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A NOVEL SOFT FRACTIONAL IDEALS APPROACH TO CHARACTERIZE DEDEKIND DOMAINS

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A solid mathematical method for handling uncertainty is offered by the Molodtsov-conceived notion of soft sets. This study investigates the benefits of using soft set theory methods, opening the door for creative approaches to soft multiplicative ideal theory research. Our findings demonstrate that a finite minimal generating set (as a soft R-submodule) for a soft fractional ideal of finite values of integral domain exists with invertible level ideals. We provide characterizations of Dedekind domains in relation to invertibility of specific soft fractional ideals and also in connection with the factorization of soft ideals as prime and maximal soft ideal products.

Keywords: Fractional ideal, Soft fractional ideal, Minimal generating set, Dedekind domain.

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1. **Introduction**

As a crucial extension to several other mathematical concepts for addressing uncertainty. Molodtsov [1] used a set-valued mapping to develop the idea of soft set theory. Jun [2] and Jun and Park [3] pioneered several directions concerning the application of soft sets within the framework of ideal theory for soft BCK/BCI algebras. Subsequently, numerous researchers have investigated various methodologies associated with soft set theory. Maji et al. [4] instigated an in-depth theoretical study of soft sets, encompassing concepts such as supersets, subsets, union and intersection operations, null soft sets, and their associated properties. Onyeozili and Gwary [5] carried out a critical and systematic research on matrix representation of soft sets, relations and functions of soft sets with their operational properties. Aktas and Çağman [6] conducted a comparative analysis between soft set theory and its counterparts—rough sets and fuzzy sets, highlighting similarities and distinctions in their foundational concepts and properties. Building on this framework, they extended the notion of soft sets to group theory by introducing the concept of soft groups and exploring their associated characteristics. Acar et al. [7] investigated and elaborated on the foundational concepts underlying soft ring theory. Sun et al. [8] instigated the notion of soft modules by integrating principles from module theory and soft set theory, and pointed out several key properties associated with this framework. Türkmen and Pancar [9] formulated various properties related to soft submodules within the context of module theory and supported their theoretical findings through illustrative examples. Atagün and Sezgin [10] explored the algebraic formulation of soft substructures within modules, fields, and rings. They introduced notions such as soft subfields, soft submodules over R-modules, soft subrings, and

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soft ideals, and provided illustrative examples to clarify these concepts. Rahimi and Alimoradi [11] instigated the notions of maximal and prime soft ideals in soft rings and shows the comparison between soft maximal ideals and soft prime ideals. Acar and Ozturk [12] instigated the ideas of maximal and prime soft ideals, soft irreducible, maximal and prime soft idealistic over BCK/BCI algebras and also demonstrated their relations. Taouti and Khan [13] introduced the concepts of soft fields and soft integral domains, along with the concept of fractional elements within soft rings. Taouti et al. [14] examined ideal-related structures in soft rings and formulated the definition of soft fractional ideals. The fractional ideal in conjunction with several basic soft operations has also been investigated by the authors. Studying domains, particularly Dedekind domains demonstrate the unique use of this fractional ideal concept.

Furthermore, Zadeh's [15] groundbreaking work on the fuzzy sets gives an acceptable foundation in his fuzzy set theory containing the most applicable concepts to deal with unforeseen problems. The revolutionary work of Zadeh led the way of fuzzifying algebraic structures. Since then, there has been an additional expansion of this work by Mordeson and Malik [16]. Lee and Mordeson [17] demonstrate that every fuzzy ideal can be uniquely factorized as a product of fuzzy maximal ideals in a Dedekind domain. For a detailed account of relevant definitions and key findings, we refer [18, 19, 20, 22, 23, 24, 25, 26].

2. **Preliminaries**

This section recalls the key definitions and results essential to our framework. Let R denote an integral domain with fraction field M and let U be the universal set.

Definition 2.1 ([1]). *A soft set (β, E) over U is a mapping from a set of parameters E into $P(U)$. i.e., $\beta : E \longrightarrow P(U)$, where $P(U)$ symbolizes the power set of U.*

In other words, A function $f_B : E \longrightarrow P(U)$ such that $f_B(x) = \phi$ if $x \notin B$, is called a soft set over U, where E is a set of parameters and $B \subseteq E$.

Definition 2.2 ([4]). *For two soft sets (δ, C) and (ω, D) over U, we say that (δ, C) is a soft subset of (ω, D) symbolized by $(\delta, C) \widetilde{\subset} (\omega, D)$, if it fulfils:*

(i) $C \subset D$

(ii) $\forall\ l \in C$, *$\delta(l)$ and $\omega(l)$ are identical approximations.*

Consider δ and ω be soft subsets of R. We write $\delta \widetilde{\subseteq} \omega$ if $\delta(k) \subseteq \omega(k)\ \forall\ k \in R$. Moreover, if $\delta \widetilde{\subseteq} \omega$ and there exists $k \in R$ such that $\delta(k) \subset \omega(k)$, then we write $\delta \widetilde{\subset} \omega$. We symbolize the image of δ by $|\Im m(\delta)|$. We say that the value of δ is finite if $|\Im m(\delta)| < \infty$. If there is a maximal element in every nonempty subset of $|\Im m(\delta)|$, then δ is said to have the sup-property.

Definition 2.3. *Let δ be a soft set over a field M. Then,*

$$\delta_\alpha = \{t \in M : \delta(t) \supseteq \alpha\}$$

is called the level set, for every $\alpha \in P(M)$.

Assume $L \subseteq R \subseteq M$, with χ_L as its characteristic function. Then, for each $\alpha \in P(M)$, the soft subset χ_L^α of M is outlined by

$$\chi_L^\alpha(t) = \begin{cases} \alpha & ; \quad if \ \ t \in M - R \\ U & ; \quad if \ \ t \in R \end{cases}$$

Definition 2.4 ([7]). *Let (ζ, A) be a soft set. The set*

$$supp(\zeta, A) = \{t \in A \mid \zeta(t) \neq \phi\}$$

is said to be support of the soft set (ζ, A). A soft set is said to be non-null if its support is not equal to the empty set.

Definition 2.5 ([7])**.** *Let (ζ, A) be a non-null soft set over R. Then, (ζ, A) is said to be a soft ring over R if $\zeta(t)$ is a subring of R, $\forall\ t \in A$.*

Definition 2.6 ([7])**.** *Let (H, C) be a soft ring over R. A non-null soft set (β, F) over R is said to be soft ideal of (H, C), symbolized by $(\beta, F)\ \widetilde{\triangleleft}\ (H, C)$, if it satisfies:*

(i) $F \subset C$

(ii) *$\beta(t)$ is an ideal of $H(t)$, $\forall\ t \in supp(\beta, F)$.*

Definition 2.7 ([14])**.** *A soft subset ζ is a soft ideal of a ring R if $\zeta(c-d) \supseteq \zeta(c) \cap \zeta(d)$ and $\zeta(cd) \supseteq \zeta(c) \cup \zeta(d)$, for all $c, d \in R$.*

A soft subset ζ of R is an ideal of R if and only if $\zeta(0) \supseteq \zeta(c)\ \forall\ c \in R$ and ζ_α is an ideal of R, $\forall\ \alpha \in P(M)$.

Definition 2.8 ([14])**.** *Let R be a ring contained in a field M and (ζ, M) be a soft subset over the field M. Then, ζ is a soft R-submodule of M if it satisfies:*

(i) $\zeta(c-d) \supseteq \zeta(c) \cap \zeta(d),\ \forall\ c, d \in M$

(ii) $\zeta(rc) \supseteq \zeta(c),\ \forall\ c \in M,\ r \in R$

(iii) $\zeta(0) = R$

A soft subset ζ of M is a soft R-submodule of M if and only if $\zeta(0) = R$ and for every $\alpha \in P(M)$, the set ζ_α forms an R-submodule of M. The associated set is symbolized by $\zeta_* = \{c \in M : \zeta(c) = \zeta(0)\}$. Let ζ and ξ be soft sets over the field M, then $\zeta \circ \xi$ and $\zeta\xi$ represent the soft sets of M, outlined by

$$(\zeta \circ \xi)(t) = \begin{cases} \widetilde{\bigcup}\{\zeta(c) \cap \xi(d) \,:\, t = cd\ ;\ c, d \in M\},\ \forall\ t \in M \\ \\ \quad 0 \quad ; \quad \text{otherwise} \end{cases}$$

$$\text{and } (\zeta\xi)(t) = \widetilde{\bigcup}\Big\{ \bigcap_{i=1}^{n} (\zeta(c_i) \cap \xi(d_i)) : c_i, d_i \in M,\ n \geq i \geq 1,\ n \in \mathbb{N}, \\ t = \sum_{i=1}^{n} c_i d_i \Big\},\ \forall\ t \in M$$

Definition 2.9 ([14])**.** *For $d \in M$ and $\zeta \in P(M)$, let d_{ζ} symbolizes the soft subset of M, outlined by*

$$d_{\zeta}(t) = \begin{cases} \zeta \quad ; \quad if\ \ t = d\ ,\forall\ \ t \in M \\ 0 \quad ; \quad otherwise \end{cases}$$

We call $d_{\zeta}(t)$ a soft singleton.

Let d_δ and e_ω be soft singletons of M and let f_β be a soft singleton of R. Then, $(d_\delta + e_\omega) = (d+e)_\gamma$ and $f_\beta d_\delta = (fd)_\alpha$, where $\gamma = \min\{\delta, \omega\}$ and $\alpha = \min\{\beta, \delta\}$.

Definition 2.10 ([14])**.** *Let M be the field of fractions of an integral domain R. A soft R-submodule ζ of M is called a soft fractional ideal of R, if there exists $d \in R$; $d \neq 0$ such that $d_R \circ \zeta \subseteq \chi_R^{\omega}$, for some $\omega \in M - R$.*

Definition 2.11 ([21])**.** *Let R be an integral domain and δ be a soft fractional ideal of R, then δ is soft invertible if there exists another soft fractional ideal δ' of R such that $\delta\delta' = \chi_R^{\alpha}$, for every $\alpha \in P(M)$.*

Theorem 2.1. ([21]) *Let R be an integral domain and M be the fraction field of R. Let δ be a soft fractional ideal of R. If δ is soft invertible, then $\cup\{\delta(m) : m \in M \setminus \delta_*\}$ exists iff δ_* is an invertible fractional ideal of R.*

Proposition 2.1. ([21]) *Consider an integral domain R and M be the fraction field of R. Let δ be a soft R-submodule of M. Then,*

(i) *δ is a soft fractional ideal of $R \Longleftrightarrow r\delta$ is soft fractional ideal of R, for any $r \in R$.*

(ii) *δ is a soft invertible and $\cup\{\delta(x) : x \in M \setminus \delta_*\}$ exists $\Longleftrightarrow r\delta$ is soft invertible and $\cup\{(r\delta)(y) : y \in M \setminus \delta_*\}$ exists.*

3. Soft fractional ideals (Soft $\mathcal{FI}s$) with minimal generating sets

Throughout this paper, the symbols $\mathcal{FI}$ and $\mathcal{FI}s$ denotes the fractional ideal and fractional ideals. In this section, we consider R be an integral domain with fraction field M and symbolizing the minimal generating set as $\mathfrak{m.g.s}$. This section will examine the special features of soft $\mathcal{FI}s$ with the notion of minimal generating set ($\mathfrak{m.g.s.}$) and helps to understand why soft set theory relies heavily on them. Our goal is to reveal the theoretical complexities and practical uses of these features by dissecting them in an understandable manner. We intend to accomplish this by highlighting the practical significance of soft $\mathcal{FI}s$ within the larger context of soft set theory in order to make them more approachable.

Definition 3.1. *Consider a collection of soft sets $\{\zeta_i \mid i = 1, ..., n\}$ of M. Then, the soft set $\sum\limits_{i=1}^{n} \zeta_i$ of M is defined by*

$$(\sum_{i=1}^{n} \zeta_i)(t) = \cup\{\cap(\zeta_i(t) : i = 1, ..., n) \mid t = \sum_{i=1}^{n} t_i,\ t_i \in M\},\ \forall\ t \in M$$

Also, the soft set $\widetilde{\bigcap}_{i\in I}\zeta_i$ of M formed by the intersection over an index set I is defined by

$$(\widetilde{\bigcap}_{i\in I}\zeta_i)(t) = \cap\{\zeta_i(t) \mid i \in I\},\ \forall\ t \in M.$$

Definition 3.2. *Consider a soft subset γ of M and $\langle\gamma\rangle$ be the intersection of all soft submodules of M containing γ. Then, $\langle\gamma\rangle$ is said to be a soft submodule of M generated by γ. Moreover, let $\mathcal{B}$ be a set of soft singletons of M such that if $k_u, k_v \in \mathcal{B}$, then $u = v > 0$. For all $k \in M$, we define a soft subset γ of M such that $k_u \in \mathcal{B}$ and is outlined by*

$$\gamma(k) = \begin{cases} u & ;\quad if\ \ k_u \in \mathcal{B} \\ 0 & ;\quad if\ \ u \notin P(M) \end{cases}$$

Then we can state that $\langle\mathcal{B}\rangle = \langle\gamma\rangle$.

Definition 3.3. *Consider $\mathcal{B}$ be a set of soft singletons of M with the condition that $v = u > 0$, if $k_v, k_u \in \mathcal{B}$ and δ be a soft $\mathcal{FI}$ of M. If $\mathcal{B}$ generates δ, then $\mathcal{B}$ is said to be a $\mathfrak{m.g.s.}$ for δ (as a soft R-submodule), if $\delta = \langle\mathcal{B}\rangle \mathbin{\widetilde{\cup}} 0_R$ and $\nexists\ \mathcal{B}' \mathbin{\widetilde{\subset}} \mathcal{B}$ such that $\delta = \langle\mathcal{B}'\rangle \mathbin{\widetilde{\cup}} 0_R$.*

Theorem 3.1. *Consider δ be a soft $\mathcal{FI}$ of R such that $\mathcal{I}m(\delta) = \{u_0, u_1, ..., u_n\}$ with the condition $u_0 \subset u_1 \subset ... \subset u_n$. Then, δ has a finite $\mathfrak{m.g.s.}$ (as a soft R-submodule), if δ_{u_i} is invertible for $i = 1, 2, ..., n$.*

Proof. For $i = 0, 1, ..., n, \delta_{u_i}$ has a finite basis as an R-module by [18](Lemma 3). Thus, for each δ_{u_i}, a finite generating set can be constructed as follows: Consider $\mathcal{F}_i = \{k_{i1}, k_{i2}, ..., k_{im_i}\} \mathbin{\widetilde{\subseteq}} \delta_{u_i}, i = 1, 2, ..., n$ be such that $\mathcal{F}_n$ is a basis of $\delta_{u_n} = \delta_*$ and $\mathcal{F}_i \mathbin{\widetilde{\subseteq}} \delta_{u_i} \setminus \delta_{u_{i+1}}$ a smallest subset such that $\mathcal{F}_n \mathbin{\widetilde{\cup}} ..... \mathbin{\widetilde{\cup}} \mathcal{F}_i$ generates δ_{u_i} for $i = 1, 2, ..., n-1$. Suppose $\mathcal{B} = \{(k_{ij})_{u_i} \mid i = 1, 2, ..., n; j = 1, 2, ..., m_i\}$. We proceed to show that $\mathcal{B}$ is a $\mathfrak{m.g.s.}$ for δ. Let $k \in M$, then

$$(\langle\mathcal{B}\rangle \mathbin{\widetilde{\cup}} 0_R)(k) = \begin{cases} \widetilde{\bigcup}\{(\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m_i}(t_{ij})_R(k_{ij})_{u_i})(k) \mid t_{ij} \in R, j = 1, ..., m_i \\ \qquad\qquad ; i = 1, ..., n\} \quad ; \quad if \quad k \in M \setminus \delta_{u_n} \\ \\ u_n \quad ; \quad if \ \ k \in \delta_{u_n} \end{cases}$$

If $k \in \delta_{u_p} \setminus \delta_{u_{p+1}}$ for some $p = 1, ..., n-1$, then the above soft union is obtained for $k = \sum_{i=1}^{p} \sum_{j=1}^{m_i} t_{ij} k_{ij}$ for some $t_{ij} \neq 0$. Thus, $(\langle \mathcal{B} \rangle \, \widetilde{\cup} \, 0_R)(k) = u_p = \delta(k)$. Therefore, $\delta = \langle \mathcal{B} \rangle \, \widetilde{\cup} \, 0_R$. Hence, $\mathcal{B}$ is a generating set for δ. We now show that $\mathcal{B}$ is minimal. For this, let $(k_{ij})_{u_i} \in \mathcal{B}$ and let $\mathcal{B}^{'} = \mathcal{B} \setminus \{(k_{ij})_{u_i}\}$. Now $k_{ij} \notin \langle \mathcal{F}_n, ..., \mathcal{F}_i \setminus \{k_{ij}\} \rangle$ and so $(\langle \mathcal{B}^{'} \rangle \, \widetilde{\cup} \, 0_R)(k_{ij}) \subset u_i = \delta(k_{ij})$. Hence, $\mathcal{B}$ is a finite $\mathfrak{m.g.s}$. of δ. ☐

Definition 3.4. *Consider δ be a soft $\mathcal{FI}$ of R. Then, it is simple to demonstrate that $\delta|_R$ is a soft ideal of R. If $\delta|_R$ is a prime (resp., maximal) soft ideal of R, then δ is said to be a prime (resp., maximal) soft $\mathcal{FI}$ of R.*

Definition 3.5. *A soft $\mathcal{FI}$ δ is called an integral soft $\mathcal{FI}$ of R, if $\delta(x) = y$, $\forall \ x \in M \setminus R$, $y \in P(M)$. Thus, the $\mathcal{I}m(\delta) = \{u_i\} \ \forall \ u_i \in P(M)$, if δ is a prime (resp., maximal) integral soft $\mathcal{FI}$ of R and $(\delta|_R)_*$ is a prime (resp., maximal) ideal of R.*

Corollary 3.1. *In an integral domain especially that of a Dedekind domain R, a $\mathfrak{m.g.s}$. with a maximum of $2n$ elements exists for every integral soft $\mathcal{FI}$ δ of R with $|\mathcal{I}m(\delta)| = n+1$, for any $n \in \mathbb{N}$.*

Proof. Consider δ be an integral soft $\mathcal{FI}$ with $\mathcal{I}m(\delta) = \{u_0, u_1, ..., u_n\}$ with the condition $u_0 \subset u_1 \subset ... \subset u_n = R$. Then, for $i = 1, 2, ..., n$, δ_{u_i} is an integral soft $\mathcal{FI}$ of R which is invertible. δ_{u_i} admits a basis of two elements for $i = 1, ..., n$ by [18](Corollary 2). Then by Theorem 3.1, $\mathcal{B} = \{(k_{ij})_{u_i} \mid i = 1, ..., n; j = 1, 2\}$ is a $\mathfrak{m.g.s}$. for δ with a maximum of $2n$ elements. i.e., $|\mathcal{B}| \leq 2n$. ☐

Proposition 3.1. *Consider δ be a soft $\mathcal{FI}$ of R. Then, δ can be factorized uniquely as product of finite valued soft invertible prime integral soft $\mathcal{FI}$s.*

Proof. Let the product of soft invertible prime integral soft $\mathcal{FI}$s be $\delta = \prod_{i=1}^{n} \zeta_i$. Let there is one more such product say $\prod_{j=1}^{m} \zeta_j^{'}$. Since ζ_i and $\zeta_j^{'}$ are finite valued, therefore

$$\prod_{i=1}^{n} (\zeta_i)_* = \left(\prod_{i=1}^{n} \zeta_i \right)_* = \left(\prod_{j=1}^{m} \zeta_j^{'} \right)_* = \prod_{j=1}^{m} (\zeta_j^{'})_*$$

Now $(\zeta_i)_*$ and $(\zeta_j^{'})_*$ are invertible prime integral ideals of R. Therefore, after a suitable arrangement, we have $(\zeta_i)_* = (\zeta_i^{'})_*$ and $n = m$ for $i = 1, 2, ..., n$ by [18](Lemma 5). For $i = 1, ..., n$, let $u_i = \zeta_i(k_i)$ and $u_i^{'} = \zeta_i^{'}(k_i^{'})$, where $k_i \in R \setminus (\zeta_i)_*$ and $k_i^{'} \in R \setminus (\zeta_i^{'})_*$. Now take a minimal element with smallest value u_i or $u_i^{'}$ in the set $\{\zeta_i \mid i = 1, 2, ..., n\} \, \widetilde{\cup} \, \{\zeta_j^{'} \mid j = 1, 2, ..., n\}$ say ζ_1. Now $\prod_{j=1}^{n} \zeta_j^{'} \, \widetilde{\subseteq} \, \zeta_1$ and so there exists some $\zeta_j^{'}$ say $\zeta_1^{'}$ such that $\zeta_1^{'} \, \widetilde{\subseteq} \, \zeta_1$. Now $\prod_{i=1}^{n} \zeta_i \, \widetilde{\subseteq} \, \zeta_1^{'}$ and so some $\zeta_i \, \widetilde{\subseteq} \, \zeta_1^{'}$ say ζ_x. Thus, by the minimality of ζ_1, we have $\zeta_x = \zeta_1^{'} = \zeta_1$. Assume δ_1 be an inverse of $\zeta_1 = \zeta_1^{'}$. Then, we have the option of choosing a soft inverse δ_1 such that $\mathcal{I}m(\delta_1) = \{u_i\}$, for every $u_i \in P(M)$. Therefore, $\delta_1 \zeta_1 = \chi_R^{(u_i)}$. Hence, $\delta_1 \zeta_1, ..., \zeta_n = \delta_1 \zeta_1^{'}, ..., \zeta_n^{'}$ or $\chi_R^{(u_i)} \zeta_2, ..., \zeta_n = \chi_R^{(u_i)} \zeta_2^{'}, ..., \zeta_n^{'}$. For $i = 1, 2, ..., n$; $j = 1, 2, ..., n$ we have $\zeta_i(k), \zeta_j^{'}(k) \supseteq u_1 \ \forall \ k \in R$ and $\zeta_i(k) = \zeta_j^{'}(k) = R$ for $k \in M \setminus R$ by the choice of ζ_1. Thus, $\forall \ u \in P(M)$ we have

$$\begin{aligned} R(\zeta_2, ..., \zeta_n)_u &= (\chi_R^{(u_1)} \zeta_2, ..., \zeta_n)_u \\ &= (\chi_R^{(u_1)} \zeta_2^{'}, ..., \zeta_n^{'})_u \\ &= R(\zeta_2^{'}, ..., \zeta_n^{'})_u \end{aligned}$$

Also $(\zeta_2,...,\zeta_n)_u = R = (\zeta_2^{'},...,\zeta_n^{'})_u$ and $(\zeta_2,...,\zeta_n)_u = M = (\zeta_2^{'},...,\zeta_n^{'})_u$, $\forall\ u \in P(M)$. Hence, $\zeta_2,...,\zeta_n = \zeta_2^{'},...,\zeta_n^{'}$. The conclusion is reached via induction, because the case $n = 1$ is instantaneous. □

Proposition 3.2. *Let δ, ω and η be finite-valued soft $\mathfrak{FI}$s of R such that for $i = 1,...,n$, ω_{u_i} is an invertible $\mathfrak{FI}$ and $\delta(k),\eta(k) \supseteq u_0$, $\forall\ k \in M$, where $\mathfrak{Im}(\omega) = \{u_i\}$, for $i = 0,1,...,n$ with the condition that $u_0 \subset u_1 \subset ... \subset u_n = R$. If $\omega\delta = \omega\eta$, then $\delta = \eta$.*

Proof. We know that $\omega_u\delta_u = \omega_u\eta_u$, for every $u \in P(M)$. Since ω_u is invertible $\mathfrak{FI}$, there exists another $\mathfrak{FI}$ $\mathcal{A}_u$ of R such that $\mathcal{A}_u\omega_u = R$, for every $u \in P(M)$. Thus,

$$\begin{aligned}\delta_u &= R\delta_u\\ &= (\mathcal{A}_u\omega_u)\delta_u\\ &= \mathcal{A}_u(\omega_u\delta_u)\\ &= \mathcal{A}_u(\omega_u\eta_u)\\ &= (\mathcal{A}_u\omega_u)\eta_u\\ &= R\eta_u\\ &= \eta_u\end{aligned}$$

and $\delta_u = M = \eta_u$, for every $u \in P(M)$. Hence, $\delta = \eta$. □

Corollary 3.2. *Consider D be a soft fractionary left (right) ideal of R. Then, D is a two-valued, if D_* is a maximal fractionary left (right) ideal of R.*

Proof. Since D_* is a maximal fractionary left ideal of R and $D_* \neq R$. Thus, $\exists\ k \in R$ such that $D(k) \neq D(l_1)$, $\forall\ l_1 \in P(M)$. Therefore, D is atleast two-valued. Let $l_1 \subseteq u \subset D(l_1)$ and $u \in \mathfrak{Im}(D)$. Then, D_u is a fractionary left ideal of R such that $D_* \mathrel{\widetilde{\subset}} D_u$. Since D_* is a maximal fractionary left ideal, $D_u = R$. Thus, if $u_1, u_2 \in \mathfrak{Im}(D)$ and $u_1 \neq D(l_1), u_2 \neq D(l_1)$, then $Du_1 = R = Du_2$. But then $u_1 = u_2$. Hence, D is two-valued. □

Proposition 3.3. *Consider a soft $\mathfrak{FI}$ δ of R which is soft invertible such that for $i = 1,...,n$, δ_{u_i} is a $\mathfrak{FI}$ of R and $\mathfrak{Im}(\delta) = \{u_i\}$ for $i = 0,1,...,n$ with the condition $u_0 \subset u_1 \subset ... \subset u_n = R$. Then, there exists $u, u \supseteq u_{n-1}$ and a unique soft $\mathfrak{FI}$ $\delta^{'}$ of R such that $\delta\delta^{'} = \chi_R^{(u)}$.*

Proof. Since δ is soft invertible, $\exists$ a soft $\mathfrak{FI}$ $\delta^{'}$ such that $\delta\delta^{'} = \chi_R^{(u)}$, for every $u \in P(M)$. Suppose $\delta^{'}$ be a soft fractionary left (right) ideal of R and $\delta_*^{'}$ is a maximal fractionary left (right) ideal of R. If $n > 1$, then by Corollary 3.2, $\delta^{'}$ is two-valued with $\mathfrak{Im}(\delta^{'}) = \{u_0^{'}, u_n\}$ with the condition $u_0^{'} = u \supseteq u_{n-1}$. Thus, by Proposition 3.2, $\delta^{'}$ is unique. Choosing $\delta^{'}$ such that $\mathfrak{Im}(\delta^{'}) = \{u_0, u_n\}$ and $\delta\delta^{'} = \chi_R^{(u_0)}$, for every $u_0 \in P(M)$, if $n = 1$. Hence, again by Proposition 3.2, $\delta^{'}$ is unique. □

4. **Characterization of Dedekind domains using soft fractional ideals (soft $\mathfrak{FI}s$)**

Algebraic theory often explores the structural properties of Dedekind domains. This section provides a fairly characterization of Dedekind domains in a very clear way by using the notion of soft $\mathfrak{FI}s$. This increases the comprehensibility of softness in Dedekind domains and emphasizes their usefulness within the foundational principles of soft set theory.

Lemma 4.1. *Every non-constant prime soft ideal is a maximal soft ideal in a Dedekind domain R.*

Proof. Consider δ be a non-constant prime soft ideal of a Dedekind domain R. Then, $\delta(0) = R$ and δ_* is a prime ideal of R. δ_* is a maximal ideal of R by [18](Theorem 10). It is clear that δ is two-valued by Corollary 3.2. Let $\mathfrak{Im}(\delta) = \{u_i\}$, for every $u_i \in P(M)$.

Suppose ω be a soft ideal of R such that $\delta \mathrel{\widetilde{\subseteq}} \omega$, then $\omega(0) = R$. Assume $k \in \delta_*$, then $R = \delta(0) = \delta(k) \subseteq \omega(k)$. Therefore, $\omega(k) = R = \omega(0)$ and thus $k \in \omega_*$. Hence, $\delta_* \mathrel{\widetilde{\subseteq}} \omega_*$. Since δ_* is a maximal ideal of R, $\omega_* = R$ or $\delta_* = \omega_*$. If $\omega_* = R$, then $\omega = \lambda_R$. Hence, δ is a maximal soft ideal of R. ☐

Lemma 4.2. *In a Dedekind domain R, every prime integral soft $\mathfrak{FI}$ is soft invertible soft $\mathfrak{FI}$.*

Proof. Consider δ be a prime integral soft $\mathfrak{FI}$. Then, by [18](Theorem 10), δ_* is an invertible prime ideal of R. Now to prove that δ is soft invertible, i.e., we have to demonstrate that $\cup\{\delta(k) \mid k \in M \setminus \delta_*\}$ exists. It is clearly visible by Proposition 2.1, that $\cup\{\delta(k) \mid k \in M \setminus \delta_*\}$ exists. Hence, δ is soft invertible. ☐

Remark 4.1. The distinctiveness of the factorization of any soft ideal into soft maximal ideals in a Dedekind domain R is obtained from Proposition 3.1 and Lemma 4.2.

Theorem 4.1. *Consider R be an integral domain. Then, the below listed conditions are satisfied:*

(i) *R is a Dedekind domain.*
(ii) *Every integral soft $\mathfrak{FI}$ δ of R such that $\cup\{\delta(k) \mid k \in M \setminus \delta_*\}$ exists is soft invertible.*
(iii) *Every soft $\mathfrak{FI}$ δ of R such that $\cup\{\delta(k) \mid k \in M \setminus \delta_*\}$ exists is soft invertible.*

Proof. First we prove that (i) $\Longrightarrow$ (iii). Assume δ be a soft $\mathfrak{FI}$ of a Dedekind domain R such that $\cup\{\delta(k) \mid k \in M \setminus \delta_*\}$ exists. Then by Theorem 2.1, δ_* is an invertible $\mathfrak{FI}$ of R. Now by Proposition 2.1, δ is soft invertible soft $\mathfrak{FI}$ of R. It is clearly visible that (iii) $\Longrightarrow$ (ii) is instant implication. Now it remains to show that (ii) $\Longrightarrow$ (i). For this, we assume that J be an ideal of R. Then, χ_J is an integral soft $\mathfrak{FI}$ of R such that $\cup\{\chi_J(k) \mid k \in M \setminus J\}$ exists. Thus by assumption, it is soft invertible. Therefore by Theorem 2.1, $(\chi_J)_* = J$ is an invertible $\mathfrak{FI}$ of R. Hence, by [18](Theorem 12), R is a Dedekind domain. ☐

Proposition 4.1. *If the unique expression for every finite-valued soft ideal δ of an integral domain R with $\delta(0) = R$ as a product of finite number of maximal soft ideals, then R is a Dedekind domain.*

Proof. Suppose J be an ideal of R. Then, χ_J is finite-valued integral soft $\mathfrak{FI}$ of R. Therefore, it is a product of finite maximal soft ideals. i.e., $\chi_J = \delta_1\delta_2...\delta_n$, δ_i is maximal, $\forall\ i = 1, ..., n$. Thus, $(\chi_J)_* = J = (\delta_1)_*(\delta_2)_*...(\delta_n)_*$ is a product of maximal ideals. Hence, R is a Dedekind domain. ☐

Corollary 4.1. *The necessary and sufficient condition for an integral domain R to be a Dedekind domain is that every integral soft $\mathfrak{FI}$ δ in R be soft invertible.*

Proof. Theorem 4.1 makes the sufficiency evident. We proceed by assuming R is a Dedekind domain for the necessity. Then, it immediately follows that R is Noetherian and every nonzero integral ideals of R are invertible [18](Theorems 12 and 13). Therefore under $\supset$, a soft ideal of R has a well ordered set of values. Thus, there is a union-property for every integral soft $\mathfrak{FI}$ δ in R. Hence, by Theorem 4.1, δ is soft invertible. ☐

Definition 4.1. *Consider δ be a soft ideal of R with the condition $\delta(0) = R$. If the soft subset δ^E of M is outlined by*

$$\delta^E(k) = \begin{cases} \delta(k) & , \quad \forall \quad k \in R \\ u_0 & , \quad \forall \quad k \in M \setminus R \end{cases}$$

where $u_0 \subseteq u_1$, $\forall\ u_1 \in \mathfrak{Im}(\delta)$. Then, we state δ^E an extended integral soft $\mathfrak{FI}$ of δ.

Proposition 4.2. *Consider δ be an extended integral soft $\mathfrak{FI}$ of R with finite values. Then, δ is a product of extended integral maximal soft $\mathfrak{FI}$s of finite number.*

Proof. Consider δ be an extended integral finite-valued soft $\mathfrak{FI}$ of R with $\delta(x) = u_0$, $\forall$ $x \in M \setminus R$. Then, $\delta|_R$ is a finite-valued soft ideal of R with $(\delta|_R)(0) = R$. Thus, there is a unique expression of $\delta|_R$ as a product of finite maximal soft ideals of R, say $\delta|_R = \delta_1\delta_2...\delta_n$, where δ_i is a maximal soft ideal of R for $i = 1, 2, ..., n$ and $\delta_i(k) \supseteq u_0$, $\forall$ $k \in R$. For $i = 1, 2, ..., n$, suppose $\delta_i^E(k) = \delta_i(k)$, $\forall$ $k \in R$ and $\delta_i^E(k) = u_0$, $\forall$ $k \in M \setminus R$. Then, $\delta = \delta_1^E\delta_2^E...\delta_n^E$. Hence, each δ_i^E is an extended integral maximal soft $\mathfrak{FI}$. ☐

Lemma 4.3. *Consider the $\mathfrak{FI}$s E, F of R such that $F \widetilde{\subseteq} E$. Then, there exists a unique $\mathfrak{FI}$ G of R such that $F = EG$.*

Proof. It is clear by [18](Theorem 11) that in a Dedekind domain, every $\mathfrak{FI}$ is invertible. Thus $\exists$ $E^{'}$, a unique $\mathfrak{FI}$ of R such that $EE^{'} = R$. Therefore, we have

$$\begin{aligned} F &= RF \\ &= (EE^{'})F \\ &= E(E^{'}F) \\ &= EG \end{aligned}$$

Hence, $\exists$ a unique $\mathfrak{FI}$ G of R such that $F = EG$, where $G = E^{'}F$. ☐

Theorem 4.2. *Consider a soft $\mathfrak{FI}$ δ of R with finite values. Then, there is an expression of δ as a product of extended integral maximal soft $\mathfrak{FI}$s of finite number and a two-valued soft $\mathfrak{FI}$.*

Proof. Consider $\mathfrak{I}m(\delta) = \{u_i\}$, $\forall$ $i = 0, 1, ..., n$ with the condition that $u_0 \subset u_1 \subset ... \subset u_n = R$. Since $\delta_{u_1} \widetilde{\cap} R \widetilde{\subseteq} \delta_{u_1}$, then by Lemma 4.3, there exists a unique $\mathfrak{FI}$ J of R such that $\delta_{u_1} \widetilde{\cap} R = J\delta_{u_1}$. If we assume $0 \neq d \in R$ such that $dJ \widetilde{\subseteq} R$, then we have a chain of R-submodules of M.

$$\text{i.e., } d(\delta_{u_1} \widetilde{\cap} R) = dJ\delta_{u_1} \widetilde{\supset} dJ\delta_{u_2} \widetilde{\supset} ..... \widetilde{\supset} dJ\delta_{u_n}$$

Define the soft subsets $\omega^{'}$ and ω outlined by

$$\omega^{'}(k) = \begin{cases} u_0 \ ; & if \quad k \in M \setminus dJ\delta_{u_1} \\ u_i \ ; & if \quad k \in dJ\delta_{u_i} \setminus dJ\delta_{u_{i+1}} \quad for \;\; i = 1, ..., n-1 \\ u_n \ ; & if \quad k \in dJ\delta_{u_n} \end{cases}$$

and

$$\omega(k) = \begin{cases} u_0 \ ; & if \quad k \in M \setminus dJ \\ R \ ; & if \quad k \in dJ \end{cases}$$

Since the level sets of $\omega^{'}$ and ω are R-submodules of M, thus $\omega^{'}$ and ω are soft $\mathfrak{FI}$s of R. Now $\omega^{'} = \omega\delta$ since $\omega^{'}_{u_i} = dJ\delta_{u_i} = \omega_{u_i}\delta_{u_i} = (\omega\delta)_{u_i}$, $\forall$ $i = 1, ..., n$ and $\omega^{'}_{u_0} = M = MM = \omega_{u_0}\delta_{u_0}$. Since $\omega^{'}_{u_1} = d(\delta_{u_1} \widetilde{\cap} R) \widetilde{\subseteq} R$, $\omega^{'}$ is an extended integral soft $\mathfrak{FI}$ of R. Thus by Proposition 4.2, $\omega^{'}$ is a product of extended integral maximal soft $\mathfrak{FI}$s of finite number. Since ω is soft invertible by Theorem 4.1 and we are able to select a soft inverse ω_1 of ω such that $\mathfrak{I}m(\omega_1) = \{u_i\}$, for every $u_i \in P(M)$. Thus $\omega_1\omega = \chi_R^{(u_0)}$, for every $u_0 \in P(M)$. Therefore, $\omega^{'}\omega_1 = (\delta\omega)\omega_1 = \delta(\omega\omega_1) = \delta\chi_R^{(u_0)}$. Thus $\forall$ $u \in P(M)$, $(\delta\chi_R^{(u_0)})_u = \delta_u R = (\omega^{'}\omega_1)_u$. So, $\delta_u = (\omega^{'}\omega_1)_u$ and $\delta_u = M = (\omega^{'}\omega_1)_u$ $\forall$ $u \in P(M)$. Hence, $\delta = \omega^{'}\omega_1$, a desired product. ☐

Corollary 4.2. *Suppose δ be a finite valued soft $\mathfrak{FI}$ of R such that $u_0 \in \mathfrak{I}m(\delta)$. Then, δ is a product of integral maximal soft $\mathfrak{FI}$s of finite number and an invertible soft $\mathfrak{FI}$.*

Proof. The proof is straight forward from Proposition 4.2 and Theorem 4.2. □

Example 4.1. Let Q denote the field of rational numbers and Z, the integers ring. Define a soft subset δ of Q as follows

$$\delta(l) = \begin{cases} 1 \quad ; \quad if \quad l \in (2/3Z) \\ \frac{1}{2} \quad ; \quad if \quad l \in (2/9Z) \setminus (2/3Z) \\ 0 \quad ; \quad if \quad l \in Q \setminus (2/9Z) \end{cases}$$

Then, clearly δ is a soft $\mathfrak{FI}$ of Z since every level set of δ is a $\mathfrak{FI}$ of Z. Now $\delta_{\frac{1}{2}} = 2/9Z$ and $\delta_{\frac{1}{2}} \mathbin{\widetilde{\cap}} Z = 2Z$. Thus, there exists a $\mathfrak{FI}$ $J = \delta_{\frac{1}{2}}^{-1}(\delta_{\frac{1}{2}} \mathbin{\widetilde{\cap}} Z) = 9Z$ such that $\delta_{\frac{1}{2}} \mathbin{\widetilde{\cap}} Z = J\delta_{\frac{1}{2}}$. Define the soft subsets $\omega^{'}$ and ω outlined by

$$\omega^{'}(l) = \begin{cases} 1 \quad ; \quad if \quad l \in 9Z \\ \frac{1}{2} \quad ; \quad if \quad l \in 3Z \setminus 9Z \\ 0 \quad ; \quad if \quad l \in Q \setminus 3Z \end{cases}$$

$$\omega(l) = \begin{cases} 1 \quad ; \quad if \quad l \in 9Z \\ 0 \quad ; \quad if \quad l \in Q \setminus 9Z \end{cases}$$

Then, $\omega^{'} = \omega\delta$. Define the soft subset ω_1 of Q outlined by

$$\omega_1(l) = \begin{cases} 1 \quad ; \quad if \quad l \in (1/9Z) \\ 0 \quad ; \quad if \quad l \in Q \setminus (1/9Z) \end{cases}$$

Then, $\omega\omega_1 = \chi_{_R}^{(\alpha)}$, $\forall\ \alpha \in P(M)$. Hence, $\delta = \omega^{'}\omega_1$ and $\omega^{'}$ is a product of integral maximal soft $\mathfrak{FI}s$ of finite number of Z.

Example 4.2. Let Q denote the field of rational numbers and Z, the integers ring. Define a soft subset δ of Q as follows

$$\Gamma(m) = \begin{cases} 1 \quad ; \quad if \quad m \in (7/11Z) \\ \frac{1}{7} \quad ; \quad if \quad m \in (7/33Z) \setminus (7/11Z) \\ 0 \quad ; \quad if \quad m \in Q \setminus (7/33Z) \end{cases}$$

Then, clearly Γ is a soft $\mathfrak{FI}$ of Z since every level set of Γ is a $\mathfrak{FI}$ of Z. Now $\Gamma_{\frac{1}{7}} = 7/33Z$ and $\Gamma_{\frac{1}{7}} \mathbin{\widetilde{\cap}} Z = 7Z$. Thus, there exists a $\mathfrak{FI}$ $L = \Gamma_{\frac{1}{7}}^{-1}(\Gamma_{\frac{1}{7}} \mathbin{\widetilde{\cap}} Z) = 33Z$ such that $\Gamma_{\frac{1}{7}} \mathbin{\widetilde{\cap}} Z = L\Gamma_{\frac{1}{7}}$. Define the soft subsets $\Omega^{'}$ and Ω outlined by

$$\Omega^{'}(m) = \begin{cases} 1 \quad ; \quad if \quad m \in 21Z \\ \frac{1}{7} \quad ; \quad if \quad m \in 7Z \setminus 21Z \\ 0 \quad ; \quad if \quad m \in Q \setminus 7Z \end{cases}$$

$$\Omega(m) = \begin{cases} 1 \quad ; \quad if \quad m \in 33Z \\ 0 \quad ; \quad if \quad m \in Q \setminus 33Z \end{cases}$$

Then, $\Omega^{'} = \Omega\Gamma$. Define the soft subset Ω_1 of Q defined by

$$\Omega_1(m) = \begin{cases} 1 \quad ; \quad if \quad m \in (1/33Z) \\ 0 \quad ; \quad if \quad m \in Q \setminus (1/33Z) \end{cases}$$

Then, $\Omega\Omega_1 = \chi_{_R}^{(\alpha)}$, $\forall\ \alpha \in P(M)$. Hence, $\Gamma = \Omega^{'}\Omega_1$ and $\Omega^{'}$ is a product of integral maximal soft $\mathfrak{FI}s$ of finite number of Z.

Remark 4.2. It is to be noted that a finite-valued soft $\mathfrak{FI}$ of R cannot be expressed uniquely into a product of two-valued soft $\mathfrak{FI}$ and extended integral maximal soft $\mathfrak{FI}s$.

Example 4.3. Let Q denote the field of rational numbers and Z, the integers ring. Define the soft subsets $\tau^{'}$ and τ_1 of Q outlined by

$$\tau^{'}(p) = \begin{cases} 1 & ; \quad if \quad p \in 18Z \\ \frac{1}{2} & ; \quad if \quad p \in 6Z \setminus 18Z \\ 0 & ; \quad if \quad p \in Q \setminus 6Z \end{cases}$$

$$\tau_1(p) = \begin{cases} 1 & ; \quad if \quad p \in (1/18Z) \\ 0 & ; \quad if \quad p \in Q \setminus (1/18Z) \end{cases}$$

If we consider δ of Example 4.1 and Γ of Example 4.2, then $\delta = \Gamma = \tau^{'}\tau_1$ and $\tau^{'}$ is a product of integral maximal soft $\mathfrak{FI}s$ of finite number of Z. This results in providing a different factorization of δ and Γ than the factorization $\delta = \omega^{'}\omega_1$ in Example 4.1 and $\Gamma = \Omega^{'}\Omega_1$ in Example 4.2.

Example 4.4. Let Q denote the field of rational numbers and Z, the integers ring. Define the soft subsets $\sigma^{'}$ and σ_1 of Q outlined by

$$\sigma^{'}(q) = \begin{cases} 1 & ; \quad if \quad q \in 42Z \\ \frac{1}{5} & ; \quad if \quad q \in 14Z \setminus 42Z \\ 0 & ; \quad if \quad q \in Q \setminus 14Z \end{cases}$$

$$\sigma_1(q) = \begin{cases} 1 & ; \quad if \quad q \in (1/66Z) \\ 0 & ; \quad if \quad q \in Q \setminus (1/66Z) \end{cases}$$

If we consider δ of Example 4.1 and Γ of Example 4.2, then $\delta = \Gamma = \sigma^{'}\sigma_1$ and $\sigma^{'}$ is a product of integral maximal soft $\mathfrak{FI}s$ of finite number of Z. This results in providing a different factorization of δ and Γ than the factorization $\delta = \omega^{'}\omega_1$ in Example 4.1 and $\Gamma = \Omega^{'}\Omega_1$ in Example 4.2.

5. **Conclusion**

This work explores the advantages of applying techniques from soft set theory, providing a foundation for innovative approaches to the study of soft multiplicative ideal theory. The notion of existence of finite minimal generating set (as a soft R-submodule) serves a pivotal role in understanding the structure of soft fractional ideals. Introducing the novel concept and demonstrate that for a finite valued soft fractional ideal of integral domain with level ideals which are invertible, there exists a finite minimal generating set (as a soft R-submodule). In this paper, by using these concepts with the newly formulated concepts, we were able to demonstrate that the invertibility of specific soft fractional ideals and the factorization of soft ideals as products of prime soft ideals and maximal soft ideals are the two aspects by which we establish the characterizations of Dedekind domains. The class of undeniable soft fractional ideals are utilized in a wide array of mathematical investigations and constructions as compared to the class of fractional ideals. It becomes evident that using soft fractional ideal procedure techniques will result in a more accurate classification. The findings of this study will spark original ideas and serve as a strong foundation for further investigation in the future research work.

6. Open Problems

The investigation of soft ideal theory inside algebraic structures has produced some intriguing findings, but there is still much more to discover. Some of our unsolved problems provide a rich environment for additional research and development. The ensuing open challenges encourage researchers to explore the intricacies of soft ideals and their applications by highlighting important avenues for further study.

1. Can the above results be generalized in prüfer domains, krull domains, mori domains, pseudo Dedekind domains, generalized greatest common divisor domains ?
2. There has been a surge in the study of fractional calculus, which is closely related to fractional ideals. Can the soft fractional ideals technique be applied to represent nonlinear models for real-world problems ?

This suggests a potential new direction for exploring soft fractional ideals in mathematical and physical contexts.

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