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ON BOUNDING PROPERTIES OF GENERALIZED ISOSCELES GEOMETRIC CONSTANT

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Motivated by the isosceles orthogonality, we introduce a new geometric constant and investigate its bounds in normed spaces and inner product spaces. We obtain the characterization of inner product spaces, uniformly non-square spaces, and super-reflexive Banach spaces. Through this geometric constant, we find that a Banach space possesses the fixed-point property. Also, we discuss its relation with a constant introduced by Gao [J. Inequal. Appl. (2006), 94982] to study the Pythagorean approach in a Banach space and find that a Banach space has a uniformly normal structure. Using the result of Gillespie and Williams [Applicable Analysis, 9(2)(1970), 121–124], we obtain that the non-expansive mapping has a fixed point. Furthermore, we examine the value of our geometric constant in function spaces L_p and sequence spaces ℓ_p.

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1. Introduction

From a geometric perspective, the examination of Banach spaces is intrinsically linked to exploring modern mathematics. Numerous characterizations of inner product spaces are among the many fascinating results it has produced, along with many fascinating open questions. Despite the fact that quite a few "natural" geometric properties may not hold in general Banach spaces unless the norm is derived from an inner product. Unlike in Euclidean geometry, there is no unique notion of orthogonality in normed linear spaces. Indeed, there exist multiple concepts of orthogonality in a normed linear space that are not generally identical. In 1935, Birkhoff [2] introduced a new notion of orthogonality, which turned out to be the most important notion of orthogonality defined for normed linear spaces. Later, in 1947, James [10] introduced isosceles orthogonality in normed space.

The quantitative analysis of the geometry of space is represented by the study of geometric constants, which is also crucial in the investigation of numerous other geometrical issues associated with functional analysis. Numerous geometric constants are linked to Banach spaces. Many geometric constants possess remarkable mathematical elegance, and there exist infinitely many relationships among them. Various specific constants have been defined by numerous scientists to investigate certain abstract aspects of Banach spaces.

Inspired by the exceptional contributions of Jordan and von Neumann regarding the characterization of inner product spaces [12], Clarkson [4] introduced the von Neumann-Jordan constant $C_{NJ}(\mathcal{X})$.

Zbăganu [18] proposed a novel constant $\mathcal{C}_z(\mathcal{X})$ and conjectured its equivalence with $\mathcal{C}_{NJ}(\mathcal{X})$ in all normed spaces. Subsequently, Alonso and Martín [1] presented a counterexample, asserting that $\mathcal{C}_{NJ}(\mathcal{X}) \neq \mathcal{C}_z(\mathcal{X})$ in general, and they examined its correlation with

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other prominent constants. In 2021, Liu et al. [14] presented a new geometric constant $\mathcal{C}_z(\varrho, \mu, \mathcal{X})$ in a Banach space $\mathcal{X}$ that was utilised to characterise the parallelogram law's generalisations. Gastinel and Joly [7] defined rectangular constant in Banach spaces using the notion of orthogonality, according to Birkhoff-James. Baronti et al. [3] introduced and studied a similar constant, based on isosceles orthogonality.

In 1965 Kirk [13] proved that reflexive Banach spaces with normal structure have the fixed point property. After this significant research, researchers extensively investigated the normal structure and related geometric properties. A refinement of this concept, uniformly normal structure, was initially investigated by Gillespie and Williams [8] (1979). They replaced uniformly convex (or reflexive and normal structure) as required by Kirk, by uniformly normal structure to obtain a fixed point theorem for non-expansive self mappings. They demonstrated that non-expansive self-mappings on closed, bounded, and convex subsets of Banach spaces with uniformly normal structure must have fixed points. The idea has subsequently become well-known as an important aspect of geometry and an effective tool in the field of fixed point theory.

The connections among various geometric features exhibit a distinct classification: all uniformly convex spaces have a uniformly normal structure, whereas the reverse is not true. Similarly, spaces that are uniformly convex in all directions have normal structure, but need not have uniformly normal structure. However, any space with uniformly normal structure will necessarily have normal structure. Maluta [15] proved that uniformly normal structure implies reflexivity.

The purpose of this study is to provide sufficient criteria for the existence of uniformly normal structure in Banach spaces, specifically using the isosceles orthogonality concept.

Motivated by these works, we introduce a new geometric constant called Generalized I-orthogonal geometric constant (GIOGC) (denoted by $\mathcal{C}_I^s(\varrho, \mu, \mathcal{X})$) and discuss its bounds in Banach spaces. Also, we obtain a characterization of a uniformly non-square Banach space using a condition on GIOGC. It is worth mentioning that every uniformly non-square Banach space has a fixed point property (see [5]). So, one can solve some fixed point problems using GIOGC in Banach spaces. Using the result of Gillespie and Williams [8], we obtain that a nonexpansive mapping has a fixed point. Additionally, we examine its connection to the parametric constant provided by Gao [6] to study the Pythagorean approach in a Banach space. We compute the values of this constant in the sequence spaces ℓ_p and the function spaces $L_p[0, 1]$.

2. **Preliminaries**

The following are some notations and definitions that will be utilized in the subsequent sections.
Let $\mathcal{X}$ be a Banach space and $\mathcal{S}_\mathcal{X}$ be a unit sphere of a Banach space $\mathcal{X}$.

Definition 2.1. [2] *In a normed linear space $\mathcal{X}$, a vector κ is said to be Birkhoff-James orthogonal (BJ-orthogonality) to a vector τ ($\kappa \perp_B \tau$) if the inequality $\|\kappa + \lambda\tau\| \geq \|\kappa\|$ holds for any real number λ.*

Definition 2.2. [10] *Let $(\mathcal{X}, \|.\|)$ be a normed linear space. Two elements κ and τ in $\mathcal{X}$ are said to be isosceles orthogonal ($\kappa \perp_I \tau$) if*

$$\|\kappa - \tau\| = \|\kappa + \tau\|.$$

Definition 2.3. *Let $\mathcal{X}$ be a Banach space.*

(i) $\mathcal{X}$ is called a uniformly non-square [11] *if there exists a $\delta > 0$ such that $\kappa, \tau \in \mathcal{S}_\mathcal{X}$, either*

$$\left\|\frac{1}{2}(\kappa + \tau)\right\| < 1 - \delta \text{ or } \left\|\frac{1}{2}(\kappa - \tau)\right\| < 1 - \delta.$$

(ii) The von-Neumann constant [4] $\mathcal{C}_{NJ}(\mathcal{X})$*, is defined as*

$$\mathcal{C}_{NJ}(\mathcal{X}) = \sup\left\{\frac{\|\kappa+\tau\|^2+\|\kappa-\tau\|^2}{2(\|\kappa\|^2+\|\tau\|^2)} : \kappa,\tau\in\mathcal{X}, (\kappa,\tau)\neq(0,0)\right\}.$$

(iii) The Zbăganu constant [18] $\mathcal{C}_z(\mathcal{X})$ *is defined as*

$$\mathcal{C}_z(\mathcal{X}) = \sup\left\{\frac{\|\kappa+\tau\|\|\kappa-\tau\|}{(\|\kappa\|^2+\|\tau\|^2)} : \kappa,\tau\in\mathcal{X}, (\kappa,\tau)\neq(0,0)\right\}.$$

(iv) The generalized Zbăganu constant [14] $\mathcal{C}_z(\varrho,\mu,\mathcal{X})$ *is defined as*

$$\mathcal{C}_z(\varrho,\mu,\mathcal{X}) = \sup\left\{\frac{2\|\varrho\kappa+\mu\tau\|\|\mu\kappa-\varrho\tau\|}{(\varrho^2+\mu^2)(\|\kappa\|^2+\|\tau\|^2)} : \kappa,\tau\in\mathcal{X}, (\kappa,\tau)\neq(0,0)\right\}.$$

The following results will be used in the sequel.

Lemma 2.1. *If* $\kappa,\tau>0$, $1\leq \mathrm{s}<\infty$*, then* $(\kappa+\tau)^{\mathrm{s}}\leq 2^{\mathrm{s}-1}(\kappa^{\mathrm{s}}+\tau^{\mathrm{s}})$.

Hilbert space is characterized by the parallelogram law, in [6], Gao explored the Pythagorean approach by introducing a parameter

$$E(\mathcal{X}) = \sup\{\|\kappa+\tau\|^2+\|\kappa-\tau\|^2 : \kappa,\tau\in\mathcal{S}_{\mathcal{X}}\},$$

for Banach spaces.

Definition 2.4. [9] *A non-empty bounded and convex subset* $\mathcal{K}$ *of a Banach space* $\mathcal{X}$ *is said to have normal structure if for every convex subset* C *of* $\mathcal{K}$ *that contains more than one point, there exists a point* $\kappa_0\in C$ *such that*

$$\sup\{\|\kappa_0-\tau\| : \tau\in C\} < \sup\{\|\kappa-\tau\| : \kappa,\tau\in C\}.$$

Definition 2.5. $\mathcal{X}$ *is said to have uniformly normal structure if there exists* $0<c<1$ *such that for any subset* $\mathcal{K}$ *as above, there exists* $\kappa_0\in\mathcal{K}$ *such that*

$$\sup\{\|\kappa_0-\tau\| : \tau\in C\} < c.\sup\{\|\kappa-\tau\| : \kappa,\tau\in C\}.$$

The following results will be required in the sequel.

Theorem 2.1. [6] *A Banach space* $\mathcal{X}$ *with* $E(\mathcal{X})<5$ *has a uniformly normal structure.*

Theorem 2.2. [17] *Let* $\mathcal{X}$ *be a Banach space. Then* $\mathcal{X}$ *is a Hillbert space that is, norms* $\|.\|$ *derives from inner product if and only if*

$$\|\kappa+\tau\|^2+\|\kappa-\tau\|^2\geq 4,$$

for all $\kappa,\tau\in\mathcal{S}_{\mathcal{X}}$.

3. Main Results

First, we introduce the Generalized I-orthogonal geometric constant (GIOGC) and discuss its bounds.

Definition 3.1. *Let* $(\mathcal{X},\|.\|)$ *be a Banach space of dimension at least 2. Then for* $\varrho,\mu>0$ *and* $\mathrm{s}\geq 1$*, the number*

$$\mathcal{C}_I^{\mathrm{s}}(\varrho,\mu,\mathcal{X}) = \sup_{\substack{\kappa,\tau\in\mathcal{X}\\(\kappa,\tau)\neq(0,0)}}\left\{\frac{\|\varrho\kappa+\mu\tau\|^{\mathrm{s}}\|\mu\kappa-\varrho\tau\|^{\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})} : \kappa\perp_I\tau\right\}, \tag{1}$$

is said to be a generalized I-orthogonal geometric constant (GIOGC).

For $\varrho = \mu$ in (1), we define

$$\mathcal{C}_I^{\mathrm{s}}(\mathcal{X}) = \mathcal{C}_I^{\mathrm{s}}(\varrho, \varrho, \mathcal{X}) = \sup_{\substack{\kappa,\tau\in\mathcal{X}\\(\kappa,\tau)\neq(0,0)}} \left\{ \frac{\|\kappa+\tau\|^{\mathrm{s}}\|\kappa-\tau\|^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})} : \kappa \perp_I \tau \right\}.$$

We observe that the following is one possible form of $\mathcal{C}_I^{\mathrm{s}}(\mathcal{X})$:

$$\mathcal{C}_I^{\mathrm{s}}(\mathcal{X}) = \sup_{\substack{\kappa,\tau\in\mathcal{X}\\(\kappa,\tau)\neq(0,0)}} \left\{ \frac{2^{2\mathrm{s}}\|\kappa\|^{\mathrm{s}}\|\tau\|^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\|\kappa+\tau\|^{2\mathrm{s}}+\|\kappa-\tau\|^{2\mathrm{s}})} : \kappa \perp_I \tau \right\}. \tag{2}$$

For $\kappa, \tau \in \mathcal{S}_{\mathcal{X}}$, we denote the generalized I-orthogonal geometric constant as follows:

$$\mathcal{C}_{IS}^{\mathrm{s}}(\varrho, \mu, \mathcal{X}) = \sup_{\kappa,\tau\in\mathcal{S}_{\mathcal{X}}} \left\{ \frac{\|\varrho\kappa+\mu\tau\|^{\mathrm{s}}\|\mu\kappa-\varrho\tau\|^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})} : \kappa \perp_I \tau \right\}.$$

$$\mathcal{C}_{IS}^{\mathrm{s}}(\mathcal{X}) = \sup_{\kappa,\tau\in\mathcal{S}_{\mathcal{X}}} \left\{ \frac{\|\kappa+\tau\|^{\mathrm{s}}\|\kappa-\tau\|^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})} : \kappa \perp_I \tau \right\}. \tag{3}$$

Next, we discuss about the lower and upper bounds of the constant $\mathcal{C}_I^{\mathrm{s}}(\varrho, \mu, \mathcal{X})$.

Theorem 3.1. *For a Banach space* $\mathcal{X}$, $\frac{1}{2^{2\mathrm{s}-2}} \leq \mathcal{C}_I^{\mathrm{s}}(\varrho, \mu, \mathcal{X}) \leq 2$.

Proof. First we assume that $\kappa \neq 0$, $\tau = 0$ with $\varrho = \mu = 1$ and then $\kappa \perp_I \tau$. So, we have

$$\begin{aligned}
\frac{\|\varrho\kappa+\mu\tau\|^{\mathrm{s}}\|\mu\kappa-\varrho\tau\|^{\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})} &= \frac{\varrho^{\mathrm{s}}\mu^{\mathrm{s}}\|\kappa\|^{2\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})\|\kappa\|^{2\mathrm{s}}}\\
&= \frac{\varrho^{\mathrm{s}}\mu^{\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})}\\
&= \frac{1}{2^{(2\mathrm{s}-2)}}.
\end{aligned}$$

So, by taking supremum over $\kappa, \tau \in \mathcal{X}$ and $(\kappa, \tau) \neq (0, 0)$, we get

$$\mathcal{C}_I^{\mathrm{s}}(\varrho, \mu, \mathcal{X}) \geq \frac{1}{2^{(2\mathrm{s}-2)}}.$$

On the other hand,

$$\begin{aligned}
&\frac{\|\varrho\kappa+\mu\tau\|^{\mathrm{s}}\|\mu\kappa-\varrho\tau\|^{\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&\leq \frac{(\|\varrho\kappa+\mu\tau\|^{2\mathrm{s}}+\|\mu\kappa-\varrho\tau\|^{2\mathrm{s}})}{2\times 2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&\leq \frac{\left[(\varrho\|\kappa\|+\mu\|\tau\|)^{2\mathrm{s}}+(\mu\|\kappa\|+\varrho\|\tau\|)^{2\mathrm{s}}\right]}{2(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&\leq \frac{2^{2\mathrm{s}-1}(\varrho^{2\mathrm{s}}\|\kappa\|^{2\mathrm{s}}+\mu^{2\mathrm{s}}\|\tau\|^{2\mathrm{s}})+2^{2\mathrm{s}-1}(\mu^{2\mathrm{s}}\|\kappa\|^{2\mathrm{s}}+\varrho^{2\mathrm{s}}\|\tau\|^{2\mathrm{s}})}{2^{(2\mathrm{s}-2)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&= \frac{2^{2\mathrm{s}-1}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}{2^{(2\mathrm{s}-2)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&= 2.
\end{aligned}$$

Therefore, $\frac{1}{2^{2\mathrm{s}-2}} \leq \mathcal{C}_I^{\mathrm{s}}(\varrho, \mu, \mathcal{X}) \leq 2$. □

For s $= 1$, we get the following result.

Corollary 3.1. *For the Banach space* $\mathcal{X}$, $1 \leq \mathcal{C}_I^{1}(\varrho, \mu, \mathcal{X}) \leq 2$.

Taking $\varrho = \mu$ in Corollary 3.1, we have the following result.

Corollary 3.2. *For a Banach space $\mathcal{X}$, $1 \leq \mathcal{C}_I^1(\mathcal{X}) \leq 2$.*

Example 3.1. *Consider a two-dimensional real sequence space $\mathcal{X} = \ell_\infty$ induced with a norm $\|(p,q)\| = \max\{|p|,|q|\}$, $p,q \in \mathcal{X}$.*

Taking $\kappa = (1,1)$ and $\tau = (1,-1)$, we have $\|\kappa\| = 1$ and $\|\tau\| = 1$ and $\kappa \perp_I \tau$. Now, $\|\varrho\kappa + \mu\tau\| = \varrho + \mu$ and $\|\mu\kappa - \varrho\tau\| = \mu + \varrho$.

Choosing $\varrho = \mu = 2$, we get GIOGC

$$\mathcal{C}_I^{\mathrm{s}}(\varrho,\mu,\mathcal{X}) = \frac{4^{\mathrm{s}}4^{\mathrm{s}}}{2^{(2\mathrm{s}-3)}(2^{2\mathrm{s}}+2^{2\mathrm{s}})(1+1)} = 2.$$

Theorem 3.2. *For a Banach space $\mathcal{X}$, $\mathcal{C}_I^{\mathrm{s}}(\varrho,\mu,\mathcal{X}) \leq \dfrac{[(\varrho+\mu)+|\varrho-\mu|]^{2\mathrm{s}}}{2^{2\mathrm{s}-2}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})}$.*

Proof. Consider

$$\begin{aligned}
&\frac{\|\varrho\kappa+\mu\tau\|^{\mathrm{s}}\|\mu\kappa-\varrho\tau\|^{\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&= \frac{\|\frac{\varrho+\mu}{2}(\kappa+\tau)+\frac{\varrho-\mu}{2}(\kappa-\tau)\|^{\mathrm{s}}\|\frac{\mu-\varrho}{2}(\kappa+\tau)+\frac{\mu+\varrho}{2}(\kappa-\tau)\|^{\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&\leq \frac{\left(\frac{\varrho+\mu}{2}\|(\kappa+\tau)\|+\frac{|\varrho-\mu|}{2}\|(\kappa-\tau)\|\right)^{\mathrm{s}}\left(\frac{|\varrho-\mu|}{2}\|(\kappa+\tau)\|+\frac{\mu+\varrho}{2}\|(\kappa-\tau)\|\right)^{\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&\leq \frac{\left(\frac{\varrho+\mu}{2}\|(\kappa+\tau)\|+\frac{|\varrho-\mu|}{2}\|(\kappa+\tau)\|\right)^{2\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&\leq \frac{\left(\frac{\varrho+\mu}{2}+\frac{|\varrho-\mu|}{2}\right)^{2\mathrm{s}}\|\kappa+\tau\|^{2\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&\leq \frac{\left(\frac{\varrho+\mu}{2}+\frac{|\varrho-\mu|}{2}\right)^{2\mathrm{s}}(\|\kappa\|+\|\tau\|)^{2\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&\leq \frac{\left(\frac{\varrho+\mu}{2}+\frac{|\varrho-\mu|}{2}\right)^{2\mathrm{s}}2^{(2\mathrm{s}-1)}(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&= \frac{[(\varrho+\mu)+|\varrho-\mu|]^{2\mathrm{s}}}{2^{2\mathrm{s}-2}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})}.
\end{aligned} \tag{4}$$

Using (1), we have

$$\mathcal{C}_I^{\mathrm{s}}(\varrho,\mu,\mathcal{X}) \leq \frac{[(\varrho+\mu)+|\varrho-\mu|]^{2\mathrm{s}}}{2^{2\mathrm{s}-2}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})}.$$

□

Now, we discuss the bounds of the constant $\mathcal{C}_I^{\mathrm{s}}(\varrho,\mu,\mathcal{X})$ in a Hilbert space.

Theorem 3.3. *For a Hilbert space $\mathcal{H}$, $\varrho,\mu \in \mathcal{H}$, we have $\frac{1}{2^{(2\mathrm{s}-2)}} \leq \mathcal{C}_I^{\mathrm{s}}(\varrho,\mu,\mathcal{H}) \leq \dfrac{[(\varrho+\mu)+|\varrho-\mu|]^{2\mathrm{s}}}{2^{3\mathrm{s}-2}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})}$.*

Proof. For $\mathrm{s} \geq 1$, using (1) and (4), we have

$$
\begin{aligned}
\frac{\|\varrho\kappa+\mu\tau\|^{\mathrm{s}}\|\mu\kappa-\varrho\tau\|^{\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})} &\leq \frac{\left(\frac{\varrho+\mu}{2}+\frac{|\varrho-\mu|}{2}\right)^{2\mathrm{s}}\|\kappa+\tau\|^{2\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&= \frac{\left(\frac{\varrho+\mu}{2}+\frac{|\varrho-\mu|}{2}\right)^{2\mathrm{s}}(\|\kappa\|^{2}+\|\tau\|^{2})^{p}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&\leq \frac{\left(\frac{\varrho+\mu}{2}+\frac{|\varrho-\mu|}{2}\right)^{2\mathrm{s}}2^{(\mathrm{s}-1)}(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&= \frac{[(\varrho+\mu)+|\varrho-\mu|]^{2\mathrm{s}}}{2^{3\mathrm{s}-2}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})}.
\end{aligned}
$$

By taking the supremum over $\varrho, \mu \in \mathcal{H}$, we get

$$
\mathcal{C}_I^{\mathrm{s}}(\varrho,\mu,\mathcal{H}) \leq \frac{[(\varrho+\mu)+|\varrho-\mu|]^{2\mathrm{s}}}{2^{3\mathrm{s}-2}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})}.
$$

Also, $\frac{1}{2^{(2\mathrm{s}-2)}} \leq \mathcal{C}_I^{\mathrm{s}}(\varrho,\mu,\mathcal{H})$. Therefore,

$$
\frac{1}{2^{(2\mathrm{s}-2)}} \leq \mathcal{C}_I^{\mathrm{s}}(\varrho,\mu,\mathcal{H}) \leq \frac{[(\varrho+\mu)+|\varrho-\mu|]^{2\mathrm{s}}}{2^{3\mathrm{s}-2}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})}.
$$

□

Remark 3.1. *Taking $\varrho = \mu$ in Theorem 3.3, we have*

(a) $\frac{1}{2^{2\mathrm{s}-2}} \leq \mathcal{C}_I^{\mathrm{s}}(\mathcal{H}) \leq \frac{1}{2^{\mathrm{s}-1}}$.
(b) $\mathcal{C}_I^{\mathrm{s}}(\mathcal{H}) = 1$ *for* $\mathrm{s} = 1$.
(c) It is easy to see that $\lim_{s\to\infty} \mathcal{C}_I^{\mathrm{s}}(\mathcal{H}) = 0$.

Theorem 3.4. *For a Hilbert space $\mathcal{H}$, $\frac{1}{2^{\mathrm{s}-1}} \leq \mathcal{C}_{IS}^{\mathrm{s}}(\varrho,\mu,\mathcal{H}) \leq \frac{\mu^{2\mathrm{s}}}{2^{\mathrm{s}-2}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})}$.*

Proof. Suppose that $\kappa, \tau \in \mathcal{S}_{\mathcal{H}}$ and $\kappa \perp_I \tau$. Using the parallelogram identity, we get $\|\kappa \pm \tau\| = \sqrt{2}$. Taking $\varrho = \mu$, we have

$$
\frac{\|\varrho\kappa+\mu\tau\|^{\mathrm{s}}\|\mu\kappa-\varrho\tau\|^{\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})} = \frac{\|\kappa+\tau\|^{2\mathrm{s}}}{2^{(2\mathrm{s}-3)}(1+1)(1+1)} = \frac{(\sqrt{2})^{2\mathrm{s}}}{2^{(2\mathrm{s}-1)}} = \frac{1}{2^{(\mathrm{s}-1)}}.
$$

By taking supremum over $\kappa, \tau \in \mathcal{S}_{\mathcal{H}}$, we get

$$
\mathcal{C}_I^{\mathrm{s}}(\varrho,\mu,\mathcal{H}) \geq \frac{1}{2^{(\mathrm{s}-1)}}.
$$

Without loss of generality, let $\mu \geq \varrho$. Using (1), (4) and for $s \geq 1$, we have

$$
\begin{aligned}
\frac{\|\varrho\kappa+\mu\tau\|^{\mathrm{s}}\|\mu\kappa-\varrho\tau\|^{\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})} &\leq \frac{\left(\frac{\varrho+\mu}{2}+\frac{|\varrho-\mu|}{2}\right)^{2\mathrm{s}}\|\kappa+\tau\|^{2\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}\\
&= \frac{\left(\frac{\varrho+\mu}{2}+\frac{\mu-\varrho}{2}\right)^{2\mathrm{s}}(\sqrt{2})^{2\mathrm{s}}}{2^{(2\mathrm{s}-3)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})(1+1)}\\
&= \frac{\mu^{2\mathrm{s}}2^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})}\\
&= \frac{1}{2^{\mathrm{s}-2}} \times \frac{\mu^{2\mathrm{s}}}{(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})}.
\end{aligned}
$$

Using (2), we have

$$\mathcal{C}^{\mathrm{s}}_{IS}(\varrho,\mu,\mathcal{H}) \le \frac{\mu^{2\mathrm{s}}}{2^{\mathrm{s}-2}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})}.$$

Hence the result. □

Theorem 3.5. *For a Hilbert space* $\mathcal{H}$, $\mathcal{C}^{\mathrm{s}}_{IS}(\mathcal{H}) = \frac{1}{2^{\mathrm{s}-1}}$.

Proof. Let $\mathcal{H}$ be a Hilbert space. Then taking $\varrho=\mu$ in Theorem 3.4, we have $\mathcal{C}^{\mathrm{s}}_{IS}(\mathcal{H}) = \frac{1}{2^{\mathrm{s}-1}}$. □

Remark 3.2. *Taking* $\mathrm{s}=1$ *in Theorem 3.5, we get* $\mathcal{C}^{1}_{IS}(\mathcal{H}) = 1$.

Theorem 3.6. *If* $\mathcal{X}$ *is a Banach space and* $\mathcal{C}^{1}_{IS}(\mathcal{X}) = 1$, *then* $\mathcal{X}$ *is a Hilbert space.*

Proof. Let, $\mathcal{C}^{1}_{IS}(\mathcal{H}) = 1$, for all $\kappa,\tau\in\mathcal{S}_{\mathcal{H}}$. Then, $\kappa+\tau \perp_I \kappa-\tau$ for all $\kappa,\tau\in\mathcal{S}_{\mathcal{H}}$. So from the definition of $\mathcal{C}^{\mathrm{s}}_{IS}$ (3), we have

$$\begin{aligned} 1 &\ge \frac{\|(\kappa+\tau)+(\kappa-\tau)\|\|(\kappa+\tau)-(\kappa-\tau)\|}{(\|\kappa+\tau\|^2+\|\kappa-\tau\|^2)} \\ &\ge \frac{2\|\kappa\|\times 2\|\tau\|^{\mathrm{s}}}{(\|\kappa+\tau\|^2+\|\kappa-\tau\|^2)} \\ &\ge \frac{4}{(\|\kappa+\tau\|^{2\mathrm{s}}+\|\kappa-\tau\|^{2\mathrm{s}})}. \end{aligned}$$

Therefore, $\|\kappa+\tau\|^2+\|\kappa-\tau\|^2 \ge 4$. Using Theorem 2.2, $\mathcal{X}$ is a Hilbert space. □

Now, we characterize uniformly non-square space in terms of GIOGC.

It is important to remember that a space $\mathcal{X}$ is not uniformly non-square if, for every $\epsilon>0$, there exists $\kappa,\tau\in\mathcal{S}_{\mathcal{X}}$ such that $1\le\|x\pm y\|\le 1+\epsilon$ (see [3]).

In the sequel, when $\mathcal{X}$ is a 2-dimensional space and $\kappa,\tau\in\mathcal{S}_{\mathcal{X}}$, we will denote by $arc(\kappa,\tau)$ the subset of $\mathcal{S}_{\mathcal{X}}$ of the points of the shorter arc with extreme κ and τ.

Theorem 3.7. *A Banach space* $\mathcal{X}$ *with*

$$\mathcal{C}^{\mathrm{s}}_{I}(\varrho,\mu,\mathcal{X}) < \frac{(3\varrho-\mu)^{\mathrm{s}}(\mu+\varrho)^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})},$$

where $\mu\ge\varrho>0$, *is a uniformly non-square.*

Proof. On the contrary, assume that $\mathcal{X}$ is not a uniformly non-square. Take $\epsilon>0$. Then there exist elements $\kappa_\epsilon,\tau_\epsilon\in\mathcal{S}_{\mathcal{X}}$ such that $\|\kappa_\epsilon+\tau_\epsilon\|\ge 2-\epsilon$. In $V_\epsilon = span[\kappa_\epsilon,\tau_\epsilon]$, there exists $w_\epsilon\in\mathcal{S}_{V_\epsilon}$ such that $\kappa_\epsilon\perp_I w_\epsilon$. We have two potential scenarios: the first scenario: $w_\epsilon\in arc(\kappa_\epsilon,\tau_\epsilon)$, therefore $\|\kappa_\epsilon+w_\epsilon\|\ge\|\kappa_\epsilon+\tau_\epsilon\|>2-\epsilon$. The second scenario: $w_\epsilon\in arc(\tau_\epsilon,-\kappa_\epsilon)$. Thus, $\|\kappa_\epsilon-w_\epsilon\|\ge\|\kappa_\epsilon-\tau_\epsilon\|>2-\epsilon$.
We have $\kappa_\epsilon\perp_I w_\epsilon$ and $\|\kappa_\epsilon\pm w_\epsilon\|>2-\epsilon$.
Without loss of generality, let $\mu\ge\varrho$. Now, $\kappa,\tau\in\mathcal{S}_{\mathcal{X}}$, we have

$$\begin{aligned} \|\varrho\kappa_\epsilon+\mu w_\epsilon\| &= \|\varrho(\kappa_\epsilon+w_\epsilon)+(\mu-\varrho)w_\epsilon\| \\ &\ge \varrho\|\kappa_\epsilon+w_\epsilon\|-(\mu-\varrho)\|w_\epsilon\| \\ &> \varrho\,(2-\epsilon)-(\mu-\varrho) \\ &= 3\varrho-\mu-\varrho\epsilon, \end{aligned}$$

and

$$\begin{aligned}\|\mu\kappa_\epsilon-\varrho w_\epsilon\| &= \|\mu(\kappa_\epsilon-w_\epsilon)+(\mu-\varrho)w_\epsilon\|\\ &\geq \mu\|\kappa_\epsilon-w_\epsilon\|-(\mu-\varrho)\|w_\epsilon\|\\ &> \mu\,(2-\epsilon)-(\mu-\varrho)\\ &= (\mu+\varrho-\mu\epsilon).\end{aligned}$$

So,

$$\|\varrho\kappa_\epsilon+\mu w_\epsilon\|^{\mathrm{s}}\|\mu\kappa_\epsilon-\varrho w_\epsilon\|^{\mathrm{s}} \geq (3\varrho-\mu-\varrho\epsilon)^{\mathrm{s}}\,(\mu+\varrho-\mu\epsilon)^{\mathrm{s}}\,.$$

Since ϵ can be arbitrarily small, then

$$\|\varrho\kappa_\epsilon+\mu w_\epsilon\|^{\mathrm{s}}\|\mu\kappa_\epsilon-\varrho w_\epsilon\|^{\mathrm{s}} \geq (3\varrho-\mu)^{\mathrm{s}}(\mu+\varrho)^{\mathrm{s}}.$$

Therefore,

$$\mathcal{C}_I^{\mathrm{s}}(\varrho,\mu,\mathcal{X}) \geq \frac{(3\varrho-\mu)^{\mathrm{s}}(\mu+\varrho)^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})},$$

which is a contradiction to the given assumption. Hence the result. □

It is important to note that every uniformly non-square Banach space has the fixed point property [5]. Thus, we can conclude the following corollary.

Corollary 3.3. *If $\mathcal{X}$ is a Banach space with $\mathcal{C}_I^{\mathrm{s}}(\varrho,\mu,\mathcal{X}) < \frac{(3\varrho-\mu)^{\mathrm{s}}(\mu+\varrho)^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})}$, for some $\varrho,\mu>0$, then $\mathcal{X}$ has the fixed point property.*

As every uniformly non-square Banach space is super-reflexive (see [16]), we have the following result.

Corollary 3.4. *If $\mathcal{C}_I^{\mathrm{s}}(\varrho,\mu,\mathcal{X}) < \frac{(3\varrho-\mu)^{\mathrm{s}}(\mu+\varrho)^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})}$, then $\mathcal{X}$ is a super-reflexive Banach space.*

Taking $\varrho=\mu$ in the Theorem 3.7, we obtain the following result.

Theorem 3.8. *A Banach space $\mathcal{X}$ with $\mathcal{C}_I^{\mathrm{s}}(\mathcal{X})<2$ is a uniformly non-square.*

Theorem 3.9. *If a Banach space $\mathcal{X}$ is not a uniformly non-square, then*

$$\mathcal{C}_I^{\mathrm{s}}(\varrho,\mu,\mathcal{X}) \leq \frac{\mu^{\mathrm{s}}\,(2\mu-\varrho)^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})},\ \textit{where } \varrho,\mu>0.$$

Proof. Assume that $\mathcal{X}$ is not uniformly non-square. Take $\epsilon>0$. Then there exist elements $\kappa_\epsilon,\tau_\epsilon \in \mathcal{S}_{\mathcal{X}}$ such that $1 \leq \|\kappa_\epsilon+\tau_\epsilon\| \leq 1+\epsilon$. In $V_\epsilon = span[\kappa_\epsilon,\tau_\epsilon]$, there exists $w_\epsilon \in \mathcal{S}_{V_\epsilon}$ such that $\kappa_\epsilon \perp_I w_\epsilon$. We have two potential scenarios: the first scenario: $w_\epsilon \in arc(\kappa_\epsilon,\tau_\epsilon)$, therefore $\|\kappa_\epsilon-w_\epsilon\| \leq \|\kappa_\epsilon-\tau_\epsilon\| \leq 1+\epsilon$. The second scenario: $w \in arc(\tau_\epsilon,-\kappa_\epsilon)$. Thus, $\|\kappa_\epsilon+w_\epsilon\| \leq \|\kappa_\epsilon+\tau_\epsilon\| \leq 1+\epsilon$.
We have $\kappa_\epsilon \perp_I w_\epsilon$ and $1 \leq \|\kappa_\epsilon \pm w_\epsilon\| \leq 1+\epsilon$.
Without loss of generality, let $\mu \geq \varrho$.
Now, $\kappa,\tau \in \mathcal{S}_{\mathcal{X}}$, we have

$$\begin{aligned}\|\varrho\kappa_\epsilon+\mu w_\epsilon\| &= \|\varrho(\kappa_\epsilon+w_\epsilon)+(\mu-\varrho)w_\epsilon\|\\ &\leq \varrho\|\kappa_\epsilon+w_\epsilon\|+(\mu-\varrho)\|w_\epsilon\|\\ &\leq \varrho\,(1+\epsilon)+(\mu-\varrho)\\ &= \mu+\varrho\epsilon.\end{aligned}$$

and

$$\begin{aligned}\|\mu\kappa_\epsilon - \varrho w_\epsilon\| &= \|\mu(\kappa_\epsilon - w_\epsilon) + (\mu-\varrho)w_\epsilon\| \\ &\le \mu\|\kappa_\epsilon - w_\epsilon\| + (\mu-\varrho)\|w_\epsilon\| \\ &= \mu\,(1+\epsilon) + (\mu-\varrho) \\ &= (2\mu - \varrho + \mu\epsilon).\end{aligned}$$

So,

$$\|\varrho\kappa_\epsilon + \mu w_\epsilon\|^{\mathrm{s}}\|\mu\kappa_\epsilon - \varrho w_\epsilon\|^{\mathrm{s}} \le (\mu+\varrho\epsilon)^{\mathrm{s}}\,(2\mu-\varrho+\mu\epsilon)^{\mathrm{s}}\,.$$

Since ϵ can be arbitrarily small, then

$$\|\varrho\kappa_\epsilon + \mu w_\epsilon\|^{\mathrm{s}}\|\mu\kappa_\epsilon - \varrho w_\epsilon\|^{\mathrm{s}} \le \mu^{\mathrm{s}}(2\mu-\varrho)^{\mathrm{s}}.$$

Therefore,

$$\mathcal{C}_I^{\mathrm{s}}(\varrho,\mu,\mathcal{X}) \le \frac{\mu^{\mathrm{s}}\,(2\mu-\varrho)^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\varrho^{2\mathrm{s}}+\mu^{2\mathrm{s}})}.$$

□

Taking $\varrho = \mu$, we have

Corollary 3.5. *If a Banach space $\mathcal{X}$ is not a uniformly non-square, then $\mathcal{C}_I^{\mathrm{s}}(\mathcal{X}) \le \frac{1}{2^{(2\mathrm{s}-1)}}$.*

For $\mathrm{s} = 1$, we get the following result.

Corollary 3.6. *If a Banach space $\mathcal{X}$ is not a uniformly non-square, then $\mathcal{C}_I^{1}(\mathcal{X}) \le \frac{1}{2}$.*

Theorem 3.10. *For a Banach space $\mathcal{X}$, $\{2^{2\mathrm{s}}\mathcal{C}_I^{\mathrm{s}}(\mathcal{X})\}^{\frac{1}{\mathrm{s}}} \le E(\mathcal{X})$.*

Proof. Assume that $\kappa,\tau \in \mathcal{X}$ and $\kappa \perp_I \tau$. Taking $u = \frac{\kappa+\tau}{2}$ and $v = \frac{\kappa-\tau}{2}$, we have $\|u\| = \|v\|$ and

$$\begin{aligned}\frac{2^{2\mathrm{s}}\|\kappa\|^{\mathrm{s}}\|\tau\|^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\|\kappa+\tau\|^{2\mathrm{s}}+\|\kappa-\tau\|^{2\mathrm{s}})} &= \frac{2^{2\mathrm{s}}\|u+v\|^{\mathrm{s}}\|u-v\|^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\|2u\|^{2\mathrm{s}}+\|2v\|^{2\mathrm{s}})} \\ &= \frac{2^{2\mathrm{s}}\|u+v\|^{\mathrm{s}}\|u-v\|^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\|2u\|^{2\mathrm{s}}+\|2v\|^{2\mathrm{s}})} \\ &\le \frac{2^{2\mathrm{s}}\|u+v\|^{2\mathrm{s}}+\|u-v\|^{2\mathrm{s}}}{2\times 2^{(2\mathrm{s}-2)}\times 2^{2\mathrm{s}}(\|u\|^{2\mathrm{s}}+\|v\|^{2\mathrm{s}})} \\ &\le \frac{\|u+v\|^{2\mathrm{s}}+\|u-v\|^{2\mathrm{s}}}{2\times 2^{(2\mathrm{s}-2)}(\|u\|^{2\mathrm{s}}+\|v\|^{2\mathrm{s}})}.\end{aligned}$$

Let $\kappa' = \frac{u}{\|u\|}$ and $\tau' = \frac{v}{\|v\|}$, then $\kappa',\tau' \in \mathcal{S}_{\mathcal{X}}$ and

$$\begin{aligned}\frac{2^{2\mathrm{s}}\|\kappa\|^{\mathrm{s}}\|\tau\|^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\|\kappa+\tau\|^{2\mathrm{s}}+\|\kappa-\tau\|^{2\mathrm{s}})} &\le \frac{\|\kappa'+\tau'\|^{2\mathrm{s}}+\|\kappa'-\tau'\|^{2\mathrm{s}}}{2\times 2^{(2\mathrm{s}-2)}(\|\kappa'\|^{2\mathrm{s}}+\|\tau'\|^{2\mathrm{s}})} \\ &= \frac{\|\kappa'+\tau'\|^{2\mathrm{s}}+\|\kappa'-\tau'\|^{2\mathrm{s}}}{2\times 2^{(2\mathrm{s}-2)}(1+1)} \\ &\le \frac{(\|\kappa'+\tau'\|^{2}+\|\kappa'-\tau'\|^{2})^{\mathrm{s}}}{2^{(2\mathrm{s})}} \\ &\le \frac{[E(\mathcal{X})]^{\mathrm{s}}}{2^{2\mathrm{s}}}.\end{aligned}$$

Using (2), we have

$$\mathcal{C}_I^{\mathrm{s}}(\mathcal{X}) \le \frac{1}{2^{2\mathrm{s}}}[E(\mathcal{X})]^{\mathrm{s}}.$$

Hence, $\{2^{2\mathrm{s}}\mathcal{C}_I^{\mathrm{s}}(\mathcal{X})\}^{\frac{1}{\mathrm{s}}} \le E(\mathcal{X})$. □

Remark 3.3. *For a Banach space $\mathcal{X}$, if $\{2^{2\mathrm{s}}\mathcal{C}_I^{\mathrm{s}}(\mathcal{X})\}^{\frac{1}{\mathrm{s}}} \le E(\mathcal{X}) \le 5$ holds. Then using Theorem 2.1, $\mathcal{X}$ has uniformly normal structure.*
Suppose that D is a non-empty, closed, bounded, and convex subset of the Banach space $\mathcal{X}$ with $[2^{2\mathrm{s}}\mathcal{C}_I^{\mathrm{s}}(\mathcal{X})]^{\frac{1}{\mathrm{s}}} \le E(\mathcal{X}) \le 5$. Further, if $f : D \to D$ is a non-expansive mapping. Then using the corresponding result of Gillespie and Williams [[8], p.123], f has a fixed point.

Next, we estimate the value of the geometric constant GIOGC in the sequence space ℓ_p. For this, we use the following well-known Clarkson inequality:

$$\|\kappa+\tau\|^p + \|\kappa-\tau\|^p \le (\|\kappa\| + \|\tau\|)^p + |\|\kappa\| - \|\tau\||^p.$$

Theorem 3.11. $2^{1-\frac{2\mathrm{s}}{q}} \le \mathcal{C}_{IS}^{\mathrm{s}}(\ell_p) \le 2^{1-\frac{2\mathrm{s}}{p}}$ *for all* $p, q \ge 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $\kappa = (1, 0, 0, 0, \ldots)$ and $\tau = (0, 1, 0, 0, \ldots) \in \mathcal{S}_{\ell_p}$. Here $\|\kappa+\tau\| = \|\kappa-\tau\| = 2^{\frac{1}{p}}$, so $\kappa \perp_I \tau$.

$$\frac{\|\kappa+\tau\|^{\mathrm{s}}\|\kappa-\tau\|^{\mathrm{s}}}{2^{(2\mathrm{s}-2)}(\|\kappa\|^{2\mathrm{s}} + \|\tau\|^{2\mathrm{s}})} = \frac{2^{\frac{2\mathrm{s}}{p}}}{2^{(2\mathrm{s}-2)}(1+1)} = \frac{2^{\frac{2\mathrm{s}}{p}}}{2^{(2\mathrm{s}-1)}} = 2^{1-2\mathrm{s}+\frac{2\mathrm{s}}{p}} = 2^{1-\frac{2\mathrm{s}}{q}}.$$

Therefore,

$$\mathcal{C}_{IS}^{s}(\ell_p) \ge 2^{1-\frac{2s}{q}}.$$

Using Clarkson inequality for all $\kappa, \tau \in \mathcal{S}_{\ell_p}$ and $\kappa \perp_I \tau$, we have

$$\|\kappa+\tau\|^p \le 2^{\frac{p-1}{p}}. \tag{5}$$

Now using (5), we have

$$\frac{\|\kappa+\tau\|^{s}\|\kappa-\tau\|^{s}}{2^{(2\mathrm{s}-2)(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}} = \frac{\|\kappa+\tau\|^{2\mathrm{s}}}{2^{(2\mathrm{s}-2)(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}} \le \frac{2^{(\frac{p-1}{p})2\mathrm{s}}}{2^{(2\mathrm{s}-2)}(1+1)} = \frac{2^{2\mathrm{s}-\frac{2\mathrm{s}}{p}}}{2^{2\mathrm{s}-1}} = 2^{1-\frac{2\mathrm{s}}{p}}.$$

Using the definition of $\mathcal{C}_{IS}^{s}$, we get $\mathcal{C}_{IS}^{s}(\ell_p) \le 2^{1-\frac{2s}{p}}$. Therefore,

$$2^{1-\frac{2s}{q}} \le \mathcal{C}_{IS}^{\mathrm{s}}(\ell_p) \le 2^{1-\frac{2\mathrm{s}}{p}}.$$

□

Remark 3.4. *Substituting* p = 2 *into Theorem 3.11, we have the result of Theorem 3.5, that is,* $\mathcal{C}_{IS}^{\mathrm{s}}(\ell_p) = 2^{1-\mathrm{s}} = \frac{1}{2^{\mathrm{s}-1}}$.

Subsequently, we evaluate the value of the geometric constant GIOGC within the function space L_p.

Theorem 3.12. $\mathcal{C}_{IS}^{\mathrm{s}}(L_p[0,1]) = \frac{1}{2^{2\mathrm{s}(1-\frac{1}{q})-1}}$ *for all* $p, q \ge 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Choose $\kappa(t) = 1$, for all $t \in [0,1]$ and $\tau(t) = \begin{cases} 1, & 0 \le t \le \frac{1}{2} \\ -1, & \frac{1}{2} \le t \le 1 \end{cases}$. Clearly $\kappa, \tau \in \mathcal{S}_{L_p}$.

Also, here $\|\kappa+\tau\| = \|\kappa-\tau\| = 2^{\frac{p-1}{p}}$, so $\kappa \perp_I \tau$.
Consider,

$$\frac{\|\kappa+\tau\|^{s}\|\kappa-\tau\|^{s}}{2^{(2\mathrm{s}-2)}(\|\kappa\|^{2\mathrm{s}} + \|\tau\|^{2\mathrm{s}})} = \frac{2^{(\frac{p-1}{p})2\mathrm{s}}}{2^{(2\mathrm{s}-2)}(1+1)} = \frac{2^{2s-\frac{2\mathrm{s}}{p}}}{2^{2\mathrm{s}-1}} = 2^{1-\frac{2\mathrm{s}}{p}}.$$

Therefore,

$$\mathcal{C}_{IS}^{s}(L_p[0,1]) \ge 2^{1-\frac{2s}{p}}.$$

Using the Clarkson inequality for all $\kappa, \tau \in \mathcal{S}_{L_p[0,1]}$ and $\kappa \perp_I \tau$, we have

$$\|\kappa+\tau\|^p \le 2^{\frac{p-1}{p}}. \tag{6}$$

Now using (6), we have

$$\frac{\|\kappa+\tau\|^{\mathrm{s}}\|\kappa-\tau\|^{\mathrm{s}}}{2^{(2\mathrm{s}-2)(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}}=\frac{\|\kappa-\mu\|^{2\mathrm{s}}}{2^{(2\mathrm{s}-2)(\|\kappa\|^{2\mathrm{s}}+\|\tau\|^{2\mathrm{s}})}}\leq\frac{2^{(\frac{p-1}{p})2\mathrm{s}}}{2^{(2\mathrm{s}-2)}(1+1)}=\frac{2^{2\mathrm{s}-\frac{2\mathrm{s}}{p}}}{2^{2\mathrm{s}-1}}=2^{1-\frac{2\mathrm{s}}{p}}.$$

Using the definition of $\mathcal{C}^{\mathrm{s}}_{IS}$, we have $\mathcal{C}^{\mathrm{s}}_{I}(L_p[0,1])\leq 2^{1-\frac{2\mathrm{s}}{p}}$. Therefore, $\mathcal{C}^{\mathrm{s}}_{IS}(L_p[0,1])=2^{1-\frac{2\mathrm{s}}{p}}$. Since $\frac{1}{p}+\frac{1}{q}=1$, we have $\mathcal{C}^{\mathrm{s}}_{IS}(L_p[0,1])=\frac{1}{2^{2\mathrm{s}(1-\frac{1}{q})-1}}$ for all $p,q\geq 1$. ☐

Remark 3.5. *(i) Taking* $\mathrm{p}=2$, *we have the result of the Theorem 3.5, that is* $\mathcal{C}^{\mathrm{s}}_{I}(L_2[0,1])=2^{1-\mathrm{s}}=\frac{1}{2^{\mathrm{s}-1}}$.

(ii) Taking $\mathrm{s}=p$ *in the Theorem 3.12, we have* $\mathcal{C}^{p}_{IS}(L_p[0,1])=\frac{1}{2}$.

4. **Conclusion**

The paper presents a new constant $\mathcal{C}^{\mathrm{s}}_{I}(\varrho,\mu,\mathcal{X})$ in a Banach space $\mathcal{X}$. We obtain its bound in Banach and Hilbert spaces. Also, we explore its relationship with other renowned geometric constants. We calculate the value of GIOGC in function spaces $L_p[0,1]$ and sequence spaces ℓ_p.

We have the following open question from our observation:

Is it possible to determine the characterization of the inner product utilizing the constant $\mathcal{C}^{\mathrm{s}}_{I}(\varrho,\mu,\mathcal{X})$ where $\mathrm{s}\geq 2$?

Which other geometric constants are more closely related to $\mathcal{C}^{\mathrm{s}}_{I}(\varrho,\mu,\mathcal{X})$ than those mentioned in the study?

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