SOME FIXED POINT RESULTS FOR QUADRILATERALS PERIMETER CONTRACTION IN b-METRIC SPACE

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In this paper, mappings called 'quadrilaterals perimeter contraction' in b-metric spaces is introduced and some fixed point results are obtained and justified with suitable examples. The quadrilaterals perimeter contraction mappings are also compared with an existing class of mappings called mappings contracting perimeters of triangles and shown that the two classes of mappings are independent of each other. As an application, the existence of the solution to a non-linear Fredholm integral equation with lipschitz condition is obtained.

Keywords: b-metric space, fixed point, quadrilaterals perimeter contraction.

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1. Introduction

In 1922, Banach [1] formulated and proved a fixed point result for contraction mappings within the framework of complete normed vector spaces (later called Banach spaces). In 1930, Caccioppoli [2] extended the result to metric spaces and showed that it could be used even if the mapping only became a contraction after a certain number of iterations. The contraction principle has been generalized in many ways over the years. Many authors obtained several fixed point results in various generalized metric spaces (refer to [3, 4, 5, 6, 7, 9, 10, 11]).

In 1989, Bakhtin [12] introduced the notion of b-metric spaces, which was formally defined by Czerwik [13] in 1993 with a view of generalizing the contraction principle. Numerous authors has also generalized fixed point theorems in b-metric spaces (refer to [14, 15, 16, 17]) . The definition of b-metric spaces has been extended in different ways (refer to [18, 19, 20, 21, 22, 23, 24]).

Petrov [25] introduced a new type of contraction mappings called "mappings contracting perimeters of triangles" (or triangles perimeter contraction) and obtained some fixed point results in metric spaces and obtained Banach's Contraction principle as a corollary. In [26] Petrov and Salimov did some generalization of Petrov's works; one may also refer to the works of [27], [28], [29], etc. In this paper, we introduced the notion of quadrilaterals perimeter contraction in a b-metric space setting based on the definition of triangles perimeter contraction in metric spaces.

2. Preliminaries

In this section, we state some definitions and results used in our subsequent discussion, starting with the definition of a b-metric space.

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Definition 2.1. [1] *Let X be a non empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric. if it satisfied the following properties.*

(i) $d(x, y) = 0$ *if and only if* $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$ *for all* $x, y, z \in X$.
Then, the triplet (X, d, s) is a called a b-metric space with coefficient s.

It is evident from the above definition that every metric is also a b-metric.

Example 2.1. *The triplet (X, d, s) is a complete b-metric space with coefficient $s = 1$, where $X = [0, \infty)$ and*

$$d(x, y) = \begin{cases} x + y, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

In [8], Khamshi and Hussain defined a topology on b-metric space as follows.

Definition 2.2. [8] *Let (X, d, s) be a b-metric space. A subset A of X is said to be open if for any $a \in A$, there exists $\varepsilon > 0$ such that $B(a, r) \subset A$ where*

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

Denote by $\mathcal{T}d$ the family of all open subsets of X in the sense of Definition 2.2, then $\mathcal{T}d$ is a topology on X (see also [15]).

In [25] Petrov defined 'mappings contracting perimeters of triangles'

Definition 2.3. [25] *Let (X, d) be a metric space with $|X| > 3$. A mapping $T : X \to X$ is called* a mapping contracting perimeters of triangles *on X if there exists $\alpha \in [0, 1)$ such that*

$$d(T(x), T(y)) + d(T(y), T(z)) + d(T(x), T(z)) \leq \alpha\big(d(x, y) + d(y, z) + d(x, z)\big)$$

for all pairwise distinct points x, y and z in X.

and proved the following result.

Theorem 2.1. [25] *Let (X, d), $|X| > 3$ be a complete metric space and let the mapping $T : X \longrightarrow X$ satisfy the following two conditions:*

(i) $T\big(T(x)\big) \neq x$ *for all* $x \in X$ *such that* $T(x) \neq x$;
(ii) *T is a mapping contracting perimeters of triangles on X.*

Then T has a fixed point. The number of fixed points is at most two.

Now, we define quadrilaterals perimeter contraction as follows.

Definition 2.4. *Let (X, d, s) be a b-metric space with $|X| \geq 4$. We say that $T : X \longrightarrow X$ is a quadrilalaters perimeter contraction on X if there exists $\alpha \in [0, 1)$, $s \geq 1$ and $s\alpha < 1$ such that the inequality*

$$\begin{aligned} &d(T(x), T(y)) + d(T(y), T(z)) + d(T(z), T(w)) + d(T(w), T(x)) \\ &\qquad \leq \alpha\big(d(x, y) + d(y, z) + d(z, w) + d(w, x)\big) \end{aligned} \tag{1}$$

holds for all pairwise distinct point w, x, y and z in X.

Remark 2.1. *Note that the requirement for w, x, y and z in X to be pairwise distinct is essential. One can see that, if any three of the points are equal, this definition is equivalent to the definition of contraction mapping. If any two of the points are equal, then the quadrilaterals perimeter contraction reduces to triangles perimeter contraction (Definition 2.3).*

Example 2.2. *Consider the complete b-metric space* (X, d, s) *where* $X = \left[0, \frac{\pi}{3}\right]$ *and d is the usual metric, i.e.,* $d(x, y) = |x - y|$ *for all* $x, y \in X$.

Define $T : X \to X$ *by* $T(x) = \cos x$ *for all x in X. Since* $\cos x \in \left[\frac{1}{2}, 1\right] \subset X$ *for all* $x \in \left[0, \frac{\pi}{3}\right]$, *the mapping T is well defined. For any pairwise distinct points* $x, y, z, w \in X$, *we have,*

$$P = d(x, y) + d(y, z) + d(z, w) + d(w, x)$$

and

$$P_T = d(\cos x, \cos y) + d(\cos y, \cos z) + d(\cos z, \cos w) + d(\cos w, \cos x).$$

On the interval $\left[0, \frac{\pi}{3}\right]$, *by Mean value Theorem, we have,*

$$|\cos u - \cos v| \leq \frac{\sqrt{3}}{2} |u - v| \quad \text{for all } u, v \in X.$$

Summing over the four sides of the quadrilateral, we get $P_T \leq \alpha P$, *where* $\alpha = \frac{\sqrt{3}}{2} \in [0, 1)$.

Hence T is a quadrilaterals perimeter contraction on (X, d, s). *Moreover, the fixed point equation* $x = \cos x$ *has a unique solution* $x^* \in (0, \frac{\pi}{3})$.

3. Main results

In this Section, we obtain some fixed point results for quadrilaterals perimeter contraction in b-metric spaces. From Definition 2.2, it is seen that the notion of continuity of functions in metric spaces can be extended to b-metric spaces.

Proposition 3.1. *Quadrilaterals perimeter contraction mapping on a b-metric space is continuous.*

Proof. Let (X, d, s) be a b-metric space with $|X| \geq 4$ and let $T : X \longrightarrow X$ be a quadrilaterals perimeter contraction on X and x_0 be any point of X. If x_0 is an isolated point of X, then, clearly, T is continuous at x_0.

On the other hand, if x_0 be an accumulation point, to prove the continuity of the mapping T, we need to show that for a given $\epsilon > 0$, there exist $\delta > 0$ such that $d\big(T(x_0), T(x)\big) < \epsilon$ whenever $d(x, x_0) < \delta$ where $x \in X$. Since x_0 is an accumulation point, for every $\delta > 0$ there exist $y \in X$ such that $d(x_0, y) < \delta$. we have,

$$\begin{aligned} d\big(T(x_0), T(x)\big) &\leq d\big(T(x_0), T(x)\big) + d\big(T(x_0), T(y)\big) + d\big(T(x_0), T(z)\big) \\ &+ d\big(T(x_0), T(w)\big) + d\big(T(x), T(y)\big) + d\big(T(y), T(z)\big) + d\big(T(z), T(w)\big) \\ &+ d\big(T(w), T(x)\big) \leq \alpha\big(d(x_0, x) + d(x_0, y) + d(x_0, z) + d(x_0, w) \\ &+ d(x, y) + d(y, z) + d(z, w) + d(w, x)\big). \end{aligned}$$

Now we apply the b-metric triangular inequalities to bound the terms on the right-hand side of the above inequality by multiples of the distances from x_0:

$$\begin{aligned} d(x, y) &\leq s\big(d(x, x_0) + d(x_0, y)\big), \\ d(y, z) &\leq s\big(d(y, x_0) + d(x_0, z)\big), \\ d(z, w) &\leq s\big(d(z, x_0) + d(x_0, w)\big), \\ d(w, x) &\leq s\big(d(w, x_0) + d(x_0, x)\big). \end{aligned}$$

Combining these estimates (and using that each of $d(x_0, y)$, $d(x_0, z)$, $d(x_0, w)$, $d(x_0, x)$ is less than δ), we get,

$$d\big(T(x_0), T(x)\big) \leq 12s\alpha\, \delta.$$

Choose $\delta = \varepsilon/(12s\alpha)$ which is possible since $\alpha > 0$). Then we get,

$$d\big(T(x_0), T(x)\big) < \varepsilon \qquad \text{whenever} \qquad d(x_0, x) < \delta.$$

Hence T is continuous at x_0. Since x_0 is arbitrary, T is continuous on X and hence the result. □

Proposition 3.2. *Triangles perimeter contraction mappings are continuous.*

Proof. Taking $w = z$ in Proposition 3.1, we get the result. □

Example 3.1. *Let (X, d, s) be a complete b-metric space with $X = [0, \infty)$ and*

$$d(x,y) = \begin{cases} x + y, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

Let $T : X \longrightarrow X$ be a mapping on X defined by $T(x) = \frac{x}{n}$, for all $n \in \mathbb{N} \setminus \{1, 2\}$.

Then d is a b-metric with $s = 2$. The given b-metric d is discontinuous. Consider the two sequences $\{x_n\}$ and $\{y_n\}$, where $x_n = 1$ and $y_n = 1 + \frac{1}{n}$. Now,

$$d(x_n, y_n) = 1 + 1 + \frac{1}{n} = 2 + \frac{1}{n}.$$

As $n \to \infty$, $d(x_n, y_n) \to 2$ and 1 is the limit point for both the sequences. Therefore,

$$2 = \lim_{n\to\infty} d(x_n, y_n) \neq d\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right) = d(1,1) = 0$$

and thus d is not continuous.

Next, we show that T satisfy condition (1) for $n = 3$. Let

$$\begin{aligned} P_c &= d\big(T(x), T(y)\big) + d\big(T(y), T(z)\big) + d\big(T(z), T(w)\big) + d\big(T(w), T(x)\big) \\ &= \frac{x}{3} + \frac{y}{3} + \frac{y}{3} + \frac{z}{3} + \frac{z}{3} + \frac{w}{3} + \frac{w}{3} + \frac{x}{3} = \frac{2}{3}(x + y + z + w) \end{aligned}$$

and,

$$P = d(x,y) + d(y,z) + d(z,w) + d(w,x) = 2(x + y + z + w).$$

Then, $\frac{P_c}{P} = \frac{1}{3} < 1$, i.e., $\alpha = \frac{1}{3} \in [0, 1)$ and $s\alpha < 1$, showing that T is a quadrilaterals perimeter contraction on the given b-metric space.

Lastly, to show the continuity of T, let x_0 be a limit point of X. Then,

$$d\big(T(x), T(x_0)\big) = d\left(\frac{x}{3}, \frac{x_0}{3}\right) = \frac{x}{3} + \frac{x_0}{3} = \frac{1}{3} d(x, x_0).$$

Thus, for every $\epsilon > 0$, we can choose $\delta = \frac{\epsilon}{3}$ so that

$$d(T(x), T(x_0)) < \epsilon \qquad \text{whenever} \qquad d(x, x_0) < \delta,$$

as required.

Theorem 3.1. *Let (X, d, s), $|X| \geq 4$ be a complete b-metric space and let the mapping $T : X \to X$ satisfy the following two conditions:*

(1) *$T\big(T(T(x))\big) \neq x$, for all $x \in X$ such that $T(x) \neq x$.*
(2) *T is a quadrilaterals perimeter contraction on X.*

Then T has a fixed point. The number of fixed points is at most three.

Proof. Let $x_0 \in X$. Define $T(x_0) = x_1$, $T(x_1) = x_2, \ldots, T(x_n) = x_{n+1}, \ldots$. Suppose that x_i is not a fixed point of the mapping T for every $i = 0, 1, \ldots$.

We claim that all x_i's are different. Since x_i is not a fixed point, then $x_i \neq x_{i+1} = T(x_i)$.

By Condition 1 above, $x_{i+3} = T(x_{i+2}) = T(T(x_{i+1})) = T(T(T(x_i))) \neq x_i$ and by the supposition that x_{i+1} is not a fixed point, we have $x_{i+1} \neq x_{i+2} = T(x_{i+1})$.

Hence, $x_i, x_{i+1}, x_{i+2}$and x_{i+3} are pairwise distinct.

Now, let

$$\begin{aligned}
p_0 &= d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_0),\\
p_1 &= d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + d(x_4, x_1),\\
p_2 &= d(x_2, x_3) + d(x_3, x_4) + d(x_4, x_5) + d(x_5, x_2),\\
p_n &= d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_n).
\end{aligned}$$

Since x_i, x_{i+1}, x_{i+2} and x_{i+3} are pairwise distinct we have

$$p_1 \leq \alpha p_0, \quad p_2 \leq \alpha p_1, \quad \ldots \quad, \quad p_n \leq \alpha p_{n-1},$$

and therefore,

$$p_0 > p_1 > \cdots > p_n > \cdots. \tag{2}$$

To see this, suppose that $j \geq 4$ is a minimal positive integer such that $x_j = x_i$ for some i such that $0 \leq i \leq j-3$. Then $x_{j+1} = x_{i+1}$ and $x_{j+2} = x_{i+2}$ so that, $p_i = p_j$, a contradiction to (2).

Further, we claim that $\{x_i\}$ is a cauchy sequence. It is clear that

$$d(x_1, x_2) \leq p_0, \quad d(x_2, x_3) \leq p_1 \leq \alpha p_0, \quad d(x_3, x_4) \leq p_2 \leq \alpha p_1 \leq \alpha^2 p_0, \ldots$$
$$d(x_n, x_{n+1}) \leq p_{n-1} \leq \alpha^{n-1} p_0, \quad d(x_{n+1}, x_{n+2}) \leq p_n \leq \alpha^n p_0.$$

By the triangle inequality,

$$d(x_n, x_{n+p}) \leq s(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p})).$$

By applying this recursively, we get

$$\begin{aligned}
d(x_n, x_{n+p}) \leq{}& s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3})\\
&+ \cdots + s^{p-1} d(x_{n+p-1}, x_{n+p}).
\end{aligned}$$

Therefore, since $s\alpha < 1$,

$$d(x_n, x_{n+p}) \leq s \sum_{k=0}^{p-1} s^k \alpha^{n+k} p_0 = s\alpha^n p_0 \sum_{k=0}^{p-1} (s\alpha)^k = s\alpha^n p_0 \cdot \frac{1-(s\alpha)^p}{1-s\alpha}.$$

As $n \to \infty$, $\alpha^n \to 0$, and we get, $d(x_n, x_{n+p}) \to 0$, showing that $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete b-metric space, $\{x_n\}$ has a limit $x \in X$. We shall show that x is a fixed point of T. By the triangle inequality, we have

$$\begin{aligned}
d\big(x, T(x)\big) &\leq s\big(d(x, x_n) + d(x_n, T(x))\big) = s(d(x, x_n) + d(T(x_{n-1}), T(x)))\\
&\leq s(d(x, x_n) + \alpha(d(x_{n-1}, x) + d(x_n, x) + d(x_{n+1}, x) + d(x_{n+2}, x))).
\end{aligned}$$

Since $d(x_n, x) \to 0$ as $n \to \infty$, taking the limits in the above inequality, we get $d(x, Tx) = 0$, i.e., $Tx = x$, as required.

To estimate the number of fixed point, Let if possible, $Tx = x$, $T(y) = y$, $T(z) = z$ and $T(w) = w$. Then, as $\alpha < 1$ we get from (1),

$$d(x, y) + d(y, z) + d(z, w) + d(w, x) < d(x, y) + d(y, z) + d(z, w) + d(w, x),$$

which is a contradiction and hence it has at most three fixed points. □

Remark 3.1. *Replacing $T(T(T(x))) \neq x$ by $T(T(x)) \neq x$ and setting any two points to be equal and taking $s = 1$ in Theorem 3.1, we obtain Theorem 2.1.*

Example 3.2. *Let $X = \{0, 1, 2, 3\}$ with the usual metric $d(x, y) = |x - y|$. Define $T : X \longrightarrow X$ by*

$$T(x) = -\tfrac{1}{3}x^3 + x^2 + \tfrac{1}{3}x.$$

Then

$$T(0) = 0, \quad T(1) = 1, \quad T(2) = 2, \quad T(3) = 1,$$

so that 0, 1 and 2 are the fixed points of T and T has exactly three fixed points.

For the only choice of four pairwise distinct points $\{0, 1, 2, 3\}$ one has

$$P = d(0, 1) + d(1, 2) + d(2, 3) + d(3, 0) = 6,$$

while the image-perimeter

$$\begin{aligned} P_c &= d(T(0), T(1)) + d(T(1), T(2)) + d(T(2), T(3)) + d(T(3), T(0)) \\ &= 1 + 1 + 1 + 1 = 4. \end{aligned}$$

Therefore $P_c \leq \frac{2}{3}P$, so that T is a quadrilaterals perimeter contraction on X with $\alpha = \frac{2}{3}$ (and $s\alpha = \frac{2}{3} < 1$). Here, T has exactly three fixed points.

We now construct a function with three fixed points which is continuous but not a quadrilaterals perimeter contraction.

Example 3.3. *Let (X, d, s) be a b-metric space with $d(x, y) = |x - y|$ on $X = [0, 2\pi]$ and let $T : X \to X$ be defined by*

$$T(x) = x + sinx, \quad \text{for all } x \text{ in } [0, 2\pi].$$

Clearly, T has three fixed points – $x = 0, \pi$ and 2π. Let

$$\begin{aligned} P &= d(x, y) + d(y, z) + d(z, w) + d(w, x) \\ &= |x - y| + |y - z| + |z - w| + |w - x|. \end{aligned}$$

Then using Mean Value Theorem, we have

$$\begin{aligned} P_c &= d\big(T(x), T(y)\big) + d\big(T(y), T(z)\big) + d\big(T(z), T(w)\big) + d\big(T(w), T(x)\big) \\ &= |T(x) - T(y)| + |T(y) - T(z)| + |T(z) - T(w)| + |T(w) - T(x)| \\ &= (1 + \cos c_1)|x - y| + (1 + \cos c_2)|y - z| + (1 + \cos c_3)|z - w| \\ &\quad + (1 + \cos c_4)|w - x|, \end{aligned}$$

for some $c_1, c_2, c_3, c_4 \in (0, 2\pi)$.

For points x, y, z and w very close to 0, we get $P_c \approx 2P$ so that T is not a quadrilaterals perimeter contraction.

Example 3.4. *Let (X, d, s) be a b-metric space with $d(x, y) = |x - y|$ on $X = [0, \infty)$ and let $T : X \to X$ be defined by*

$$T(x) = \frac{x}{n}, \quad \text{for all } n \text{ in } \mathbb{N} \setminus \{1\}.$$

Clearly, (X, d) is a complete metric space. We check that T is a contraction. For $x, y \in X$,

$$d\big(T(x), T(y)\big) = \left|\frac{x}{n} - \frac{y}{n}\right| = \frac{1}{n}|x - y| = \alpha\, d(x, y),$$

where $\alpha = \frac{1}{n} \in (0, 1)$ since $n > 1$. Thus T is a contraction.

To verify Condition 1 of Theorem 3.1, assume that $Tx \neq x$. Then

$$T(T(T(x))) = \frac{1}{n}\left(\frac{1}{n}\left(\frac{x}{n}\right)\right) = \frac{x}{n^3}.$$

If $T(T(T(x))) = x$, then $\frac{x}{n^3} = x$. Hence $(1 - \frac{1}{n^3})x = 0$, giving $x = 0$, which is a fixed point of T. Thus Condition 1 of Theorem 3.1 holds.

For Condition 2, let w, x, y and z be distinct points of X. Then

$$\begin{aligned} &d(T(x),T(y)) + d(T(y),T(z)) + d(T(z),T(w)) + d(T(w),T(x)) \\ &= \tfrac{1}{n}\big(|x-y| + |y-z| + |z-w| + |w-x|\big) \\ &= \alpha\big(d(x,y) + d(y,z) + d(z,w) + d(w,x)\big), \end{aligned}$$

with $\alpha = \frac{1}{n} \in (0,1)$ and $s\alpha < 1$. Hence Condition 2 also holds.

By Theorem 3.1, T has at most three fixed points. Solving $T(x) = x$, i.e. $\frac{x}{n} = x$, gives $(\frac{1}{n} - 1)x = 0$, so $x = 0$ is the only fixed point.

Therefore T satisfies the hypotheses of Theorem 3.1 and has a unique fixed point $x = 0$.

Corollary 3.1. *Let (X,d,s), $|X| \geq 4$ be a complete b-metric space and let $T : X \longrightarrow X$ be a contraction mapping. Then T admits a unique fixed point.*

Proof. Let $T : X \longrightarrow X$ be a contraction, i.e., $d(Tx,Ty) \leq \alpha d(x,y)$ for some $\alpha \in [0,1)$. It is easy to see that T is a quadrilaterals perimeter contraction.

Let us assume that $T(T(Tx)) = x$ with $Tx \neq x$. Now,

$$\begin{aligned} d\big(x,T(x)\big) &= d\big(T(T(T(x))),T(x)\big) \leq \alpha d\big(T(T(x)),x\big) \\ &\leq s\alpha\big(d(T(T(x)),T(x)) + d(T(x),x)\big) \leq s\alpha\big(\alpha d(T(x),x) + d(T(x),x)\big), \end{aligned}$$

i.e.,

$$(1 - s\alpha(1+\alpha))\, d\big(T(x),x\big) \leq 0.$$

Since $(1 - s\alpha(1+\alpha)) \neq 0$ (because $s\alpha < 1$), we get $T(x) = x$, a contradiction. By Theorem 3.1, T has a fixed point. Suppose that x and y are two distinct fixed points, then since T is a contraction, we have, $(1-\alpha)d(x,y) \leq 0$, i.e., $d(x,y) = 0$, or, $x = y$, contradicting the assumption that x and y are distinct. Hence T has a unique fixed point. ☐

Example 3.5. *Let (X,d,s) be a complete b-metric space where $X = [0,\infty)$ and*

$$d(x,y) = \begin{cases} x+y, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

and $T : X \longrightarrow X$ be defined by $T(x) = \frac{x}{3}$.

Clearly, d is a b-metric with coefficient $s = 2$. And,

$$\begin{aligned} P_c &= d\big(T(x),T(y)\big) + d\big(T(y),T(z)\big) + d\big(T(z),T(w)\big) + d\big(T(w),T(x)\big) \\ &= 2\left(\frac{x}{3} + \frac{y}{3} + \frac{z}{3} + \frac{w}{3}\right) = \frac{1}{3}P. \end{aligned}$$

Hence, $P_c \leq \alpha P$, where $\alpha = \frac{1}{3} \in (0,1)$and $s\alpha < 1$.

Therefore T is a quadrilaterals perimeter contraction on X with contraction constant $\alpha = \frac{1}{3}$. And, T has a unique fixed point $x = 0$.

Theorem 3.2. *Let (X,d,s), $|X| \geq 4$ be a complete b-metric space and let $T : X \to X$ be a quadrilaterals perimeter contraction on X such that for every $\varepsilon > 0$ there exists $\delta > 0$ with the property*

$$P(x,y,z,w) < \delta \quad \Rightarrow \quad P\big(T(x),T(y),T(z),T(w)\big) < \varepsilon,$$

for all pairwise distinct $x,y,z,w \in X$, where

$$P(x,y,z,w) = d(x,y) + d(y,z) + d(z,w) + d(w,x).$$

Then T has a fixed point and the number of fixed points is at most three.

Proof. Let $x_0 \in X$ and define the iterative sequence $x_{n+1} = T(x_n)$ for $n \geq 0$. For each $n \geq 0$ and x_i not a fixed point for all $i \in \mathbb{N}$, let

$$P_n = d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_n).$$

By assumption, whenever P_n is small, P_{n+1} becomes even smaller. Thus $\{P_n\}$ is a nonincreasing sequence of nonnegative real numbers. Hence $\{P_n\}$ converges to some $L \geq 0$.

Suppose $L > 0$. Then choose $\varepsilon = L/2 > 0$. By the above given property of T, there exists $\delta > 0$ such that

$$P(x, y, z, w) < \delta \quad \Rightarrow \quad P\big(T(x), T(y), T(z), T(w)\big) < \varepsilon.$$

Since $P_n \to L$, for sufficiently large n we have $P_n < \delta$, which would imply $P_{n+1} < \varepsilon = L/2 < L$, a contradiction to the convergence of P_n to L.

Therefore $L = 0$ and so $P_n \to 0$.

In particular, $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. We now show that $\{x_n\}$ is a Cauchy sequence. For $p \geq 1$,

$$\begin{aligned} d(x_n, x_{n+p}) &\leq s\big(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p})\big) \\ &\leq s\, d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{p-1} d(x_{n+p-1}, x_{n+p}). \end{aligned}$$

Then, each $d(x_k, x_{k+1}) \to 0$, the right-hand side tends to 0 as $n \to \infty$ and thus, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ with $x_n \to x$.

We shall show that x is a fixed point of T. We have,

$$d(x, T(x)) \leq s\big(d(x, x_n) + d(x_n, T(x))\big) = s\big(d(x, x_n) + d(T(x_{n-1}), Tx)\big).$$

By the defining property of T, $d(T(x_{n-1}), T(x)) \to 0$ as $n \to \infty$. Also $d(x, x_n) \to 0$. Thus $d(x, T(x)) = 0$, i.e. $T(x) = x$.

Finally, suppose T has four distinct fixed points x, y, z, w. Then

$$P(T(x), T(y), T(z), T(w)) = P(x, y, z, w),$$

which contradict the perimeter contraction property. Therefore T has at most three fixed points. □

We now construct a mapping on a b-metric space which is a quadrilaterals perimeter contraction but not a triangles perimeter contraction.

Example 3.6. *Let $X = \{0, 1, 2, 3\}$ with the usual metric $d(x, y) = |x - y|$. Define a mapping $T : X \longrightarrow X$ by*

$$T(0) = 0, \quad T(1) = 2, \quad T(2) = 1, \quad T(3) = 1.$$

Consider the only set of four pairwise distinct points $\{0, 1, 2, 3\}$. Then, the perimeter of the quadrilateral formed by these points is

$$P = d(0, 1) + d(1, 2) + d(2, 3) + d(3, 0) = 1 + 1 + 1 + 3 = 6.$$

The perimeter of the image quadrilateral is

$$\begin{aligned} P_c &= d\big(T(0), T(1)\big) + d\big(T(1), T(2)\big) + d\big(T(2), T(3)\big) + d\big(T(3), T(0)\big) \\ &= d(0, 2) + d(2, 1) + d(1, 1) + d(1, 0) = 2 + 1 + 0 + 1 = 4. \end{aligned}$$

Hence, $P_c = \frac{2}{3}P$, so that the inequality $P_c \leq \alpha P$ holds for $\alpha = \frac{2}{3} < 1$ and thus, T is a quadrilaterals perimeter contraction on X.

Now, consider the triple $(0, 1, 2)$. Its original perimeter is

$$P_\triangle = d(0, 1) + d(1, 2) + d(0, 2) = 1 + 1 + 2 = 4,$$

and the image perimeter is

$$\begin{aligned} P_{c,\triangle} &= d\big(T(0),T(1)\big) + d\big(T(1),T(2)\big) + d\big(T(0),T(2)\big) \\ &= d(0,2) + d(2,1) + d(0,1) = 2 + 1 + 1 = 4. \end{aligned}$$

Thus, $P_{c,\triangle} = P_\triangle$, so that there is no $\alpha < 1$ satisfying

$$P_{c,\triangle} \le \alpha P_\triangle.$$

Therefore, T is not a triangles perimeter contraction, but it is a quadrilaterals perimeter contraction.

In the above example, we show that quadrilaterals perimeter contraction need not be triangles perimeter contraction. Next, we show that triangles perimeter contraction need not be quadrilaterals perimeter contraction.

Example 3.7. *Consider the b-metric space (X,d,s) where $X = \{0,1,2,3\}$, d is the discrete metric given by*

$$d(u,v) = \begin{cases} 0, & u = v, \\ 1, & u \ne v. \end{cases}$$

Define the mapping $T : X \longrightarrow X$ by

$$T(0) = 0, \qquad T(1) = 1, \qquad T(2) = 0, \qquad T(3) = 1.$$

Then, T is a contracting perimeters of triangles. For, if a, b and c are any three pairwise distinct points in X, the original perimeter is

$$P_\triangle = d(a,b) + d(b,c) + d(a,c) = 1 + 1 + 1 = 3.$$

Since T takes only the two values 0 and 1, the set $\{Ta, Tb, Tc\}$ contains at most two distinct elements. Hence the image perimeter satisfies $P_{c,\triangle} \le 2$.

Thus, for $\alpha = \frac{2}{3} < 1$, $P_{c,\triangle} \le 2 = \frac{2}{3} \cdot 3 = \alpha\, P_\triangle$ and hence T is a triangles perimeter contraction with $\alpha = \frac{2}{3}$.

To see that T is not a quadrilaterals perimeter contraction, consider the four pairwise distinct points $0, 1, 2, 3$. Their original quadrilateral perimeter is

$$P = d(0,1) + d(1,2) + d(2,3) + d(3,0) = 1 + 1 + 1 + 1 = 4.$$

The image points are

$$T(0) = 0,\ T(1) = 1,\ T(2) = 0,\ T(3) = 1,$$

so the image perimeter is

$$P_c = d(0,1) + d(1,0) + d(0,1) + d(1,0) = 1 + 1 + 1 + 1 = 4.$$

For $\alpha = \frac{2}{3}$ we have $P_c \le \alpha P = \frac{8}{3}$ but $P_c = 4 > \frac{8}{3}$ and thus, T does not *contract the perimeter of this quadrilateral.*

Hence T is a triangles perimeter contraction but not quadrilaterals perimeter contraction.

4. **Application**

In this section, we prove the existence of the solution to a non-linear Fredholm integral equation with lipschitz condition by applying our result.

Consider the non-linear integral equation

$$x(t) = g(t) + \int_a^b K(t,s,x(s))\, ds, \quad t \in [a,b], \tag{3}$$

where $X = C([a,b],\mathbb{R})$ is the space of continuous real-valued functions on $[a,b]$.

Theorem 4.1. *Consider the non-linear integral equation* (3) *and suppose that* $f : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous and satisfies the following condition:*

(i) *for all* $(t, x), (t, y) \in [a, b] \times \mathbb{R}$,
$|K(t, x(t)) - K(t, y(t))| \leq L\,|x(t) - y(t)|$,
where $L \geq 0$ *is the Lipschitz constant.*

(ii) *such that for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *with the property*

$$P(x, y, z, w) < \delta \quad \Rightarrow \quad P(T(x), T(y), T(z), T(w)) < \varepsilon,$$

for all pairwise distinct $x, y, z, w \in X$*, where*

$$P(x, y, z, w) = d(x, y) + d(y, z) + d(z, w) + d(w, x).$$

Then, it has at most three solution with $L(b - a) < 1$.

Proof. Consider a complete b-metric space $(X, d, 1)$ with usual metric and Let $T : X \to X$ be the operator defined by

$$(Tx)(t) = g(t) + \int_a^b K(t, s, x(s))\, ds, \quad t \in [a, b].$$

Since K and g are continuous and $x \in X$ is continuous, it follows from standard properties of integrals that $Tx \in X$. Hence T maps X into itself.

For $x, y \in X$ and each $t \in [a, b]$, using condition (i),

$$\begin{aligned}
|(Tx)(t) - (Ty)(t)| &= \left| \int_a^b \big(K(t, s, x(s)) - K(t, s, y(s))\big)\, ds \right| \\
&\leq \int_a^b |K(t, s, x(s)) - K(t, s, y(s))|\, ds \\
&\leq \int_a^b L|x(s) - y(s)|\, ds \\
&\leq L(b - a)|x - y| \\
&\leq L(b - a)d(x, y)
\end{aligned}$$

Then,

$$P(T(x), T(y), T(z), T(w)) \leq L(b - a)P(x, y, z, w)$$

follow by conditions (ii). By theorem 3.2, T has a fixed point. □

Conclusions

Quadrilaterals perimeter contraction mappings are introduced and some fixed point results are obtained. It is also shown that Quadrilaterals perimeter contraction mappings are continuous in a b-metric space setting. With suitable examples, it is also shown that triangles perimeter contraction need not be Quadrilaterals perimeter contraction, and vice versa. Finally, the existence of the solution to a non-linear Fredholm integral equation with Lipschitz condition is obtained as an application. A generalized polygonal perimeter contraction fixed point results may be studied in a similar manner in a more generalized space setting.

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