FIXED POINT RESULTS ON A CLOSED BALL IN A COMPLETE CONTINUOUS CONTROLLED METRIC SPACE WITH APPLICATIONS

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In this study, complete continuous controlled metric spaces have been discussed. Existence of the fixed points for the mappings satisfying contractive conditions on closed balls in such spaces have been ensured. Additionally, a concrete example is provided to validate our results. Finally, the practical implications of the findings have been explored by applying results to solve Fredholm integral equation of second kind.

Keywords: fixed point, complete continuous controlled metric space, closed ball, Fredholm integral equation

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1. Introduction and Preliminaries

Fixed point theory is one of the significant subfields within functional analysis. Fixed point theory has vast applications in mathematical modeling in economics, computer science, decision theory, psychology, ecology and game theory, see [8, 9, 12]. For instance, in game theory, fixed point theorems such as those by Bohnenblust and Karlin [9] are fundamental to proving the existence of Nash equilibria, while in economics, they are essential for establishing general equilibrium models as discussed by Aubin [8]. This theory help to determine stable population distributions and equilibrium states in ecology [5, 10, 22].

The idea of a b-metric space [11] is the extension of standard metric space which is obtained by relaxing the triangular inequality. Hit et al. [13] studied generalizations of b-metric spaces together with related fixed point theorems and Karapinar et al. [15] studied Meir–Keeler type contraction mappings in generalized b-metric spaces. In addition, Navascués and Mohapatra [21] investigated fixed point dynamics in a new type of contraction in b-metric spaces. Souayah and Mlaiki [31] contributed to Sb-metric spaces. Van An et al. [33] established important topological properties including a Stone-type theorem on b-metric spaces.

The idea of a controlled metric space is introduced by Mlaiki et al. [19] which is the extension of a b-metric space. In metric spaces, the distance function is continuous with respect to each variable, however controlled metric spaces do not guarantee such continuity, in general. In a controlled metric space, closed balls are not necessarily closed sets unless a continuity condition on the controlled metric is satisfied. Alamgir et al. [4] studied controlled rectangular metric spaces, relaxing the rectangular inequality while maintaining controlled structures, and provided applications to integral equations. Ahmad [3] and Al-Mazrooei [18]

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analyzed fixed points in a controlled framework with applications to real-world problems. Azmi [28] contributed new fixed point results in double controlled metric type spaces, further expanding the theoretical landscape. These collective contributions [2, 4, 3, 28, 18] highlight the active development and broad applicability of controlled and double controlled metric spaces in modern fixed point theory.

Many researchers study fixed point results in which conditions hold for the elements within a closed ball and applied their results to find the fixed point of mappings for which the traditional approaches failed, see [7, 14, 17, 20, 25, 27, 28, 30]. This localized approach is particularly valuable when global contractive conditions are too restrictive or when the mapping is only well-behaved on a bounded region of interest. Arshad et al. [7] initiated work on dominated mappings on closed balls in ordered dislocated metric spaces, while Hussain et al. [14] combined closed ball methods with graphic contractions. Mudhesh et al. [20] developed new contractive conditions specifically on closed balls and [25, 27, 28, 30] contributed foundational results on dominated mappings, multivalued mappings, and graph structures within closed ball frameworks.

In this paper, fixed point results on closed ball in a complete continuous controlled metric space is studied. The concept of this manuscript is motivated by Van et al. [33]. Our results generalized the results in [19] and [33]. The following concepts will be used in our main results.

Definition 1.1. Let Z be a non-empty set and $\alpha : Z\times Z\to [1,\infty)$. A controlled metric is a function $q : Z \times Z \to [0,\infty)$ such that for all $l, m, n \in Z$:

1. $q(l,m) = 0 \Leftrightarrow l = m$,
2. $q(l,m) = q(m,l)$,
3. $q(l,m) \leq \alpha(l,n)q(l,n) + \alpha(n,m)q(n,m)$.

The pair (Z,q) is called controlled metric type space.

Example 1.1. Let $Z = \{0,1,2\}$ and define a function $q : Z \times Z \to \mathbb{R}$ by

$$q(0,0) = q(1,1) = q(2,2) = 0,$$

$$q(0,1) = q(1,0) = 1.5,\ q(0,2) = q(2,0) = 0.4,\ q(1,2) = q(2,1) = 0.5.$$

Clearly, q is not a metric because the triangle inequality is not satisfied. Indeed, $q(0,1) = 1.5$, while

$$q(0,2) + q(2,1) = 0.4 + 0.5 = 0.9.$$

Hence, $q(0,1) > q(0,2) + q(2,1)$, so q is not a metric on Z. Now define the control function $\alpha : Z \times Z \to [1,\infty)$ by

$$\alpha(0,0) = \alpha(1,1) = \alpha(2,2) = 1,$$

$$\alpha(0,1) = \alpha(1,0) = 2,\ \alpha(0,2) = \alpha(2,0) = 2,\ \alpha(1,2) = \alpha(2,1) = 1.$$

Then for all $l, m, n \in Z$, we have $q(l,m) \leq \alpha(l,n)q(l,n) + \alpha(n,m)q(n,m)$. Hence, q is not a metric, but it is a controlled metric.

Remark 1.1. If for all $l, m \in Z$, there exist a parameter $g \geq 1$ such that $\alpha(l,m) = g$ in Definition 1, then (Z,q) is a b-metric space. Consequently, every b-metric space is a controlled metric type space.

Definition 1.2. Let (Z, q) be a controlled metric space. The function $q : Z \times Z \to [0, \infty)$ is called continuous if it is continuous with respect to the convergence induced by q, that is, whenever $\lim_{n\to\infty}(a_n, b_n) = (a, b)$ in (Z, q), then

$$\lim_{n\to\infty} q(a_n, b_n) \to q(a, b).$$

Definition 1.3. Let (Z, q) be a continuous controlled metric space. A sequence $\{a_n\}$ in (Z, q) is considered to be

1. convergent to $a \in Z$, if $\lim_{p\to\infty} q(a_p, a) = 0$, or any $\epsilon > 0$, $\exists\ p_\circ \in \mathbb{N}$, such $\forall\ p > p_\circ$, $q(a_p, a) < \epsilon$.
2. Cauchy sequence if for $k, p \in \mathbb{N}$ with $k > p$, $\lim_{p,k\to\infty} q(a_k, a_p) = 0$ or for any $\epsilon > 0$, there exists $k_\circ \in \mathbb{N}$, for all $p, k \in \mathbb{N}$ with $k > p \geq k_\circ$, $q(a_k, a_p) \leq \epsilon$.
3. if all Cauchy sequences converge to $a \in Z$ then (Z, q) is known as complete continuous controlled metric space.

Definition 1.4. Let (Z, q) be a continuous controlled metric space. Then, the closed ball centered at $a_\circ$ with radius r is defined as $\overline{B_q(a_\circ, r)} = \{a \in Z : q(a_\circ, a) \leq r\}$.

Proposition 1.1. Let (Z, d) be a complete continuous controlled metric space. Then the closed ball $\overline{B_q(a_\circ, r)}$ is a closed set in Z.

2. **Main Result**

Now, we proceed to establish our main result, which extends both the principle of Banach contraction and the Kannan fixed point theorem within the setting of a complete continuous controlled metric space.

Theorem 2.1. *Let (Z, q) be a complete continuous controlled metric space. Let $\mathcal{S} : \overline{B_q(a_\circ, r)} \to Z$ be a mapping such that*

$$q(\mathcal{S}a, \mathcal{S}b) \leq \kappa q(a, b) \tag{1}$$

and

$$q(a_\circ, \mathcal{S}a_\circ) \leq \frac{(1 - \xi\kappa)}{\xi} r \tag{2}$$

for all $a, b \in \overline{B_q(a_\circ, r)}$, where $\kappa \in (0, 1)$. Moreover, take $a_n = \mathcal{S}a_{n-1}$ for $n \in \mathbb{N}$. Suppose that the controlled function satisfies

$$\alpha(a_i, a_j) \leq \xi \text{ for all } i, j \in \mathbb{N} \text{ where } \xi \in [1, \tfrac{1}{\kappa}) \tag{3}$$

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\alpha(a_{i+1}, a_{i+2})\,\alpha(a_{i+1}, a_m)}{\alpha(a_i, a_{i+1})} < \frac{1}{\kappa}. \tag{4}$$

Then, there exist a point $a^* \in \overline{B_q(a_\circ, r)}$ such that a_n converges to a^*. In addition,

$$\lim_{n\to\infty} \alpha(a_n, a^*) \text{ and } \lim_{n\to\infty} \alpha(a^*, a_n) \text{ exist and are finite.} \tag{5}$$

Then, a^* is a unique common fixed point of $\mathcal{S}$ in $\overline{B_q(a_\circ, r)}$.

Proof. Consider the sequence $a_n = \mathcal{S}^n a_\circ$ such that $a_1 \in \mathcal{S}a_\circ$ for $a_\circ \in \overline{B_q(a_\circ, r)}$ and using (1), we have

$$q(a_n, a_{n+1}) = q(\mathcal{S}a_{n-1}, \mathcal{S}a_n) \leq \kappa q(a_{n-1}, a_n) \cdots \leq \kappa^n q(a_\circ, a_1).$$

As given in (2) $q(a_\circ, \mathcal{S}a_\circ) \leq \frac{(1-\xi\kappa)}{\xi} r$ and since $\xi \geq 1$ and $\kappa < 1$, we get $q(a_\circ, a_1) \leq r$. This implies that $a_1 \in \overline{B_q(a_\circ, r)}$. By using the triangle property, (1) and (2)

$$\begin{aligned} q(a_\circ, a_2) &\leq \alpha(a_\circ, a_1)q(a_\circ, a_1) + \alpha(a_1, a_2)q(a_1, a_2) \\ &\leq \alpha(a_\circ, a_1)q(a_\circ, a_1) + \alpha(a_1, a_2)q(\mathcal{S}a_\circ, \mathcal{S}a_1) \\ &\leq \xi q(a_\circ, a_1) + \xi\kappa q(a_\circ, a_1) \\ &\leq (\xi + \xi\kappa)q(a_\circ, a_1) \\ &\leq \xi(1+\kappa)\frac{(1-\xi\kappa)}{\xi} r \\ &\leq (1+\kappa)(1-\xi\kappa)r. \end{aligned}$$

Since $\xi \geq 1$ and $\kappa \in (0,1)$, we have

$$\begin{aligned} \xi(1+\kappa) &> 1, \\ 1 - \xi(1+\kappa) &< 0, \\ (1-\xi) - \xi\kappa &< 0, \\ \kappa(1-\xi) - \xi\kappa^2 &< 0, \\ 1 + \kappa(1-\xi) - \xi\kappa^2 &< 1, \\ (1+\kappa)(1-\xi\kappa)r &< r. \end{aligned}$$

Thus, we have $q(a_\circ, a_2) \leq r$ and it follows that $a_2 \in \overline{B_q(a_\circ, r)}$. Now, we suppose that $a_3, a_4 \cdots a_n \in \overline{B_q(a_\circ, r)}$ for some $n \in \mathbb{N}$. Now

$$\begin{aligned} q(a_\circ, a_{n+1}) &\leq \alpha(a_\circ, a_1)q(a_\circ, a_1) + \alpha(a_1, a_{n+1})q(a_1, a_{n+1}) \\ &\leq \alpha(a_\circ, a_1)q(a_\circ, a_1) + \alpha(a_1, a_{n+1})\alpha(a_1, a_2)q(a_1, a_2) \\ &\quad + \alpha(a_1, a_{n+1})\alpha(a_2, a_{n+1})q(a_2, a_{n+1}) \\ &\leq \alpha(a_\circ, a_1)q(a_\circ, a_1) + \alpha(a_1, a_{n+1})\alpha(a_1, a_2)q(a_1, a_2) \\ &\quad + \alpha(a_1, a_{n+1})\alpha(a_2, a_{n+1})\alpha(a_2, a_3)q(a_2, a_3) \\ &\quad + \cdots + \prod_{j=1}^{n} \alpha(a_j, a_{n+1})q(a_n, a_{n+1}) \\ &\leq \alpha(a_\circ, a_1)q(a_\circ, a_1) + \sum_{i=1}^{n-1} \Big(\prod_{j=1}^{i} \alpha(a_j, a_{n+1})\Big)\alpha(a_i, a_{i+1})q(a_i, a_{i+1}) \\ &\quad + \prod_{j=1}^{n} \alpha(a_j, a_{n+1})q(a_n, a_{n+1}) \\ &\leq \xi q(a_\circ, a_1) + \sum_{i=1}^{n-1} \xi^{i+1}\kappa^i q(a_\circ, a_1) + (\xi\kappa)^n q(a_\circ, a_1) \\ &\leq \xi q(a_\circ, a_1) + \xi \sum_{i=1}^{n-1} (\xi\kappa)^i q(a_\circ, a_1) + (\xi\kappa)^n q(a_\circ, a_1) \\ &\leq q(a_\circ, a_1)\Big(\xi + \xi \sum_{i=1}^{n-1} (\xi\kappa)^i\Big) + (\xi\kappa)^n q(a_\circ, a_1) \\ &\leq \xi q(a_\circ, a_1) \sum_{i=0}^{n-1} (\xi\kappa)^i + (\xi\kappa)^n q(a_\circ, a_1). \end{aligned}$$

The series $\sum_{i=0}^{n-1}(\xi\kappa)^i$ converges as $\xi\kappa < 1$ and $\sum_{i=0}^{n-1}(\xi\kappa)^i = \frac{1-(\xi\kappa)^n}{1-\xi\kappa}$. Thus, we have

$$\begin{aligned} q(a_\circ, a_{n+1}) &\leq \xi\frac{(1-\xi\kappa)}{\xi}r\frac{1-(\xi\kappa)^n}{1-\xi\kappa} + (\xi\kappa)^n\frac{(1-\xi\kappa)}{\xi}r \\ &\leq r(1-(\xi\kappa)^n) + (\xi\kappa)^n\frac{(1-\xi\kappa)}{\xi}r \\ &\leq r\Big(1-(\xi\kappa)^n + \frac{(\xi\kappa)^n}{\xi} - (\xi\kappa)^n\kappa\Big) \\ &\leq r\Big(1-(\xi\kappa)^n\big(1-\tfrac{1}{\xi}+\kappa\big)\Big). \end{aligned}$$

Since $\kappa \in (0,1)$ and $\xi \geq 1$, then $(1-\frac{1}{\xi}+\kappa) > 0$. Since $(\xi\kappa)^n < 1$, we have

$$\begin{aligned} -(\xi\kappa)^n\big(1-\tfrac{1}{\xi}+\kappa\big) &< 0, \\ 1-(\xi\kappa)^n\big(1-\tfrac{1}{\xi}+\kappa\big) &< 1, \\ r\Big(1-(\xi\kappa)^n\big(1-\tfrac{1}{\xi}+\kappa\big)\Big) &< r. \end{aligned}$$

This implies $q(a_\circ, a_{n+1}) \leq r$. Thus, $a_{n+1} \in \overline{B_q(a_\circ, r)}$ and hence $a_n \in \overline{B_q(a_\circ, r)}$, for all $n \in \mathbb{N}$. By using techniques analogous to those in [19], it can be established that the sequence a_n is Cauchy in the controlled metric space. Thus, we have $\lim_{n,m\to\infty} q(a_n, a_m) = 0$. Since every closed ball in a continuous controlled metric space is closed (Proposition 1) so $(\overline{B_q(a_\circ, r)}, q)$ is a complete continuous controlled metric space so every Cauchy sequence a_n converges to point in a $\overline{B_q(a_\circ, r)}$. Thus, there exist a^* such that $\lim_{n\to\infty} q(a_n, a^*) = 0$. Now, it can be easily proved that a^* is a unique fixed point of $\mathcal{S}$ as in [19]. □

Example 2.1. Let $Z = \mathbb{R}$. Define $q : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ by

$$q(x,y) = \begin{cases} |x-y|^2, & \text{if } x, y \in [-3,3], \\ (1+|x|+|y|)\,|x-y|, & \text{otherwise.} \end{cases}$$

Define the control function $\alpha : \mathbb{R}\times\mathbb{R} \to [1,\infty)$ by

$$\alpha(x,y) = 1+|x|+|y|.$$

Then (Z, q) is a controlled metric space. By taking $a_0 = 0$ and $r = 4$. Then the closed ball is

$$\overline{B_q(0,4)} = \{x \in \mathbb{R} : q(0,x) \leq 4\}.$$

For $x \in [-3,3]$, we have $q(0,x) = |x|^2 \leq 4$, which implies $|x| \leq 2$. Hence,

$$\overline{B_q(0,4)} = [-2,2].$$

Define the mapping $S : \overline{B_q(0,4)} \to \mathbb{R}$ by

$$S(x) = \begin{cases} \dfrac{x}{4}, & |x| \leq 2, \\ 2x-3, & x > 2, \\ 2x+3, & x < -2. \end{cases}$$

For any $x \in [-2,2]$, we have $S(x) = \frac{x}{4} \in [-\frac{1}{2}, \frac{1}{2}] \subset [-2,2]$. Thus,

$$S(\overline{B_q(0,4)}) \subseteq \overline{B_q(0,4)}.$$

For all $x, y \in [-2, 2]$, we obtain

$$q(Sx, Sy) = \left|\frac{x}{4} - \frac{y}{4}\right|^2 = \frac{1}{16}|x - y|^2 = \frac{1}{16}q(x, y),$$

so that $\kappa = \frac{1}{16} \in (0, 1)$. Also,

$$q(a_0, Sa_0) = q(0, 0) = 0.$$

Choose $\xi = 5 \in [1, 1/\kappa) = [1, 16)$. Then

$$\frac{1 - \xi\kappa}{\xi} r = \frac{1 - 5 \cdot \frac{1}{16}}{5} \cdot 4 = \frac{11}{20},$$

and clearly $0 \leq \frac{11}{20}$. Define the iterative sequence $a_n = S^n(0)$. Then $a_n = 0$ for all $n \in \mathbb{N}$. Hence,

$$\alpha(a_i, a_j) = \alpha(0, 0) = 1 \leq \xi = 5.$$

Moreover,

$$\frac{\alpha(a_{i+1}, a_{i+2})\alpha(a_{i+1}, a_m)}{\alpha(a_i, a_{i+1})} = \frac{1 \cdot 1}{1} = 1 < \frac{1}{\kappa} = 16.$$

Thus, all the conditions of Theorem 1 are satisfied. Therefore, S has a unique fixed point in $B_q(0, 4)$, namely $x = 0$. Let $x = 4$ and $y = 3.5$. Then, since $x, y \notin [-3, 3]$, we have

$$q(4, 3.5) = (1 + 4 + 3.5)\,|4 - 3.5| = 8.5 \times 0.5 = 4.25.$$

Moreover, $S(4) = 5, \quad S(3.5) = 4$. Thus,

$$q(S4, S3.5) = (1 + 5 + 4)\,|5 - 4| = 10 \times 1 = 10.$$

Therefore,

$$\frac{q(Sx, Sy)}{q(x, y)} = \frac{10}{4.25} > 1.$$

Hence, there does not exist $\kappa < 1$ such that

$$q(Sx, Sy) \leq \kappa q(x, y) \quad \text{for all } x, y \in \mathbb{R}.$$

This shows that the Banach contraction principle fails on the whole space, while Theorem 1 guarantees the existence of a unique fixed point in the closed ball.

Corollary 2.1. *Let (Z, q) be a complete metric space. Let $\mathcal{S} : \overline{B_q(a_\circ, r)} \to Z$ be a mapping such that*

$$q(\mathcal{S}a, \mathcal{S}b) \leq \kappa q(a, b)$$

and

$$q(a_\circ, \mathcal{S}a_\circ) \leq (1 - \kappa)r$$

for all $a, b \in \overline{B_q(a_\circ, r)}$, with $\kappa \in (0, 1)$. Then, $\mathcal{S}$ has unique fixed point in $\overline{B_q(a_\circ, r)}$.

Corollary 2.2. *Let (Z, q) be a complete continuous b-metric space with parameter ξ. Let $\mathcal{S} : \overline{B_q(a_\circ, r)} \to Z$ be a mapping such that*

$$q(\mathcal{S}a, \mathcal{S}b) \leq \kappa q(a, b)$$

and

$$q(a_\circ, \mathcal{S}a_\circ) \leq \frac{(1 - \xi\kappa)}{\xi} r$$

for all $a, b \in \overline{B_q(a_\circ, r)}$, with $\kappa \in (0, 1)$ and $\xi\kappa < 1$. Then, $\mathcal{S}$ has unique fixed point in $\overline{B_q(a_\circ, r)}$.

Theorem 2.2. *Let (Z, q) be a complete continuous controlled metric space. Consider the mapping $\mathcal{S} : \overline{B_q(a_\circ, r)} \to Z$ such that*

$$q(\mathcal{S}a, \mathcal{S}b) \leq \kappa\big[q(a, \mathcal{S}a) + q(b, \mathcal{S}b)\big] \tag{6}$$

and

$$q(a_\circ, \mathcal{S}a_\circ) \leq \frac{(1 - \xi\theta)}{\xi} r \tag{7}$$

for all $a, b \in \overline{B_q(a_\circ, r)}$, where $\theta = \frac{\kappa}{1-\kappa}$ and $\kappa \in (0, \frac{1}{2})$.

Moreover, take $a_n = \mathcal{S}a_{n-1}$ for $n \in \mathbb{N}$. Suppose that the controlled function satisfies

$$\alpha(a_i, a_j) \leq \xi \text{ for all } i, j \in \mathbb{N} \text{ where } \xi \in [1, \tfrac{1}{\theta}) \tag{8}$$

and

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\alpha(a_i, a_{n+1})\alpha(a_{i+1}, a_m)}{\alpha(a_i, a_{i+1})} < \frac{1}{\theta}. \tag{9}$$

Then, there exist a point $a^* \in \overline{B_q(a_\circ, r)}$ such that a_n converges to a^*. In addition,

$$\lim_{n \to \infty} \alpha(a_n, a^*) \text{ and} \lim_{n \to \infty} \alpha(a^*, a_n) \text{ exist and are finite.} \tag{10}$$

Then, a^* is a unique fixed point of $\mathcal{S}$ in $\overline{B_q(a, r)}$.

Proof. Consider the sequence $a_n = \mathcal{S}^n a_\circ$ such that $a_1 \in \mathcal{S}a_\circ$ for $a_\circ \in \overline{B_q(a_\circ, r)}$ and using (6), we have

$$\begin{aligned} q(a_n, a_{n+1}) &= q(\mathcal{S}a_{n-1}, \mathcal{S}a_n) \\ &\leq \kappa\big[q(a_{n-1}, a_n) + q(a_n, a_{n+1})\big] \\ &\leq \frac{\kappa}{1-\kappa} q(a_{n-1}, a_n). \end{aligned}$$

As $\frac{\kappa}{1-\kappa} = \theta$, then, $\theta \in (0, 1)$ and continuing similar way

$$q(a_n, a_{n+1}) \leq \theta q(a_{n-1}, a_n) \leq \theta^2 q(a_{n-2}, a_{n-1}) \cdots \leq \theta^n q(a_\circ, a_1).$$

The remaining part is same as the theorem 1 and a_n is a Cauchy sequence as proved in [19]. So, we have

$$\lim_{n,m \to \infty} q(a_n, a_m) = 0.$$

Since every closed ball in a continuous controlled metric space is closed (Proposition 1) so $(\overline{B_q(a_\circ, r)}, q)$ is a complete continuous controlled metric space so every Cauchy sequence converges to point in $\overline{B_q(a_\circ, r)}$. Thus, there exist a^* such that $\lim_{n \to \infty} q(a_n, a^*) = 0$. Now, for fixed point

$$\begin{aligned} q(a^*, \mathcal{S}a^*) &\leq \alpha(a^*, a_n) q(a^*, a_n) + \alpha(a_n, \mathcal{S}a^*) q(a_n, \mathcal{S}a^*) \\ &\leq \alpha(a^*, a_n) q(a^*, a_n) + \kappa\alpha(a_n, \mathcal{S}a^*)\big[q(a_{n-1}, a_n) + q(a^*, \mathcal{S}a^*)\big] \\ &\leq \alpha(a^*, a_n) q(a^*, a_n) + \kappa\alpha(a_n, \mathcal{S}a^*)\big[q(a_{n-1}, a^*) + q(a^*, \mathcal{S}a^*)\big]. \end{aligned}$$

Taking $n \to \infty$ and using (10), we get $q(a^*, \mathcal{S}a^*) = 0$. Finally, suppose that two fixed point a^* and v of $\mathcal{S}$. Then,

$$\begin{aligned} q(a^*, v) &= q(\mathcal{S}a^*, \mathcal{S}v) \\ &\leq \kappa\big[q(a^*, \mathcal{S}a^*) + q(v, \mathcal{S}v)\big] \\ &\leq \kappa\big[q(a^*, a^*) + q(v, v)\big] \end{aligned}$$

As $\kappa \in (0, \frac{1}{2})$, so we get $q(a^*, v) = 0$. Thus, $\mathcal{S}$ has a unique fixed point. □

Example 2.2. Let $Z = \mathbb{R}$. Define $q : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by $q(x, y) = |x - y|^2$ and $\alpha : \mathbb{R} \times \mathbb{R} \to [1, \infty)$ by $\alpha(x, y) = 1 + |x| + |y|$ for all $x, y \in \mathbb{R}$. Then (Z, q, α) is a controlled metric space. Take $a_0 = 0$ and $r = 3$. Then

$$\overline{B_q(0, 3)} = \{x \in \mathbb{R} : q(0, x) = x^2 \leq 3\} = [-\sqrt{3}, \sqrt{3}].$$

Define $S : \overline{B_q(0, 3)} \to \mathbb{R}$ by

$$S(x) = \begin{cases} \dfrac{x+2}{5}, & |x| \leq \sqrt{3}, \\ 5x - 8, & x > \sqrt{3}, \\ 5x + 8, & x < -\sqrt{3}. \end{cases}$$

For $x \in [-\sqrt{3}, \sqrt{3}]$, $S(x) = (x + 2)/5 \in [(-\sqrt{3} + 2)/5, (\sqrt{3} + 2)/5] \subset [-\sqrt{3}, \sqrt{3}]$. Hence $S(B_q(0, 3)) \subseteq B_q(0, 3)$ and for $x, y \in [-\sqrt{3}, \sqrt{3}]$,

$$q(Sx, Sy) = \left| \frac{x+2}{5} - \frac{y+2}{5} \right|^2 = \frac{|x - y|^2}{25} = \frac{1}{25} q(x, y),$$

so $\kappa = 1/25 = 0.04 \in (0, 1/2)$. Also we have $q(a_0, Sa_0) = q(0, 0.4) = 0.16$. Let $\xi = \sup\{\alpha(x, y) : x, y \in B_q(0, 3)\} = 1 + 2\sqrt{3} \approx 4.464$. Let $\theta = \frac{\kappa}{1-\kappa} = \frac{0.04}{0.96} = \frac{1}{24} \approx 0.04167$. Then $\xi\theta \approx 0.186 < 1$. Now

$$\frac{1 - \xi\theta}{\xi} \cdot r = \frac{1 - 0.186}{4.464} \cdot 3 \approx 0.547,$$

and $0.16 \leq 0.547$, so condition (7) holds. For conditions (8) and (9), $a_0 = 0$, we have $a_1 = 0.4$, $a_2 = 0.48$, $a_3 = 0.496$, $\ldots$, all in $[0, 0.5] \subset B_q(0, 3)$. Thus $\alpha(a_i, a_j) \leq 1 + 0.5 + 0.5 = 2 \leq \xi$, so (8) holds. Moreover,

$$\sup \lim_{m\to\infty} \frac{\alpha(a_i, a_{n+1})\alpha(a_{i+1}, a_m)}{\alpha(a_i, a_{i+1})} \leq \frac{\xi \cdot \xi}{1} = \xi^2 \approx 19.93 < \frac{1}{\theta} = 24,$$

so (9) holds. All conditions of Theorem 2 are satisfied. Solving $S(x) = x$ on the ball gives $x = 1/2$, which is the unique fixed point and for $x = 3$, $y = 2.5$ gives $q(3, 2.5) = 0.25$, $S(3) = 7$, $S(2.5) = 4.5$, $q(7, 4.5) = 6.25$, $q(Sx, Sy)/q(x, y) = 25 > 1$ which shows the failure of Banach contraction.

Corollary 2.3. *Let (Z, q) be a complete continuous controlled metric space with $\overline{B_q(a_\circ, r)}$ as a compact in (Z, q) and let $\mathcal{S} : (Z, q) \to (Z, q)$ be a mapping such that it satisfies conditions (6), (7), (8) and (9) of Theorem 2. Then, $\mathcal{S}$ has a unique fixed point in $\overline{B_q(a_\circ, r)}$.*

Corollary 2.4. *Let (Z, q) be a complete continuous b-metric space with parameter ξ. Let $\mathcal{S} : \overline{B_q(a_\circ, r)} \to Z$ be a mapping such that*

$$q(\mathcal{S}a, \mathcal{S}b) \leq \kappa\big[q(a, \mathcal{S}a) + q(b, \mathcal{S}b)\big]$$

and

$$q(a_\circ, \mathcal{S}a_\circ) \leq \frac{(1 - \xi\theta)}{\xi} r$$

for all $a, b \in \overline{B_q(a_\circ, r)}$, with $\kappa \in (0, \frac{1}{2})$, $\theta \in (0, 1)$ and $\xi\theta < 1$. Then, $\mathcal{S}$ has unique fixed point in $\overline{B_q(a_\circ, r)}$.

3. **Applications**

In this section, we explore the existence of a unique solution for Fredholm integral equation

$$a(\tau) = \int_0^1 K(\tau, t, a(t))dt, \text{ for all } \tau \in [0,1], \tag{11}$$

where the Lipschitz kernel K is continuous. Let $Z = C[0,1]$ be a set of real continuous functions defined on $[0,1]$ and

$$q(a,b) = \sup_{\tau\in[0,1]} |a(\tau) - b(\tau)|^m$$

for all $a, b \in C[0,1]$, $m \geq 1$ and parameter $\xi = 2^{m-1}$. Define the closed ball $\overline{B_q(a_\circ, r)} = \{a^* \in C[0,1] : q(a_\circ, a^*) \leq r\}$ where $a^* \in C[0,1]$ and $r > 0$. It is clear that the space (Z, d) is a continuous b-metric space with a parameter $\xi = 2^{m-1}$.

Theorem 3.1. *Consider the integral equation (11) and suppose that*

(i) $K : [0,1] \times [0,1] \times C[0,1] \to \mathbb{R}$ *is continuous for all* $t \in [0,1]$ *and there exists* $\kappa \in (0,1)$ *such that*

$$|K(\tau, t, a(t)) - K(\tau, t, b(t))| \leq \kappa^{\frac{1}{m}} |a(t) - b(t)|$$

for all, $a, b \in C[0,1]$ *such that* $a(t)$*,* $b(t) \in [a_\circ(t) - r^{\frac{1}{m}}, a_\circ(t) + r^{\frac{1}{m}}]$ *for all* $t \in [0,1]$ *and* $a_0 \in C[0,1]$.

(ii) There exist $\left|a_\circ(t) - \int_0^1 K(\tau, t, a_\circ(t))dt\right|^m \leq \frac{1-\xi\kappa}{\xi} r$*, where* $\xi = 2^{m-1}$*,* $r > 0$ *and* $\xi\kappa < 1$*. Then,* $a^* \in C[0,1]$ *is the unique solution of (11) such that* $a^*(t) \in [a_\circ(t) - r^{\frac{1}{m}}, a_\circ(t) + r^{\frac{1}{m}}]$.

Proof. Let $S : C[0,1] \to C[0,1]$ be a mapping such that

$$\mathcal{S}a(\tau) = \int_0^1 K(\tau, t, a(t))dt, \text{ for all } \tau \in [0,1].$$

Also, let $a, b \in C[0,1]$ such that $a(t), b(t) \in [a_\circ(t) - r^{\frac{1}{m}}, a_\circ(t) + r^{\frac{1}{m}}]$, we have

$$\begin{aligned}|\mathcal{S}a(\tau) - \mathcal{S}b(\tau)|^m &= \left|\int_0^1 [K(\tau, t, a(t)) - K(\tau, t, b(t))]dt\right|^m \\ &\leq \int_0^1 |[K(\tau, t, a(t)) - K(\tau, t, b(t))]|^m \, dt\end{aligned}$$

By condition (i), we have

$$|\mathcal{S}a(\tau) - \mathcal{S}b(\tau)|^m \leq \int_0^1 \left|\kappa^{\frac{1}{m}} [a(t) - b(t)]\right|^m dt$$

Now, by taking sup, we have

$$\sup_{\tau\in[0,1]} |\mathcal{S}a(\tau) - \mathcal{S}b(\tau)|^m \leq \int_0^1 \left|\kappa^{\frac{1}{m}} [a(t) - b(t)]\right|^m dt,$$

$$\begin{aligned}\sup_{\tau\in[0,1]} |\mathcal{S}a(t) - \mathcal{S}b(t)|^m &\leq \kappa \sup_{t\in[0,1]} |a(t) - b(t)|^m \int_0^1 dt \\ &\leq \kappa \sup_{t\in[0,1]} |a(t) - b(t)|^m .\end{aligned}$$

This implies $q(\mathcal{S}a, \mathcal{S}b) \leq \kappa q(a,b)$ for all $a(t), b(t) \in [a_\circ(t) - r^{\frac{1}{m}}, a_\circ(t) + r^{\frac{1}{m}}]$. Now, by condition (ii), we have

$$\left| a_\circ(t) - \int_0^1 K(\tau, t, a_\circ(t))dt \right|^m \leq \frac{1-\xi\kappa}{\xi} r,$$

$$\sup_{t\in[0,1]} \left| a_\circ(t) - \int_0^1 K(\tau, t, a_\circ(t))dt \right|^m \leq \frac{1-\xi\kappa}{\xi} r.$$

$$\sup_{t\in[0,1]} |a_\circ(t) - \mathcal{S}a_\circ(t)|^m \leq \frac{1-\xi\kappa}{\xi} r$$

and this implies $q(a_\circ, \mathcal{S}a_\circ) \leq \frac{1-\xi\kappa}{\xi} r$. Since $\xi\kappa < 1$, and all the conditions of Corollary 2 are satisfied, a^* is the unique fixed point of $\mathcal{S}$ in $C[0,1]$. Thus, there exist a unique a^* such that $a^*(t) = \mathcal{S}a^*(t) = \int_0^1 K(\tau, t, a^*(t))dt$ and $a^* \in C[0,1]$ is the unique solution of (11) such that $a^*(t) \in [a_\circ(t) - r^{\frac{1}{m}}, a_\circ(t) + r^{\frac{1}{m}}]$. □

4. **Conclusion**

In this paper, we established new fixed point results on a closed ball in a complete continuous controlled metric space. The obtained theorems guarantee the existence and uniqueness of fixed points under suitable contractive conditions without requiring the mapping to be contractive on the whole space. A concrete example was provided to verify the validity of the theoretical results. Furthermore, the main results were successfully applied to a Fredholm integral equation, showing their practical usefulness. In the future, these results can be extended to more generalized structures such as fuzzy controlled metric spaces, double controlled metric space [2] and controlled rectangular metric space [4].

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