let for any $i \geq 0$, $\widehat{d}(\widehat{f}^i(G(z)), G(x_i)) < r_i'\eta$. Then $\widehat{d}(\pi(f^i(z)), \pi(x_i)) < r_i'\eta$ and when $n > N$, $\frac{1}{n}\Sigma_{i=0}^{n-1} r_i' < 1$. Thus, $\pi(f^i(z)) \in B(\pi(x_i), r_i'\eta) \Rightarrow f^i(z) \in \pi^{-1}(B(\pi(x_i), r_i'\eta)) \subset B(x_i, r_i'\epsilon) \Rightarrow d(f^i(z), x_i) < r_i'\epsilon \Rightarrow \frac{1}{n}\Sigma_{i=0}^{n-1} d(f^i(z), x_i) < \frac{1}{n}\Sigma_{i=0}^{n-1} r_i'\epsilon \Rightarrow \lim_{n\to\infty}\frac{1}{n}\Sigma_{i=0}^{n-1} d(f^i(z), x_i) = 0$. So, f has G-AASP.

$\Rightarrow$: Suppose that f has G-AASP. Let $\epsilon > 0$. Since X is compact and π is a strong covering map, there exists $\delta > 0$ such that for any $r > 0$, $d(x, y) < r\delta$ implies $\widehat{d}(\pi(x), \pi(y)) < r\epsilon$, $\forall x, y \in X$. And there exists $\eta > 0$ such that for any $r > 0$ and any $x \in X$, there exists $x_\alpha \in X$ such that

$$\pi_{-1} \mid_{B(x_\alpha, r\alpha)} (B(\pi(x), r\eta)) \subset B(x, r\epsilon).$$

Let $\{G(x_i)\}_{i=0}^{\infty}$ be an asymptotic average pseudo orbit of $\widehat{f}$. Then

$$\lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n-1} d(\widehat{f}(G(x_i)), G(x_{i+1})) = 0.$$

For $\eta > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$$\frac{1}{n}\Sigma_{i=0}^{n-1} d(\widehat{f}(G(x_i)), G(x_{i+1})) < \eta.$$

Suppose that for any $i \geq 0$, $d(\widehat{f}(G(x_i)), G(x_{i+1})) = \widehat{d}(\pi(f(x_i)), \pi(x_{i+1})) < r_i\eta$. Then for any $n > N$, $\Sigma_{i=0}^{n-1}\frac{r_i}{n} < 1$. And for any $i \geq 0$, there exists $x_{i+1,\alpha} \in X$ such that

$$x_{i+1} \in \pi^{-1} \mid_{B(x_{i+1,\alpha}, r\alpha)} (B(\pi(f(x_i)), r_i\eta)) \subset B(f(x_i), r_i\epsilon).$$

Then $d(f(x_i), x_{i+1}) < r_i\epsilon$, $\forall i \geq 0$. Thus, $\lim_{n\to\infty}\frac{1}{n}\Sigma_{i=0}^{n-1} d(f(x_i), x_{i+1}) = 0$. That is $\{x_i\}_{i=0}^{\infty}$ is an asymptotic average pseudo orbit of f.

Since f has G-AASP, there exist $z \in X$ and $\{t_i\}_{i=0}^{\infty} \subset G$ such that

$$\lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n-1} d(f^i(z), t_i x_i) = 0.$$

Then for $\delta > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $\frac{1}{n}\Sigma_{i=0}^{n-1} d(f^i(z), t_i x_i) < \delta$. Let for any $i \geq 0$, $d(f^i(z), t_i x_i) < r_i'\delta$, then $\widehat{d}(\pi(f^i(z)), \pi(t_i x_i)) < r_i'\epsilon \Rightarrow \widehat{d}(\widehat{f}^i(G(z)), G(x_i)) < r_i'\epsilon$ $\Rightarrow \frac{1}{n}\Sigma_{i=0}^{n-1}\widehat{d}(\widehat{f}^i(G(z)), G(x_i)) < \epsilon$. Thus,

$$\lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n-1}\widehat{d}(\widehat{f}^i(G(z)), G(x_i)) = 0.$$

So, $\widehat{f}$ has AASP.

Wu [12] proved that $\overline{d}$-shadowing of $f : X \to X$ is iteratively invariant. Now we show that $G - \overline{d}$-shadowing of $f : X \to X$ is also iteratively invariant.

Theorem 3.3. *Let (X,d) be a compact metric G-space, then the following statements are equivalent.*

(1) *f has $G-\overline{d}$-shadowing,*
(2) *f^k has $G-\overline{d}$-shadowing for any $k\in\mathbb{N}$,*
(3) *f^k has $G-\overline{d}$-shadowing for some $k\in\mathbb{N}$,*

Proof. (1) $\Rightarrow$ (2): Suppose that $f:X\to X$ has $G-\overline{d}$-shadowing property. Let $k\in\mathbb{N}$. By Theorem 3.1, $\widehat{f}:X/G\to X/G$ has $\overline{d}$-shadowing property. By Lemma 2.1 of [12], $\widehat{f}^k$ has $\overline{d}$-shadowing property. By Theorem 3.1, f^k has $G-\overline{d}$-shadowing property.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1): Suppose that f^k has $G-\overline{d}$-shadowing property for some $k\in\mathbb{N}$. By Theorem 3.1, $\widehat{f}^k$ has $\overline{d}$-shadowing property. By Lemma 2.1 of [12], $\widehat{f}$ has $\overline{d}$-shadowing property. By Theorem 3.1, $\widehat{f}$ has $\overline{d}$-shadowing property. □

□

4. G-transitive properties and transitive properties

Ekta Shah[10] proved that if $f:X\to X$ is G-transitive then $\widehat{f}:X/G\to X/G$ is transitive. Now we study the converse.

Theorem 4.1. *Suppose that $\pi:X\to X/G$ is a homeomorphic map. If $\widehat{f}:X/G\to X/G$ is transitive then $f:X\to X$ is transitive. In particular, f is G-transitive.*

Proof. Let $U,V\subset X$ be two nonempty open sets. Then $\pi(U)$ and $\pi(V)$ are two nonempty open sets of X/G. Let $x\in U, y\in V,\epsilon>0$ such that $B(x,\epsilon)\subset U$ and $B(y,\epsilon)\subset V$. Since π is homeomorphic and X is compact, there exists $\delta>0$ such that for any $G(u),G(v)\in X/G$, $d(G(u),G(v))<\delta$ implies $d(\pi^{-1}(G(u)),\pi^{-1}(G(v)))=d(u,v)<\epsilon$. Since $\widehat{f}$ is transitive, there exists $n>0$ such that $\widehat{f}^n(B(G(x),\delta))\bigcap B(G(y),\delta)\neq\emptyset$. Then there exists $G(z)\in X/G$ such that $d(G(x),G(z))<\delta$ and $d(\widehat{f}^n(G(z)),G(y))<\delta$. Furthermore, $d(x,z)<\epsilon$ and $d(f^n(z),y)<\epsilon$. That is, $f^n(B(x,\epsilon))\bigcap B(y,\epsilon)\neq\emptyset$. So, f is transitive. Particularly, it is G-transitive. □

Theorem 4.2. *Suppose that $\pi:X\to X/G$ is a homeomorphic map. $f:X\to X$ is G-totally chain transitive if and only if $\widehat{f}:X/G\to X/G$ is totally chain transitive.*

Proof. Necessity: Suppose that f is G-totally chain transitive. Let $n\in\mathbb{N}$, $G(x),G(y)\in X/G$ and $\epsilon>0$. Then $x,y\in X$. Since π is continuous, there exists $\delta>0$ such that for any $u,v\in X$, $d(u,v)<\delta$ implies $d(G(u),G(v))<\epsilon$. Since f is G-totally chain transitive, f^n is G-chain transitive. Then there exists a (G,δ)-chain of f^n from x to y, denoted it by $\{x_0^n,x_1^n,x_2^n,\cdots,x_k^n\}$. And there exists $\{g_{n,i}\}_{i=0}^{k-1}\subset G$ such that for any $0\le i\le k-1$, $d(g_{n,i}f^n(x_i^n),x_{i+1}^n)<\delta$. Then for any $0\le i\le k-1$, $d(G(g_{n,i}f^n(x_i^n)),G(x_{i+1}^n))<\epsilon$. Since $G(g_{n,i}f^n(x_i^n))=G(f^n(x_i^n))=\widehat{f}^n(G(x_i^n))$,

$$d(\widehat{f}^n(G(x_i^n)),G(x_{i+1}^n))<\epsilon.$$

That is $\{G(x_0^n),G(x_1^n),\cdots,G(x_k^n)\}$ is a ϵ-chain of $\widehat{f}^n$ from $G(x)$ to $G(y)$. So, $\widehat{f}$ is totally chain transitive.

Sufficiency: Suppose that $\widehat{f}$ is totally chain transitive. Let $n\in\mathbb{N}$, $x,y\in X$ and $\epsilon>0$. Then $\pi(x)=G(x),\pi(y)=G(y)\in X/G$. Since Since π is homeomorphic and X is compact, there exists $\delta>0$ such that for any $G(u),G(v)\in X/G$, $d(G(u),G(v))<\delta$ implies $d(\pi^{-1}(G(u)),\pi^{-1}(G(v)))=d(u,v)<\epsilon$. Since $\widehat{f}$ is totally chain transitive, there exists a δ-chain of $\widehat{f}^n$ from $G(x)$ to $G(y)$, denoted it by $\{G(x_0),G(x_1),G(x_2),\cdots,G(x_k)\}$.

Then for any $0 \le i \le k-1$, $d(\widehat{f}^n(G(x_i)), G(x_{i+1})) < \delta$. Furthermore, $d(f^n(x_i), x_{i+1}) < \epsilon$, $\forall 0 \le i \le k-1$. That is $\{x_0, x_1, \cdots, x_k\}$ is a (G, ϵ)-chain of f^n from x to y. So, f is totally chain transitive. Particularly, it is G-totally chain transitive □

By [10] and the proof of Theorem 4.1 and Theorem 4.2, we can similarly obtain the following theorem.

Theorem 4.3. *Suppose that $\pi : X \to X/G$ is a homeomorphic map. $f : X \to X$ is G-chain mixing (weakly mixing, mixing, totally transitive, respectively) if and only if $\widehat{f} : X/G \to X/G$ is chain mixing (weakly mixing, mixing, totally transitive, respectively).*

Theorem 4.4. *If $f : X \to X$ has $G - \overline{d}$-shadowing property, then $f : X \to X$ is G-chain mixing.*

Proof. Suppose that $f : X \to X$ has $G - \overline{d}$-shadowing property. By Theorem 3.1, $\widehat{f} : X/G \to X/G$ has $\overline{d}$-shadowing property. Then $\widehat{f} : X/G \to X/G$ is chain mixing. So, f is G-chain mixing. □

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