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# SOLVING SOME DIFFERENTIAL EQUATIONS WITH NEUTRAL DEVIATION ARGUMENTS USING SEVERAL GENERALIZATIONS OF DARBO THEOREM

Farnaz Soltanpour[1], Hamid Majani[2], Ali Shole Haghighi[3]

*The target of this article is to provide a few extensions of Darbo theorem using the techniques of family of measures of noncompactness (FMN) in a Fréchet space. Several interconnections among assumptions imposed in the proved theorems are indicated. We also show the applicability of the mentioned results to solve some differential equations with neutral deviated arguments.*

**Keywords:** family of measure of noncompactness, Fréchet space, system of differential equations.

**MSC2020:** 47H10, 47H08, 39B72.

## 1. Introduction

The study of neutral differential equations has attracted much attention due to its wide application in various scientific fields, including control theory, mathematical biology, and engineering. These equations, which involve both current state and time-delayed or advanced terms, pose distinct analytical challenges, making the development of efficient solution methods crucial. Hence, some researchers have focused on the existence of solutions for functional differential equations under particular assumptions [1, 5, 9, 13]. These approaches typically use fixed point (FP) results and compactness arguments which, although effective, often require strict conditions on the function spaces and associated operators. For example, Darbo [8, 1955] has established a well-known theorem as follows:

**Theorem 1.1.** *Let $\mathcal{D}$ be a nonempty, closed, bounded, and convex subset of a Banach space $\mathfrak{B}$, and let $\Im : \mathcal{D} \to \mathcal{D}$ be a continuous mapping. Suppose there exists a constant $\delta \in [0, 1)$ such that for every bounded set $Y \subseteq \mathcal{D}$,*

$$\mu(\Im Y) \leq \delta\, \mu(Y),$$

*where $\mu$ is a measure of noncompactness (e.g., the Kuratowski or Hausdorff measure). Then $\Im$ has at least one fixed point in $\mathcal{D}$.*

Some researchers have offered a more flexible framework using FMN, which allows for solutions in wider functional spaces with fewer assumptions. For important works on MN and FMN, see [2, 3, 4, 6, 7, 12, 15, 16].

This paper first establishes several FP results in the Fréchet space of continuous functions on the real half-axis, which extend existing methods in Darbo theorems in Banach

[1]Assistant Professor, Department of Mathematics, Faculty of Mathematical Sciences and Computer, Shahid Chamran University of Ahvaz, Ahvaz, Iran, e-mail: `f-soltanpour@stu.scu.ac.ir`

[2]Associate Professor, Department of Mathematics, Faculty of Mathematical Sciences and Computer, Shahid Chamran University of Ahvaz, Ahvaz, Iran, Corresponding author's e-mail: `h.majani@scu.ac.ir`, `majani.hamid@gmail.com`

[3]Assistant Professor, Department of Mathematics, Karaj Branch, IAU, Iran, e-mail: `a_sholehaghighi@yahoo.com`

spaces, and then presents a novel approach for proving the existence of solutions to neutral differential equations with deviated arguments, leveraging the concept of an FMN. Unlike some traditional and recent techniques, our method is based on more general conditions, which expands the scope of solvable problems while reducing the reliance on restrictive assumptions. Further, our approach improves upon previous work by simplifying the existence proof and eliminating the need for complex and sometimes cumbersome assumptions used in previous studies. This simplification not only increases the accessibility of the theory, but also improves its computational feasibility, making it a more practical tool for applied mathematicians and engineers. This advancement is expected to open new avenues for further research and practical applications in areas such as stability analysis, control systems, and population dynamics. This paper is divided into four sections. The next section provides the preliminaries necessary to achieve our goal. Section 3 is served as the main results of this article. The last section demonstrates the existence of a solution to some differential equations, which is organized into two parts.

## 2. **Preliminaries**

In 1980, Banaś and Goebel introduced the concept of the MN in Banach spaces as follows and enriched this field of study.

**Definition 2.1** ([7])**.** Presume $\mathfrak{B}$ is a Banach space, $\mathfrak{M}_{\mathfrak{B}}$ is the collection of nonempty bounded subsets of $\mathfrak{B}$, and $\mathfrak{N}_{\mathfrak{B}}$ is the subfamily composed of all relatively compact subsets of $\mathfrak{B}$. A function $\mu : \mathfrak{M}_{\mathfrak{B}} \to [0, \infty)$ is called an MN in $\mathfrak{B}$ when the following properties are held:

1. $\emptyset \neq \ker \mu = \{\mathcal{I} \in \mathfrak{M}_{\mathfrak{B}} : \mu(\mathcal{I}) = 0\}$ is a subset of $\mathfrak{N}_{\mathfrak{B}}$;
2. For all subsets $\mathcal{I}$ and $\mathcal{J}$ with $\mathcal{I} \subset \mathcal{J}$, $\mu(\mathcal{I}) \leq \mu(\mathcal{J})$;
3. For each subset $\mathcal{I}$ of $\mathfrak{B}$, $\mu(\bar{\mathcal{I}}) = \mu(\mathcal{I})$ in which $\bar{\mathcal{I}}$ denotes the closure of $\mathcal{I}$;
4. For each subset $\mathcal{I}$ of $\mathfrak{B}$, $\mu(\mathrm{Conv}\mathcal{I}) = \mu(\mathcal{I})$ in which $\mathrm{Conv}\mathcal{I}$ represents the closed convex hull of $\mathcal{I}$;
5. For two subsets $\mathcal{I}$ and $\mathcal{J}$ of $\mathfrak{B}$ and for $\omega \in [0, 1]$,
$$\mu(\omega\mathcal{I} + (1 - \omega)\mathcal{J}) \leq \omega\mu(\mathcal{I}) + (1 - \omega)\mu(\mathcal{J});$$
6. If $\{\mathcal{I}_n\}$ is a sequence of closed sets in $\mathfrak{M}_{\mathfrak{B}}$ with $\mathcal{I}_{n+1} \subset \mathcal{I}_n$ for $n = 1, 2, \cdots$ and $\lim\limits_{n\to\infty} \mu(\mathcal{I}_n) = 0$, then $\mathcal{I}_\infty = \bigcap\limits_{n=1}^{\infty} \mathcal{I}_n$ is nonempty.

Here, we introduce admissible sets that play a crucial role in construction of an FMN.

**Definition 2.2** ([4])**.** Presume $\mathcal{M}$ is a class of subsets of a Fréchet space $\mathcal{E}$, a locally convex space being complete regarding a translation-invariant metric, and $\mathfrak{N}_{\mathcal{E}}$ is the subfamily containing all relatively compact sets. $\mathcal{M}$ is named an admissible set whenever $\mathrm{Conv}(\mathcal{I})$ and $\bar{\mathcal{I}} \in \mathcal{M}$ for any $\mathcal{I} \in \mathcal{M}$, and $\mathfrak{N}_{\mathfrak{B}} \cap \mathcal{M} \neq \emptyset$.

Presume $\mathcal{E}$ is a Fréchet space and $\mathcal{M}$ is an admissible set of $\mathcal{E}$ with $\mathcal{D} \in \mathcal{M}$. In addition, $\mathfrak{N}_{\mathcal{E}}$ refers to the subfamily including all relatively compact subsets of $\mathcal{E}$. For a $\mathcal{I} \subset \mathcal{E}$, $\bar{\mathcal{I}}$ and $\mathrm{Conv}\mathcal{I}$ are considered as the closure of $\mathcal{I}$ and the closed convex hull of $\mathcal{I}$, respectively.

**Definition 2.3** ([12])**.** Presume $\mathcal{E}$ is a Fréchet space. A family of functions $\{\mu_{\mathcal{U}}\}_{\mathcal{U}\geq 0}$ with $\mu_{\mathcal{U}} : \mathcal{M} \to [0, \infty)$ is called an FMN in $\mathcal{E}$ if the following are met:

(1) $\emptyset \neq \ker\{\mu_{\mathcal{U}}\} = \{\mathcal{I} \in \mathcal{M} : \mu_{\mathcal{U}}(\mathcal{I}) = 0 \text{ for all } \mathcal{U} \geq 0\}$ and $\ker\{\mu_{\mathcal{U}}\} \subset \mathfrak{N}_{\mathcal{E}}$;
(2) For any $\mathcal{I} \subset \mathcal{J}$, $\mu_{\mathcal{U}}(\mathcal{I}) \leq \mu_{\mathcal{U}}(\mathcal{J})$ for all $\mathcal{U} \geq 0$;
(3) $\mu_{\mathcal{U}}(\mathrm{Conv}\mathcal{I}) = \mu_{\mathcal{U}}(\mathcal{I})$ for all $\mathcal{U} \geq 0$;
(4) If $\{\mathcal{I}_n\}$ is a sequence of closed sets in $\mathcal{M}$ with $\mathcal{I}_{n+1} \subset \mathcal{I}_n$ for $n = 1, 2, \cdots$ and $\lim\limits_{n\to\infty} \mu_{\mathcal{U}}(\mathcal{I}_n) = 0$ for any $\mathcal{U} \geq 0$, then $\mathcal{I}_\infty = \bigcap\limits_{n=1}^{\infty} \mathcal{I}_n$ is nonempty.

Note that $\ker\{\mu_{\mathcal{U}}\}$ in Definition 2.3(1) is the kernel of the FMN and $\mathcal{I}_\infty$ in Definition 2.3(4) belongs to this kernel. In fact, it follows from $\mu_{\mathcal{U}}(\mathcal{I}_\infty) \le \mu_{\mathcal{U}}(\mathcal{I}_n)$ for any $\mathcal{U} \ge 0$ and $n \in \mathbb{N}$ that $\mu_{\mathcal{U}}(\mathcal{I}_\infty) = 0$. Hence, we deduce $\mathcal{I}_\infty \in \ker\{\mu_{\mathcal{U}}\}$.

Here, a famous theorem in topology, needed in our main theorem, is recalled.

**Theorem 2.1** ([1]). *Tychonoff Theorem. Presume $\mathfrak{B}$ is a Hausdorff locally convex linear topological space and $\mathcal{D}$ is a convex subset of $\mathfrak{B}$. Also, suppose $\mathcal{H} : \mathcal{D} \to \mathfrak{B}$ is a continuous mapping so that $\mathcal{H}(\mathcal{D}) \subseteq \mathcal{A} \subseteq \mathcal{D}$ in which $\mathcal{A}$ is compact. Then $\mathcal{H}$ possesses an FP.*

## 3. Main theorems

In the sequel, presume $\mathfrak{N} \in \mathcal{M}$ is a nonempty, closed, and convex subset of a Fréchet space $\mathcal{E}$.

**Theorem 3.1.** *Presume $\mathcal{H} : \mathfrak{N} \to \mathfrak{N}$ is a continuous operator satisfying*

$$\Omega_{\mathcal{U}}(\mu_{\mathcal{U}}(\mathcal{H}(\mathfrak{L}))) \le \Omega_{\mathcal{U}}(\mu_{\mathcal{U}}(\mathfrak{L})) - \Upsilon_{\mathcal{U}}(\mu_{\mathcal{U}}(\mathfrak{L})) \tag{1}$$

*for any subset $\mathfrak{L}$ of $\mathfrak{N}$ and for each $\mathcal{U} \ge 0$ in which $\{\mu_{\mathcal{U}}\}_{\mathcal{U}\ge 0}$ is an arbitrary FMN in $\mathcal{E}$ and $\Omega_{\mathcal{U}}, \Upsilon_{\mathcal{U}} : \mathcal{R}_+ \to \mathcal{R}_+$ are two functions provided that $\Omega_{\mathcal{U}}$ is lower semicontinuous and $\Upsilon_{\mathcal{U}}$ is continuous on $\mathcal{R}_+$. Additionally, presume $\Upsilon_{\mathcal{U}}(0) = 0$ and $\Upsilon_{\mathcal{U}}(t) > 0$ for all $t > 0$ and $\mathcal{U} \ge 0$. Then $\mathcal{H}$ possesses an FP in $\mathfrak{N}$.*

*Proof.* Consider the sequence $\{\mathfrak{N}_n\}$ by $\mathfrak{N}_0 = \mathfrak{N}$ and $\mathfrak{N}_n = Conv(\mathcal{H}\mathfrak{N}_{n-1})$ for $n = 1, 2, \cdots$. If there is an integer $N \ge 0$ so that $\mu_{\mathcal{U}}(\mathfrak{N}_N) = 0$ for all $\mathcal{U} \ge 0$, then $\mathfrak{N}_N$ is relatively compact. In this case, Theorem 2.1 induces $\mathcal{H}$ has an FP in $\mathfrak{N}$. So, assume $\mu_{\mathcal{U}}(\mathfrak{N}_n) \ne 0$ for $n = 1, 2, \cdots$ and some $\mathcal{U} \ge 0$. By (1), we get

$$\Omega_{\mathcal{U}}(\mu_{\mathcal{U}}(\mathfrak{N}_{n+1})) = \Omega_{\mathcal{U}}(\mu_{\mathcal{U}}(Conv(\mathcal{H}\mathfrak{N}_n))) = \Omega_{\mathcal{U}}(\mu_{\mathcal{U}}(\mathcal{H}\mathfrak{N}_n)) \le \Omega_{\mathcal{U}}(\mu_{\mathcal{U}}(\mathfrak{N}_n)) - \Upsilon_{\mathcal{U}}(\mu_{\mathcal{U}}(\mathfrak{N}_n)). \tag{2}$$

Since $\{\mu_{\mathcal{U}}(\mathfrak{N}_n)\}$ is nonincreasing and nonnegative for $\mathcal{U} \ge 0$, we infer $\mu_{\mathcal{U}}(\mathfrak{N}_n) \to r$ when $n$ tends to infinity in which $r$ is a nonnegative real number. On the other hand, if follows from (2) that

$$\limsup_{n\to\infty} \Omega_{\mathcal{U}}(\mu_{\mathcal{U}}(\mathfrak{N}_{n+1})) \le \limsup_{n\to\infty} \Omega_{\mathcal{U}}(\mu_{\mathcal{U}}(\mathfrak{N}_n)) - \limsup_{n\to\infty} \Upsilon_{\mathcal{U}}(\mu_{\mathcal{U}}(\mathfrak{N}_n))$$

for each $\mathcal{U} \ge 0$, which yields $\Omega_{\mathcal{U}}(r) \le \psi_{\mathcal{U}}(r) - \Upsilon_{\mathcal{U}}(r)$ for $\mathcal{U} \ge 0$. Consequently, $\Upsilon_{\mathcal{U}}(r) = 0$, implying $r = 0$. Hence, for each $\mathcal{U} \ge 0$, $\mu_{\mathcal{U}}(\mathfrak{N}_n) \to 0$ as $n \to \infty$. Now, as $\mathfrak{N}_{n+1} \subseteq \mathfrak{N}_n$, if follows from Definition 2.3(4) that $\mathfrak{N}_\infty = \bigcap_{n=1}^{\infty} \mathfrak{N}_n$ is a nonempty, closed, and convex and $\mathfrak{N}_\infty \subset \mathfrak{N}$. Further, $\mathfrak{N}_\infty$ is invariant under $\mathcal{H}$, and $\mathfrak{N}_\infty \in \ker\{\mu_{\mathcal{U}}\}$ for each $\mathcal{U} \ge 0$. Thus, applying Theorem 2.1, the proof ends. □

**Theorem 3.2.** *Assume $\mathcal{H} : \mathfrak{N} \to \mathfrak{N}$ is a continuous operator fulfilling $\mu_{\mathcal{U}}(\mathcal{H}(\mathfrak{L})) \le \varrho_{\mathcal{U}}(\mu_{\mathcal{U}}(\mathfrak{L}))$ for any subset $\mathfrak{L}$ of $\mathfrak{N}$ and for each $\mathcal{U} \ge 0$ in which $\{\mu_{\mathcal{U}}\}_{\mathcal{U}\ge 0}$ is an arbitrary FMN on $\mathcal{E}$ and $\varrho_{\mathcal{U}} : \mathcal{R}_+ \to \mathcal{R}_+$ is a continuous function. Then $\mathcal{H}$ possesses an FP in $\mathfrak{N}$.*

*Proof.* It is enough to take $\Omega_{\mathcal{U}}(t) = t$ and $\Upsilon_{\mathcal{U}}(t) = t - \varrho_{\mathcal{U}}(t)$ for all $\mathcal{U} \ge 0$ in Theorem 3.1. □

In recent decade, many researchers have discussed the equivalence of the existence and uniqueness of $n$-tuples FPs and usual FPs for multidimensional mappings in [10, 11, 14]. Following their ideas, we present some useful theorems regarding an FMN on a finite product space.

**Theorem 3.3** ([7]). *Suppose* $\{\mu^1_{\mathcal{U}}\}_{\mathcal{U}\geq 0}, \cdots, \{\mu^n_{\mathcal{U}}\}_{\mathcal{U}\geq 0}$ *are FMN on Fréchet spaces* $\mathcal{E}_1, \cdots, \mathcal{E}_n$*, respectively. Also, presume* $\mathcal{H} : \mathcal{R}^n_+ \to \mathcal{R}_+$ *is a convex function satisfying* $\mathcal{H}(q_1, \cdots, q_n) = 0$ *iff* $q_i = 0$ *for* $i = 1, \cdots, n$*. Then* $\mu_{\mathcal{U}}(\mathfrak{L}) = \mathcal{H}(\mu^1_{\mathcal{U}}(\mathfrak{L}_1), \cdots, \mu^n_{\mathcal{U}}(\mathfrak{L}_n))$ *defines an FMN on* $\mathcal{E}_1 \times \cdots \times \mathcal{E}_n$ *for* $\mathcal{U} \geq 0$ *in which* $\mathfrak{L}_i$ *represents the natural projections (in short, NPs) of* $\mathfrak{L}$ *into* $\mathcal{E}_i$ *for each* $i = 1, \cdots, n$*.*

**Example 3.1.** *Presume* $\{\mu_{\mathcal{U}}\}_{\mathcal{U}\geq 0}$ *is an FMN on a Fréchet space* $\mathcal{E}$ *and*

$$\begin{cases} \mathcal{H}_1(q_1, \cdots, q_n) = \max\{q_1, \cdots, q_n\} \\ \mathcal{H}_2(q_1, \cdots, q_n) = \sum\limits_{i=1}^{n} q_i \end{cases}$$

*for* $(q_1, \cdots, q_n) \in \mathcal{R}^n_+$*. Then the conditions of Theorem 3.3 fulfill* [2]*. Now, presume* $\mathcal{M}$ *is an admissible set in Fréchet space* $\mathcal{E}$ *and* $\mathcal{M}_{\mathcal{E}^n} = \{\mathfrak{L} \subset \mathcal{E}^n | \mathfrak{L}_i \in \mathcal{M}\}$*, where* $\mathfrak{L}_i$ *are the NPs of* $\mathfrak{L}$ *for* $i = 1, \cdots, n$*. It is evident that the conditions of Definition 2.2 are met. Therefore, the mappings* $\tilde{\mu}^1_{\mathcal{U}} : \mathcal{M}_{\mathcal{E}^n} \to [0, \infty)$ *and* $\tilde{\mu}^2_{\mathcal{U}} : \mathcal{M}_{\mathcal{E}^n} \to [0, \infty)$ *are defined by*

$$\begin{cases} \tilde{\mu}^1_{\mathcal{U}}(\mathfrak{L}) = \max\{\mu_{\mathcal{U}}(\mathfrak{L}_1), \cdots, \mu_{\mathcal{U}}(\mathfrak{L}_n)\}, \\ \tilde{\mu}^2_{\mathcal{U}}(\mathfrak{L}) = \sum\limits_{i=1}^{n} \mu_{\mathcal{U}}(\mathfrak{L}_i), \end{cases}$$

*are two FMNs in* $\mathcal{E}^n$ *for* $\mathcal{U} \geq 0$ *and* $\mathfrak{L}_i$ *represents the NPs of* $\mathfrak{L}$ *for* $i = 1, \cdots, n$*.*

**Theorem 3.4.** *Presume* $\mathcal{H}_\iota : \mathfrak{N}^n \to \mathfrak{N}$ *is a continuous operator satisfying*

$$\Omega_{\mathcal{U}}(\mu_{\mathcal{U}}(\mathcal{H}_\iota(\mathfrak{L}))) \leq \Omega_{\mathcal{U}}(\mu_{\mathcal{U}}(\mathfrak{L})) - \Upsilon_{\mathcal{U}}(\mu_{\mathcal{U}}(\mathfrak{L})) \tag{3}$$

*for any subset* $\mathfrak{L}$ *of* $\mathfrak{N}^n$ *and for each* $\mathcal{U} \geq 0$ *in which* $\{\mu_{\mathcal{U}}\}_{\mathcal{U}\geq 0}$ *is an FMN in* $\mathcal{E}$*,* $\mathfrak{L}_i$ *represents the NPs of* $\mathfrak{L}$ *into* $\mathfrak{N}$ *for* $\iota = 1, \cdots, n$ *and* $\Omega_{\mathcal{U}}, \Upsilon_{\mathcal{U}} : \mathcal{R}_+ \to \mathcal{R}_+$ *are given functions such that* $\Omega_{\mathcal{U}}$ *is lower semicontinuous and* $\Upsilon$ *is continuous. Also, assume* $\Upsilon_{\mathcal{U}}(0) = 0$ *and* $\Upsilon_{\mathcal{U}}(t) > 0$ *for all* $t > 0$ *and* $\mathcal{U} \geq 0$*. Then there is* $(q_1, \cdots, q_n) \in \mathfrak{N}^n$ *so that* $\mathcal{H}_\iota(q_1, \cdots, q_n) = q_\iota$ *for* $\iota = 1, \cdots, n$*.*

*Proof.* Take the continuous operator $\tilde{\mathcal{H}} : \mathfrak{N}^n \to \mathfrak{N}^n$ by

$$\tilde{\mathcal{H}}(q_1, \cdots, q_n) = (\mathcal{H}_1(q_1, \cdots, q_n), \mathcal{H}_2(q_1, \cdots, q_n), \cdots, \mathcal{H}_n(q_1, \cdots, q_n)).$$

By Example 3.1, $\tilde{\mu}_{\mathcal{U}}(\mathfrak{L}) = \max\{\mu_{\mathcal{U}}(\mathfrak{L}_1), \cdots, \mu_{\mathcal{U}}(\mathfrak{L}_n)\}$ defines an FMN on $\mathcal{E}^n$ for any subset $\mathfrak{L}$ in an admissible set $\mathcal{M}_{\mathcal{E}^n}$ of $\mathcal{E}^n$ and each $\mathcal{U} \geq 0$ in which $\mathfrak{L}_\iota$ is the NPs of $\mathfrak{L}$ for $\iota = 1, \cdots, n$. Using (3), we get

$$\begin{aligned} \Omega_{\mathcal{U}}(\mu_{\mathcal{U}}(\tilde{\mathcal{H}}(\mathfrak{L}))) &= \Omega_{\mathcal{U}}(\max\{\mu_{\mathcal{U}}(\mathcal{H}_1(\mathfrak{L}), \cdots, \mu_{\mathcal{U}}(\mathcal{H}_n(\mathfrak{L}))\}) \\ &\leq \Omega_{\mathcal{U}}(\max\{\mu_{\mathcal{U}}(\mathfrak{L}_1), \cdots, \mu_{\mathcal{U}}(\mathfrak{L}_n)\}) - \Upsilon_{\mathcal{U}}(\max\{\mu_{\mathcal{U}}(\mathfrak{L}_1), \cdots, \mu_{\mathcal{U}}(\mathfrak{L}_n)\}) \\ &\leq \Omega_{\mathcal{U}}(\tilde{\mu}_{\mathcal{U}}(\mathfrak{L})) - \Upsilon_{\mathcal{U}}(\tilde{\mu}_{\mathcal{U}}(\mathfrak{L})). \end{aligned}$$

Now, the proof is followed by Theorem 3.1. □

**Theorem 3.5.** *Let* $\mathcal{H}_\iota : \mathfrak{N}^n \to \mathfrak{N}$ *be a continuous operator for* $\iota = 1, \cdots, n$ *such that*

$$\mu_{\mathcal{U}}(\mathcal{H}_\iota(\mathfrak{L}_1 \times \cdots \times \mathfrak{L}_n)) \leq \varrho_{\mathcal{U}}(\max\{\mu_{\mathcal{U}}(\mathfrak{L}_1), \cdots, \mu_{\mathcal{U}}(\mathfrak{L}_n)\}) \tag{4}$$

*for every bounded subset* $\mathfrak{L}$ *of* $\mathfrak{N}^n$ *and each* $\mathcal{U} \geq 0$ *in which* $\{\mu_{\mathcal{U}}\}_{\mathcal{U}\geq 0}$ *is an arbitrary FMN on* $\mathcal{E}$*,* $\mathfrak{L}_\iota$ *represents the NPs of* $\mathfrak{L}$ *into* $\mathfrak{N}$ *for* $\iota = 1, \cdots, n$ *and* $\varrho_{\mathcal{U}} : \mathcal{R}_+ \to \mathcal{R}_+$ *is a continuous function on* $\mathcal{R}_+$ *for* $\mathcal{U} \geq 0$*. Then there is* $(q_1, \cdots, q_n) \in \mathfrak{N}^n$ *so that* $\mathcal{H}_\iota(q_1, \cdots, q_n) = q_\iota$ *for* $\iota = 1, \cdots, n$*.*

*Proof.* Take the continuous operator $\tilde{\mathcal{H}} : \mathfrak{N}^n \to \mathfrak{N}^n$ by

$$\tilde{\mathcal{H}}(q_1, \cdots, q_n) = (\mathcal{H}_1(q_1, \cdots, q_n), \mathcal{H}_2(q_1, \cdots, q_n), \cdots, \mathcal{H}_n(q_1, \cdots, q_n)).$$

By Example 3.1, $\tilde{\mu}_{\mathcal{U}}(\mathfrak{L}) = \max\{\mu_{\mathcal{U}}(\mathfrak{L}_1), \cdots, \mu_{\mathcal{U}}(\mathfrak{L}_n)\}$ defines an FMN on $\mathcal{E}^n$ for any subset $\mathfrak{L}$ in an admissible set $\mathcal{M}_{\mathcal{E}^n}$ of $\mathcal{E}^n$ and each $\mathcal{U} \geq 0$ in which $\mathfrak{L}_\iota$ is the NPs of $\mathfrak{L}$ for $\iota = 1, \cdots, n$. Using (4), we get

$$\begin{aligned} \tilde{\mu}_{\mathcal{U}}(\tilde{\mathcal{H}}(\mathfrak{L})) &\leq \tilde{\mu}_{\mathcal{U}}(\mathcal{H}_1(\mathfrak{L}) \times \cdots \times \mathcal{H}_n(\mathfrak{L})) \\ &= \max\{\mu_{\mathcal{U}}(\mathcal{H}_1(\mathfrak{L})), \cdots, \mu_{\mathcal{U}}(\mathcal{H}_n(\mathfrak{L}))\} \\ &\leq \varrho_{\mathcal{U}}(\max\{\mu_{\mathcal{U}}(\mathfrak{L}_1), \cdots, \mu_{\mathcal{U}}(\mathfrak{L}_n)\}) \\ &= \varrho_{\mathcal{U}}(\tilde{\mu}_{\mathcal{U}}(\mathfrak{L})). \end{aligned}$$

Now, the proof is followed by Theorem 3.2. □

## 4. Application

Up until now, numerous studies have been conducted by researchers to demonstrate the existence of solutions for functional differential equations with deviated arguments of neutral type [2, 4, 9, 12, 16]; however, we present a novel approach for proving the existence of solutions to these equations with deviated arguments. The application in this section is divided into two parts: differential equations and system od differential equations.

### 4.1. Differential equations

Here, we consider $C(\mathcal{R}_+) = \{q : \mathcal{R}_+ \to \mathcal{R}, q \text{ is continuous}\}$, a more restrictive function space, and systems with weaker conditions. Presume $C(\mathcal{R}_+)$ is equipped with the family of seminorms $|q|_{\mathcal{U}} = \sup\{|q(b)| : b \in [0, \mathcal{U}]\}$ for $\mathcal{U} \geq 0$. Then $C(\mathcal{R}_+)$ with the metric $d(q,p) = \sup\{\frac{1}{2^{\mathcal{U}}} \min\{1, |q-p|_{\mathcal{U}}\} : \mathcal{U} \geq 0\}$ is a Fréchet space. Also, $\emptyset \neq \mathfrak{L} \subset C(\mathcal{R}_+)$ is named bounded when $\sup\{|q|_{\mathcal{U}} : q \in \mathfrak{L}\} < \infty$ for $\mathcal{U} \geq 0$. Let us now review three facts that are required.

(1) A sequence $(q_n)$ converges to $q \in C(\mathcal{R}_+)$ iff $(q_n)$ is uniformly convergent to $q$ on a compact subset of $\mathcal{R}_+$.
(2) A family $\mathfrak{L} \subset C(\mathcal{R}_+)$ is relatively compact iff, for any $\mathcal{U} > 0$, the restriction to $[0, \mathcal{U}]$ of each function from $\mathfrak{L}$ constructs an equicontinuous and uniformly bounded set.
(3) An operator $\mathcal{H} : (C(\mathcal{R}_+))^n \to C(\mathcal{R}_+)$ is continuous iff $\mathcal{H}|_{\mathcal{U}} : (C[0, \mathcal{U}])^n \to C[0, \mathcal{U}]$ is continuous for all $\mathcal{U} \geq 0$ in which $\mathcal{H}|_{\mathcal{U}}(q_1, \cdots, q_n)(b) = \mathcal{H}(q_1, \cdots, q_n)(b)$ for $b \in [0, \mathcal{U}]$.

Moreover, let us define an FMN in $C(\mathcal{R}_+)$ by

$$\mathcal{M}_{C(\mathcal{R}_+)} = \{\mathfrak{L} \subset C(\mathcal{R}_+) : \mathfrak{L}|_{\mathcal{U}} \text{ is bounded for all } \mathcal{U} \geq 0\},$$

where $\mathfrak{L}|_{\mathcal{U}}$ is the restriction of all functions from $\mathfrak{L}$ to $[0, \mathcal{U}]$. Obviously, all conditions of Definition 2.2 are fulfilled. Fix a positive number $\mathcal{U} > 0$ and a $\mathfrak{L} \neq \emptyset$ of $\mathcal{M}_{C(\mathcal{R}_+)}$, which is an admissible set of $C(\mathcal{R}_+)$. For $q \in \mathfrak{L}$ and $\xi > 0$, the modulus of continuity of $q$ on $[0, \mathcal{U}]$, denoted by $w^{\mathcal{U}}(q, \xi)$, is

$$w^{\mathcal{U}}(q, \xi) = \sup\{|q(b) - q(a)| : a, b \in [0, \mathcal{U}], |b - a| \leq \xi\}.$$

Additionally, take

$$w^{\mathcal{U}}(\mathfrak{L}, \xi) = \sup\{w^{\mathcal{U}}(q, \xi) : q \in \mathfrak{L}\} \quad \text{and} \quad w_0^{\mathcal{U}}(\mathfrak{L}) = \lim_{\xi \to 0} w^{\mathcal{U}}(\mathfrak{L}, \xi).$$

$\{w_0^{\mathcal{U}}\}_{\mathcal{U} \geq 0}$ forms an FMN in $C(\mathcal{R}_+)$ [12]. Now, we discuss the solvability of differential equation

$$\omega'(t) = \mathfrak{h}(t, \omega'(b(t)), \omega(B(t)), \mathcal{G}(\omega(t))), \tag{5}$$

where $t \in [0, \infty)$. Presume $\omega = \omega(t)$ is a solution of (5) satisfying the initial condition $\omega(0) = 0$. (5) will be investigated under the following assumptions:

($i$) The function $\mathfrak{h} : \mathcal{R}_+ \times \mathcal{R}^3 \to \mathcal{R}$ is continuous and there exist nondecreasing continuous functions $\varphi, \psi, \varrho : \mathcal{R} \to \mathcal{R}$ with $\psi(0) = 0$ so that

$$|\mathfrak{h}(t, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3) - \mathfrak{h}(t, \mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3)| \leq \varrho(|\mathfrak{a}_1 - \mathfrak{b}_1|) + \varphi(|\mathfrak{a}_2 - \mathfrak{b}_2|) + \psi(|\mathfrak{a}_3 - \mathfrak{b}_3|)$$

for all $t \in \mathcal{R}_+$ and $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3 \in \mathcal{R}$. Also, take $M = \sup\{|\mathfrak{h}(t, 0, 0, 0)| : t \in \mathcal{R}_+\}$.

($ii$) The functions $\mathcal{G} : \mathcal{R} \to \mathcal{R}$ and $b, B : \mathcal{R}_+ \to \mathcal{R}_+$ are continuous with $B(0) = 0$ and $b(0) = 0$.

($iii$) There is a positive sequence solution of $\varphi(|B|_{\mathcal{U}} r_{\mathcal{U}}) + \varrho(r_{\mathcal{U}}) + \psi(|\mathcal{G}(\mathcal{U} r_{\mathcal{U}})|) + M \leq r_{\mathcal{U}}$ for $\mathcal{U} \geq 0$.

Now, put $\upsilon(t) = \omega'(t)$. Then (5) can be substituted equivalently by the following integral equation

$$\upsilon(t) = \mathfrak{h}(t, \upsilon(b(t)), \int_0^{B(t)} \upsilon(s)ds, \mathcal{G}(\int_0^t \upsilon(s)ds)), \qquad t \in \mathcal{R}_+. \tag{6}$$

**Theorem 4.1.** *Under assumptions* ($i$)-($iii$), (6) *possesses a solution in* $C(\mathcal{R}_+)$.

*Proof.* Consider the operator $\mathcal{H}$ on $C(\mathcal{R}_+)$ by

$$(\mathcal{H}\upsilon)(t) = \mathfrak{h}(t, \upsilon(b(t)), \int_0^{B(t)} \upsilon(s)ds, \mathcal{G}(\int_0^t \upsilon(s)ds)), \qquad t \geq 0.$$

Obviously, $\mathcal{H}\upsilon$ is continuous on $\mathcal{R}_+$. In the following steps, we check the validity of hypotheses of Theorem 3.2.

Step 1. We first demonstrate $\mathcal{H}|_{\mathcal{U}} : C[0, \mathcal{U}] \to C[0, \mathcal{U}]$ is continuous for $\mathcal{U} \geq 0$. For this, fix $\mathcal{U} \geq 0$ and take $\varkappa > 0$ and $u \in C(\mathcal{R}_+)$ so that

$$\begin{aligned}
&|(\mathcal{H}u)(t) - (\mathcal{H}\upsilon)(t)| \\
&= |\mathfrak{h}(t, u(b(t)), \int_0^{B(t)} u(s)ds, \mathcal{G}(\int_0^t u(s)ds)) - \mathfrak{h}(t, \upsilon(b(t)), \int_0^{B(t)} \upsilon(s)ds, \mathcal{G}(\int_0^t \upsilon(s)ds))| \\
&\leq \varrho(|u(b(t)) - \upsilon(b(t))|) + \varphi(\int_0^{B(t)} |u(s) - \upsilon(s)|ds) + \psi(|\mathcal{G}(\int_0^t u(s)ds) - \mathcal{G}(\int_0^t \upsilon(s)ds)|) \\
&\leq \varrho(\varkappa) + \varphi(\varkappa B_{\mathcal{U}}) + \psi(|\mathcal{G}(\int_0^t u(s)ds) - \mathcal{G}(\int_0^t \upsilon(s)ds)|) \\
&\leq \varrho(\varkappa) + \varphi(\varkappa B_{\mathcal{U}}) + \psi(\nu^{\mathcal{U}}(\mathcal{G}, \varkappa))
\end{aligned}$$

for all $\upsilon \in C(\mathcal{R}_+)$ with $|u - \upsilon|_{\mathcal{U}} \leq \varkappa$ in which

$$\begin{aligned}
\nu^{\mathcal{U}}(\mathcal{G}, \varkappa) &= \sup\{|\mathcal{G}(\int_0^t u(s)ds) - \mathcal{G}(\int_0^t \upsilon(s)ds)| : |u - \upsilon|_{\mathcal{U}} < \varkappa\}, \\
B_{\mathcal{U}} &= \sup\{|B(t)| : \ t \in [0, \mathcal{U}]\}.
\end{aligned}$$

As $\mathcal{G}, \psi$ and integral are continuous, we deduce $\psi(\nu^{\mathcal{U}}(\mathcal{G}, \varkappa)) \to 0$ as $\varkappa \to 0$.

Step 2. Presume $t \in [0, \mathcal{U}]$ and $u \in C(\mathcal{R}_+)$.

$$\begin{aligned}
|\mathcal{H}u(t)| &= |\mathfrak{h}(t, u(b(t)), \int_0^{B(t)} u(s)ds, \mathcal{G}(\int_0^t u(s)ds))| \\
&\leq |\mathfrak{h}(t, u(b(t)), \int_0^{B(t)} u(s)ds, \mathcal{G}(\int_0^t u(s)ds)) - \mathfrak{h}(t, 0, 0, 0)| + |\mathfrak{h}(t, 0, 0, 0)| \\
&\leq \varrho(|u(b(t))|) + \varphi(\int_0^{B(t)} |u(s)|ds) + \psi(|\mathcal{G}(\int_0^t u(s)ds)|) + |\mathfrak{h}(t, 0, 0, 0)| \\
&\leq \varrho(|u|_{\mathcal{U}}) + \varphi(|B|_{\mathcal{U}}|u|_{\mathcal{U}}) + \psi(|\mathcal{G}(\mathcal{U}|u|_{\mathcal{U}})|) + M.
\end{aligned}$$

Thus,

$$|\mathcal{H}u|_{\mathcal{U}} \leq \varrho(|u|_{\mathcal{U}}) + \varphi(|B|_{\mathcal{U}}|u|_{\mathcal{U}}) + \psi(|\mathcal{G}(\mathcal{U}|u|_{\mathcal{U}})|) + M. \tag{7}$$

Define $C = \{u \in C(\mathcal{R}_+)||u|_{\mathcal{U}} < r_{\mathcal{U}},\ \mathcal{U} \geq 0\}$. By (7), $\mathcal{H}$ transforms $C$ into $C$.

Step 3. Fix $\mathcal{U} > 0$ and let $\mathfrak{s}_1, \mathfrak{s}_2 \in [0, \mathcal{U}]$, $|\mathfrak{s}_1 - \mathfrak{s}_2| \leq \varkappa$ and $u \in C$. Then

$$\begin{aligned}
&|\mathcal{H}u(\mathfrak{s}_1) - \mathcal{H}u(\mathfrak{s}_2)| \\
&= |\mathfrak{h}(\mathfrak{s}_1, u(b(\mathfrak{s}_1)), \int_0^{B(\mathfrak{s}_1)} u(s)ds, \mathcal{G}(\int_0^{\mathfrak{s}_1} u(s)ds)) - \mathfrak{h}(\mathfrak{s}_2, u(b(\mathfrak{s}_2)), \int_0^{B(\mathfrak{s}_2)} u(s)ds, \mathcal{G}(\int_0^{\mathfrak{s}_2} u(s)ds))| \\
&\leq |\mathfrak{h}(\mathfrak{s}_1, u(b(\mathfrak{s}_1)), \int_0^{B(\mathfrak{s}_1)} u(s)ds, \mathcal{G}(\int_0^{\mathfrak{s}_1} u(s)ds)) - \mathfrak{h}(\mathfrak{s}_1, u(b(\mathfrak{s}_2)), \int_0^{B(\mathfrak{s}_2)} u(s)ds, \mathcal{G}(\int_0^{\mathfrak{s}_2} u(s)ds))| \\
&\quad + |\mathfrak{h}(\mathfrak{s}_1, u(b(\mathfrak{s}_2)), \int_0^{B(\mathfrak{s}_2)} u(s)ds, \mathcal{G}(\int_0^{\mathfrak{s}_2} u(s)ds)) - \mathfrak{h}(\mathfrak{s}_2, u(b(\mathfrak{s}_2)), \int_0^{B(\mathfrak{s}_2)} u(s)ds, \mathcal{G}(\int_0^{\mathfrak{s}_2} u(s)ds))| \\
&\leq \varrho(|u(b(\mathfrak{s}_1)) - u(b(\mathfrak{s}_2))|) + \varphi(|\int_{B(\mathfrak{s}_1)}^{B(\mathfrak{s}_2)} u(s)ds|) + \psi(|\mathcal{G}(\int_0^{\mathfrak{s}_1} u(s)ds) - \mathcal{G}(\int_0^{\mathfrak{s}_2} u(s)ds)|) \\
&\quad + |\mathfrak{h}(\mathfrak{s}_1, u(b(\mathfrak{s}_2)), \int_0^{B(\mathfrak{s}_2)} u(s)ds, \mathcal{G}(\int_0^{\mathfrak{s}_2} u(s)ds)) - \mathfrak{h}(\mathfrak{s}_2, u(b(\mathfrak{s}_2)), \int_0^{B(\mathfrak{s}_2)} u(s)ds, \mathcal{G}(\int_0^{\mathfrak{s}_2} u(s)ds))| \\
&\leq \varrho(|u(b(\mathfrak{s}_1)) - u(b(\mathfrak{s}_2))|) + \varphi(|\int_{B(\mathfrak{s}_1)}^{B(\mathfrak{s}_2)} u(s)ds|) + \psi(|\mathcal{G}(\int_0^{\mathfrak{s}_1} u(s)ds) - \mathcal{G}(\int_0^{\mathfrak{s}_2} u(s)ds)|) + \nu^{\mathcal{U}}(\mathfrak{h}, \varkappa) \\
&\leq \varrho(|u(b(\mathfrak{s}_1)) - u(b(\mathfrak{s}_2))|) + \varphi(\nu^{\mathcal{U}}(B, \varkappa)|u|_{\mathcal{U}}) + \psi(\nu^{\mathcal{U}}(\mathcal{G}, \varkappa)) + \nu^{\mathcal{U}}(\mathfrak{h}, \varkappa),
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{r}_{\mathcal{U}} &= \{\sup\{|u|_{\mathcal{U}} \in \mathfrak{L}\}, \\
\nu^{\mathcal{U}}(\mathfrak{h}, \varkappa) &= \sup\{|\mathfrak{h}(\mathfrak{s}_1, x, y, z) - \mathfrak{h}(\mathfrak{s}_2, x, y, z)| : |x| \leq \mathfrak{r}_{\mathcal{U}}, |y| \leq X_{\mathcal{U}}, |z| \leq |\mathcal{G}|_{\mathcal{U}}, \\
&\qquad |\mathfrak{s}_1 - \mathfrak{s}_2| < \varkappa, \mathfrak{s}_1, \mathfrak{s}_2 \in [0, \mathcal{U}]\}, \\
\nu^{\mathcal{U}}(b, \varkappa) &= \sup\{|b(\mathfrak{s}_1) - b(\mathfrak{s}_2)| : |\mathfrak{s}_1 - \mathfrak{s}_2| < \varkappa, \mathfrak{s}_1, \mathfrak{s}_2 \in [0, \mathcal{U}]\}, \\
\nu^{\mathcal{U}}(B, \varkappa) &= \sup\{|B(\mathfrak{s}_1) - B(\mathfrak{s}_2)| : |\mathfrak{s}_1 - \mathfrak{s}_2| < \varkappa, \mathfrak{s}_1, \mathfrak{s}_2 \in [0, \mathcal{U}]\}, \\
X_{\mathcal{U}} &= \sup\{|\int_0^{B(t)} u(s)ds| : u \in \mathfrak{L}, t \in [0, \mathcal{U}]\}.
\end{aligned}$$

Thus,

$$w^{\mathcal{U}}(\mathcal{H}(\mathfrak{L}, \varkappa)) \leq \varrho(w^{\mathcal{U}}(u, \nu^{\mathcal{U}}(b, \varkappa)) + \varphi(\nu^{\mathcal{U}}(B, \varkappa)|u|_{\mathcal{U}}|) + \psi(\nu^{\mathcal{U}}(\mathcal{G}, \varkappa)) + \nu^{\mathcal{U}}(\mathfrak{h}, \varkappa).$$

Since $\mathfrak{h}$ is uniform continuous on $[0, \mathcal{U}] \times [-r_{\mathcal{U}}, r_{\mathcal{U}}] \times [-X_{\mathcal{U}}, +X_{\mathcal{U}}] \times [-|\mathcal{G}|_{\mathcal{U}}, +|\mathcal{G}|_{\mathcal{U}}]$, we have $\nu^{\mathcal{U}}(\mathfrak{h}, \varkappa) \to 0$ as $\varkappa \to 0$. Also, By the continuity of $B$ and $\mathcal{G}$, $\nu^{\mathcal{U}}(B, \varkappa), \nu^{\mathcal{U}}(\mathcal{G}, \varkappa) \to 0$ as $\varkappa \to 0$. Moreover, as $\varphi$ and $\psi$ are continuous, $\varphi(\nu^{\mathcal{U}}(B, \varkappa)), \psi(\nu^{\mathcal{U}}(\mathcal{G}, \varkappa)) \to 0$ as $\varkappa \to 0$. Hence,

$$w_0^{\mathcal{U}}(\mathcal{H}(\mathfrak{L})) \leq \varrho(w_0^{\mathcal{U}}(\mathfrak{L})).$$

Finally, using Theorem 3.2, our claim will be obtained. □

### 4.2. System of differential equations

As an application of Theorem 3.5, we show the existence of solutions for a system of differential equation. For this, consider the following system

$$\begin{cases} \omega'(t) = \mathfrak{h}_1(t, \omega'(b(t)), \omega(B(t)), \varpi'(b(t)), \varpi(B(t))), \\ \varpi'(t) = \mathfrak{h}_2(t, \varpi'(b(t)), \varpi(B(t)), \omega'(b(t)), \omega(B(t))), \end{cases} \tag{8}$$

under assumptions:

(*i*) $\mathfrak{h}_\iota : \mathcal{R}_+ \times \mathcal{R}^4 \to \mathcal{R}$ is continuous and there exist nondecreasing continuous functions $\varphi, \psi, \varrho : \mathcal{R} \to \mathcal{R}$ with $\psi(0) = 0$ so that

$$|\mathfrak{h}_\iota(t, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4) - \mathfrak{h}_\iota(t, \mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3, \mathfrak{b}_4)| \leq \varrho(\max\{|\mathfrak{a}_1 - \mathfrak{b}_1|, |\mathfrak{a}_3 - \mathfrak{b}_3|\}) + \varphi(|\mathfrak{a}_2 - \mathfrak{b}_2|) + \psi(|\mathfrak{a}_4 - \mathfrak{b}_4|)$$

for any $t \in \mathcal{R}_+$, $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3, \mathfrak{b}_4 \in \mathcal{R}$ and $\iota = 1, 2$. Also, take

$$M = \sup\{|\mathfrak{h}(t, 0, 0, 0, 0)| : t \in \mathcal{R}_+\}.$$

(*ii*) $b, B : \mathcal{R}_+ \to \mathcal{R}_+$ are continuous with $B(0) = 0$ and $b(0) = 0$.

(*iii*) There is a positive sequence solution $r_{\mathcal{U}}$ of $\varrho(r_{\mathcal{U}}) + \varphi(|B|_{\mathcal{U}} r_{\mathcal{U}}) + \psi(||B|_{\mathcal{U}} r_{\mathcal{U}}|) + M \leq r_{\mathcal{U}}$ for $\mathcal{U} \geq 0$.

Now, let us put $\upsilon(t) = \omega'(t)$ and $\nu(t) = \varpi'(t)$. Then (8) can be substituted equivalently by the following system of integral equations:

$$\begin{cases} \upsilon(t) = \mathfrak{h}_1(t, \upsilon(b(t)), \int_0^{B(t)} \upsilon(s)ds, \nu(b(t)), \int_0^{B(t)} \nu(s)ds), & t \in \mathcal{R}_+, \\ \nu(t) = \mathfrak{h}_2(t, \nu(b(t)), \int_0^{B(t)} \nu(s)ds, \upsilon(b(t)), \int_0^{B(t)} \upsilon(s)ds), & t \in \mathcal{R}_+. \end{cases} \tag{9}$$

**Theorem 4.2.** *Under assumptions* (*i*)-(*iii*), (8) *possesses a solution in* $C(\mathcal{R}_+) \times C(\mathcal{R}_+)$.

*Proof.* Take the continuous operator $\mathcal{H}_\iota$ on $C(\mathcal{R}_+) \times C(\mathcal{R}_+)$ by

$$(\mathcal{H}_\iota(\upsilon, \nu))(t) = \mathfrak{h}_\iota(t, \upsilon(b(t)), \int_0^{B(t)} \upsilon(s)ds, \nu(b(t)), \int_0^{B(t)} \nu(s)ds), \qquad t \geq 0,$$

for $\iota = 1, 2$. In the following steps, we check the validity of hypotheses of Theorem 3.5.

Step 1. We first demonstrate $\mathcal{H}_\iota|_{\mathcal{U}} : C[0, \mathcal{U}] \times C[0, \mathcal{U}] \to C[0, \mathcal{U}]$ is continuous for $\mathcal{U} \geq 0$. For this, fix $\mathcal{U} \geq 0$ and take $u, v \in C(\mathcal{R}_+)$, $\varkappa > 0$, and $m, n \in C(\mathcal{R}_+)$ with $|u - m|_{\mathcal{U}} \leq \varkappa$ and $|v - n|_{\mathcal{U}} \leq \varkappa$. Then we get

$$\begin{aligned} &|(\mathcal{H}_\iota(u, v))(t) - (\mathcal{H}_\iota(m, n))(t)| \\ &= |\mathfrak{h}_\iota(t, u(b(t)), \int_0^{B(t)} u(s)ds, v(b(t)), \int_0^{B(t)} v(s)ds) - \mathfrak{h}_\iota(t, m(b(t)), \int_0^{B(t)} m(s)ds, n(b(t)), \int_0^{B(t)} n(s)ds)| \\ &\leq \varrho(\max\{|u(b(t)) - m(b(t))|, |v(b(t)) - n(b(t))|\}) + \varphi(\int_0^{B(t)} |u(s) - m(s)|ds) + \psi(\int_0^{B(t)} |v(s) - n(s)|) \\ &\leq \varrho(\varkappa) + \varphi(\varkappa B_{\mathcal{U}}) + \psi(\varkappa B_{\mathcal{U}}), \end{aligned}$$

where $B_{\mathcal{U}} = \sup\{|B(t)| : t \in [0, \mathcal{U}]\}$. As $\varrho, \varphi, \psi$ are continuous, we deduce $\mathcal{H}_\iota$ is a continuous function from $C(\mathcal{R}_+) \times C(\mathcal{R}_+)$ into $C(\mathcal{R}_+)$.

Step 2. Let $t \in [0, \mathcal{U}]$ and $(u, v) \in C(\mathcal{R}_+) \times C(\mathcal{R}_+)$. Then

$$\begin{aligned} |\mathcal{H}_\iota(u, v)(t)| &= |\mathfrak{h}_\iota(t, u(b(t)), \int_0^{B(t)} u(s)ds, v(b(t)), \int_0^{B(t)} v(s)ds)| \\ &\leq \varrho(\max\{|u(b(t))|, |v(b(t))|\}) + \varphi(\int_0^{B(t)} |u(s)|ds) + \psi(\int_0^{B(t)} |v(s)|ds) + |\mathfrak{h}_\iota(t, 0, 0, 0, 0)| \\ &\leq \varrho(|u|_{\mathcal{U}}) + \varphi(|B|_{\mathcal{U}} |u|_{\mathcal{U}}) + \psi(|B|_{\mathcal{U}} |v|_{\mathcal{U}}) + M. \end{aligned}$$

Thus,

$$|\mathcal{H}_\iota(u, v)|_{\mathcal{U}} \leq \varrho(|u|_{\mathcal{U}}) + \varphi(|B|_{\mathcal{U}} |u|_{\mathcal{U}}) + \psi(|B|_{\mathcal{U}} |v|_{\mathcal{U}}) + M. \tag{10}$$

Define $C = \{(u, v) \in C(\mathcal{R}_+) \times C(\mathcal{R}_+) |\ |u|_{\mathcal{U}} < r_{\mathcal{U}}, |v|_{\mathcal{U}} < r_{\mathcal{U}}, \mathcal{U} \geq 0\}$. It follows from (10) that $\mathcal{H}_\iota$ transforms $C \times C$ into $C$.

Step 3. Fix a positive number $\mathcal{U} > 0$ and let $\mathfrak{s}_1, \mathfrak{s}_2 \in [0, \mathcal{U}]$, $|\mathfrak{s}_1 - \mathfrak{s}_2| \le \varkappa$ and $(u, v) \in C \times C$.

$$\begin{aligned}
|\mathcal{H}_\iota(u,v)(\mathfrak{s}_1) - \mathcal{H}_\iota(u,v)(\mathfrak{s}_2)| &= |\mathfrak{h}_\iota(\mathfrak{s}_1, u(b(\mathfrak{s}_1)), \int_0^{B(\mathfrak{s}_1)} u(s)ds, v(b(\mathfrak{s}_1)), \int_0^{B(\mathfrak{s}_1)} v(s)ds) \\
&\quad - \mathfrak{h}_\iota(\mathfrak{s}_2, u(b(\mathfrak{s}_2)), \int_0^{B(\mathfrak{s}_2)} u(s)ds, v(b(\mathfrak{s}_2)), \int_0^{B(\mathfrak{s}_2)} v(s)ds)| \\
&\le |\mathfrak{h}_\iota(\mathfrak{s}_1, u(b(\mathfrak{s}_1)), \int_0^{B(\mathfrak{s}_1)} u(s)ds, v(b(\mathfrak{s}_1)), \int_0^{B(\mathfrak{s}_1)} v(s)ds) \\
&\quad - \mathfrak{h}_\iota(\mathfrak{s}_1, u(b(\mathfrak{s}_2)), \int_0^{B(\mathfrak{s}_2)} u(s)ds, v(b(\mathfrak{s}_2)), \int_0^{B(\mathfrak{s}_2)} v(s)ds)| \\
&\quad + |\mathfrak{h}_\iota(\mathfrak{s}_1, u(b(\mathfrak{s}_2)), \int_0^{B(\mathfrak{s}_2)} u(s)ds, v(b(\mathfrak{s}_2)), \int_0^{B(\mathfrak{s}_2)} v(s)ds) \\
&\quad - \mathfrak{h}_\iota(\mathfrak{s}_2, u(b(\mathfrak{s}_2)), \int_0^{B(\mathfrak{s}_2)} u(s)ds, v(b(\mathfrak{s}_2)), \int_0^{B(\mathfrak{s}_2)} v(s)ds)| \\
&\le \varrho(\max\{|u(b(\mathfrak{s}_1)) - u(b(\mathfrak{s}_2))|, |v(b(\mathfrak{s}_1)) - v(b(\mathfrak{s}_2))|\}) \\
&\quad + \varphi(|\int_{B(\mathfrak{s}_1)}^{B(\mathfrak{s}_2)} u(s)ds|) + \psi(|\int_{B(\mathfrak{s}_1)}^{B(\mathfrak{s}_2)} v(s)ds|) + \nu^{\mathcal{U}}(\mathfrak{h}_\iota, \varkappa) \\
&\le \varrho(\max\{|u(b(\mathfrak{s}_1)) - u(b(\mathfrak{s}_2))|, |v(b(\mathfrak{s}_1)) - v(b(\mathfrak{s}_2))|\}) \\
&\quad + \varphi(\nu^{\mathcal{U}}(B, \varkappa)|u|_{\mathcal{U}}) + \psi(\nu^{\mathcal{U}}(B, \varkappa)|v|_{\mathcal{U}}) + \nu^{\mathcal{U}}(\mathfrak{h}_\iota, \varkappa),
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{r}_{\mathcal{U}} &= \max\{\sup\{|u|_{\mathcal{U}} \in \mathfrak{L}_1\}, \sup\{|v|_{\mathcal{U}} \in \mathfrak{L}_2\}\}, \\
\nu^{\mathcal{U}}(\mathfrak{h}_\iota, \varkappa) &= \sup\{|\mathfrak{h}_\iota(\mathfrak{s}_1, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4) - \mathfrak{h}_\iota(\mathfrak{s}_2, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4)| : \\
&\qquad |\mathfrak{a}_2|, |\mathfrak{a}_4| \le \mathfrak{s}_{\mathcal{U}}, |\mathfrak{a}_1|, |\mathfrak{a}_3| \le \mathfrak{r}_{\mathcal{U}}, |\mathfrak{s}_1 - \mathfrak{s}_2| < \varkappa, \mathfrak{s}_1, \mathfrak{s}_2 \in [0, \mathcal{U}]\}, \\
\nu^{\mathcal{U}}(b, \varkappa) &= \sup\{|b(\mathfrak{s}_1) - b(\mathfrak{s}_2)| : |\mathfrak{s}_1 - \mathfrak{s}_2| < \varkappa, \mathfrak{s}_1, \mathfrak{s}_2 \in [0, \mathcal{U}]\}, \\
\nu^{\mathcal{U}}(B, \varkappa) &= \sup\{|B(\mathfrak{s}_1) - B(\mathfrak{s}_2)| : |\mathfrak{s}_1 - \mathfrak{s}_2| < \varkappa, \mathfrak{s}_1, \mathfrak{s}_2 \in [0, \mathcal{U}]\}, \\
\mathfrak{s}_{\mathcal{U}} &= \max\{\sup\{|\int_0^{B(t)} u(s)ds| : u \in \mathfrak{L}_1, t \in [0, \mathcal{U}]\}, \\
&\qquad \sup\{|\int_0^{B(t)} v(s)ds| : v \in \mathfrak{L}_2, t \in [0, \mathcal{U}]\}\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
w^{\mathcal{U}}(\mathcal{H}_\iota(\mathfrak{L}_1 \times \mathfrak{L}_2, \varkappa)) &\le \varrho(\max\{w^{\mathcal{U}}(u, \nu^{\mathcal{U}}(b, \varkappa)), w^{\mathcal{U}}(v, \nu^{\mathcal{U}}(b, \varkappa))\}) + \varphi(\nu^{\mathcal{U}}(B, \varkappa)|u|_{\mathcal{U}}|) \\
&\quad + \psi(\nu^{\mathcal{U}}(B, \varkappa)|v|_{\mathcal{U}}) + \nu^{\mathcal{U}}(\mathfrak{h}_\iota, \varkappa).
\end{aligned}$$

Since $\mathfrak{h}_\iota$ is uniform continuous on $[0, \mathcal{U}] \times [-\mathfrak{r}_{\mathcal{U}}, \mathfrak{r}_{\mathcal{U}}] \times [-\mathfrak{s}_{\mathcal{U}}, \mathfrak{s}_{\mathcal{U}}] \times [-\mathfrak{r}_{\mathcal{U}}, +\mathfrak{r}_{\mathcal{U}}] \times [-\mathfrak{s}_{\mathcal{U}}, \mathfrak{s}_{\mathcal{U}}] \times$, we have $\nu^{\mathcal{U}}(\mathfrak{h}_\iota, \varkappa) \to 0$ as $\varkappa \to 0$. Also, by continuity of $B$, $\nu^{\mathcal{U}}(B, \varkappa) \to 0$ as $\varkappa \to 0$. Further, as $\varphi$ and $\psi$ are continuous, $\varphi(\nu^{\mathcal{U}}(B, \varkappa)), \psi(\nu^{\mathcal{U}}(B, \varkappa)) \to 0$ as $\varkappa \to 0$. Thus,

$$w_0^{\mathcal{U}}(\mathcal{H}_\iota(\mathfrak{L}_1 \times \mathfrak{L}_2)) \le \varrho(\max\{w_0^{\mathcal{U}}(\mathfrak{L}_1), w_0^{\mathcal{U}}(\mathfrak{L}_2)\}).$$

Finally, using Theorem 3.5, our claim will be obtained. □

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