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# AN EXTENSION OF HIGHER ORDER RIESZ TRANSFORMS ASSOCIATED WITH BESSEL OPERATORS

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*In this paper, we introduce the class of fractional integral operators generated by the generalized shift operator with the kernel satisfying the Dini condition. Furthermore, the focus of this work is that these operators are obtained via a high-order Riesz-Bessel transform and are related to the generalized shift operator. Subsequently, we prove the boundedness of fractional singular integral operators with a kernel associated with non-negative, regular, and compactly supported functions in* BMO *spaces.*

**Keywords:** Riesz Bessel transform, Generalized shift operator, Laplace Bessel operator, BMO spaces

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## 1. **Introduction and main results**

In [24], Y.Xin and S.Yang studied the classical Calderon Zygmund singular integral operators known as

$$I(f)(x) := \text{ p.v. } \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^n} f(x-y)dy := \lim_{\varepsilon\to 0^+} \int_{|y|\geq\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy, \tag{1}$$

in BMO space. In this paper, rather than considering the singular integral operator defined by equation (1), we will present a boundedness estimate for the fractional singular integral redefined by a higher-order singular integral operator. This operator is known as the Riesz-Bessel transform on $\mathbb{R}^n_+$. Specifically, the Riesz Bessel transform for any $y \in \mathbb{R}^n_+$, $\gamma > 0$ and a function $f$ on $\mathbb{R}^n_+$ are defined by

$$\begin{aligned} R^{(k)}_\gamma(f)(x) :=& \text{ p.v.} \int_{\mathbb{R}^n_+} \frac{\Omega(y)}{|y|^{n+k+\gamma}} T^y f(x) y_n^\gamma dy, \quad 1 \leq k \leq n \\ =& \text{ p.v.} \int_{\mathbb{R}^n_+} K(y) T^y f(x) y_n^\gamma dy, \quad \gamma > 0 \end{aligned} \tag{2}$$

where $T^y$ generalized translate operator associated with Laplace Bessel differential operator (see [6, 18]). It should be noted that in (1) the function $f(x-y)$, which corresponds to the ordinary translation, is replaced by the generalized translation operator $T^y f(x)$. In the other important case $\Omega$ is related to a homogeneous polynomial $P_k$ of order $k$ and we will show it as $\Omega(y) = P_k(y)|y|^{-k}$. Therefore, a new singular integral operator of the Calderon-Zygmund type has been adapted. This singular integral operator is the high-order

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Riesz-Bessel transform give in (2), and its kernel is defined as $K(y) = \Omega(y)|y|^{-Q}$. For simplicity throughout our study, we will take $Q = n + \gamma$.

If $K(x)$ is such that the $L_1$-modulus of continuity $w(t)$ of its characteristic $\Omega$ satisfies the Dini condition. Therefore, we should also note that, for the purposes of this study, we are also assuming that $\Omega$ satisfies the following Dini-type condition:

$$\int_0^1 \frac{\omega(\delta)}{\delta} d\delta < \infty, \tag{3}$$

where $\omega$ is defined as

$$\omega(\delta) := \sup \left\{ |T^\xi \Omega(x) - \Omega(x)| : |x - \xi| \leq \delta, |x| = |\xi| = 1 \right\}$$

for any $0 < \delta < \infty$. In this paper, the Dini-type condition is replaced by the following continuity condition: For $\xi \in S^{n-1}$, $0 < \delta < l/n$, we denotes usual

$$w(t) = \sup_{|t| \leq t} \int_{S^{n-1}} |T^\xi \Omega(\theta) - \Omega(\theta)| d\theta \leq Cn\delta \int_{S^{n-1}} |\Omega(\theta)| d\theta, \tag{4}$$

where C is independent of dimension. Condition (4) allows specification of the constant, and it is shown below that the high order Riesz Bessel transform satisfies this condition with $\Omega(y) = P_k(y)|y|^{-k}$. Then the singular integral operator (2) exists a.e. for $f \in L_{p,\gamma}(\mathbb{R}^n)$ ( see [1, 11]). However, note that $\Omega$ satisfies the following conditions.

i) $\Omega$ is a homogeneous function of degree zero on $\mathbb{R}^n \backslash \{0\}$, that is, for any given $\lambda \in (0, \infty)$ and $x \in \mathbb{R}^n \backslash \{0\}$,

$$\Omega(\lambda x) = \Omega(x). \tag{5}$$

ii) $\Omega \in L^1(\mathbb{S}^{n-1})$ satisfies the cancellation condition

$$\int_{\mathbb{S}^n_+} \Omega(\theta)(\theta')^\gamma d\theta = 0, \tag{6}$$

where $\theta = y/|y|$ and $\mathbb{S}^n_+ := \left\{ y \in \mathbb{R}^n_+ : |y| = 1 \right\}$ denotes the unit sphere in $\mathbb{R}^n_+$.

The Laplace-Bessel differential operator $\Delta\gamma$ which is an important technical tool in harmonic analysis is defined by

$$\Delta\gamma = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + \left( \frac{\partial^2}{\partial x_n^2} + \frac{\gamma}{x_n} \frac{\partial}{\partial x_n} \right).$$

This operator is a hybrid differential operator which is obtained by applying the Laplace differential operator in the first $n-1$ variable and the Bessel differential operator in the last variable.

Denote by $T^y$ the generalized translate operator, acting according to the law

$$T^y f(x) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma(\gamma/2)\Gamma\left(\frac{1}{2}\right)} \int_0^\pi f\left(x' - y', \sqrt{(x_n^2 - 2x_n y_n \cos\theta + y_n^2)}\right) \sin^{\gamma-1}\theta d\theta,$$

where $x = (x', x_n), y = (y', y_n), x', y' \in \mathbb{R}^{n-1}$ and

$$\left( \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma(\gamma/2)\Gamma\left(\frac{1}{2}\right)} \right)^{-1} = \int_0^\pi \sin^{\gamma-1}\theta d\theta.$$

Note that the generalized translation operator $T^y$ is closely related with the Laplace-Bessel differential operator $\Delta_\gamma$ (see [4, 6, 18, 20]).

Denote $\mathbb{R}_{+}^{n}=\{x\in\mathbb{R}^{n}: x=(x_{1},\cdots,x_{n-1},x_{n}), x_{n}>0\}$. The Lebesgue measure of a measurable set $B\subset\mathbb{R}_{+}^{n}$ is denoted by

$$|B|_{\gamma}=\int_{B}y_{n}^{\gamma}dy,$$

and $\gamma>0$ is a fixed parameter. Suppose that $B(x,r)=\{y\in\mathbb{R}_{+}^{n}:|x-y|<r\}$ denotes the ball of radius $r>0$ centered at $x\in\mathbb{R}_{+}^{n}$. It is known that $|B(0,r)|_{\gamma}=r^{n+\gamma}\omega(n,\gamma)$ where $\omega(n,\gamma)=|B(0,1)|_{\gamma}$.

Let $L_{p,\gamma}\equiv L_{p,\gamma}(\mathbb{R}^{n}),\quad 1\le p<\infty$ be the space of all measurable functions on $\mathbb{R}^{n}$ with the norm

$$\|f\|_{L_{p,\gamma}}=\left(\int_{\mathbb{R}^{n}}|f(x)|^{p}(x_{n})^{\gamma}\ \mathrm{d}x\right)^{\frac{1}{p}}<\infty.$$

In the case $p=\infty$, the space $L_{\infty,\gamma}(\mathbb{R}^{n})$ is defined by means of the usual modification

$$\|f\|_{L_{\infty}}=\operatorname{ess\,sup}|f(x)|,\quad x\in\mathbb{R}^{n}.$$

The weak- $L_{p,\gamma}$ space $WL_{p,\gamma}\equiv WL_{p,\gamma}(\mathbb{R}^{n})$ is defined by

$$\|f\|_{WL_{p,\gamma}}=\sup_{r>0}r\left|\{x\in\mathbb{R}^{n}:|f(x)|>r\}\right|_{\gamma}^{1/p},\quad 1\le p<\infty.$$

Let $0<\alpha<Q+k$ and $\Omega\in L^{1}(\mathbb{S}^{n-1})$ satisfy (5) and (6). Now, for the boundedness of the high order Riesz Bessel transform generated by the generalized translate operator $T^{y}$ given in (2), we consider the following singular integral operator

$$R_{\gamma}^{(k)*}(f)(x):=\ \text{p.v.}\int_{\mathbb{R}_{+}^{n}}\frac{\Omega(y)}{|y|^{Q+k-\alpha}}T^{y}f(x)y_{n}^{\gamma}dy,\quad 1\le k\le n \tag{7}$$

on the space $\mathrm{BMO}(\mathbb{R}_{+}^{n})$. In our study, we investigate the boundedness of the singular integral operator generated by the generalized translate operator given in (7). This is a fractional singular intagral operator and formally, the Calderón-Zygmund type singular integral operator $R_{\gamma}^{(k)*}$ as in (7) becomes the Riesz Bessel transform $R_{\gamma}^{(k)}$ when $\alpha=0$.

Before stating the main results of our study, we first briefly recall partial known results on the boundedness of the singular integral operator $R_{\gamma}^{(k)*}$ as in (7).

Assume further that $\Omega\in L^{\infty}(\mathbb{S}^{n-1})$ and there exists a positive constant $A$ such that

$$\|\Omega\|_{L^{\infty}(\mathbb{S}^{n-1})}\le A. \tag{8}$$

Using the boundedness results of fractional integral operators (see, [12, 13]), we find that, for any given $0<\alpha<Q+k$,

$$\|R_{\gamma}^{(k)*}f\|_{L_{q,\gamma}(\mathbb{R}_{+}^{n})}\le C\|f\|_{L_{p,\gamma}(\mathbb{R}_{+}^{n})}, \tag{9}$$

where $1<p<\frac{Q}{\alpha}$ and $1<q<\infty$ is given by $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q+k}$. Here, the positive constant $C$ depends on $n,p,\gamma$, and $\alpha$, and the constant $C\to\infty$ when $\alpha\to 0$.

Moreover, for the extreme case $p=(Q+k)/\alpha$, it is easy to verify that high order Riesz Bessel potential operator $R_{\gamma}^{(k)*}$ is not bounded from $L_{(Q+k)/\alpha,\gamma}(\mathbb{R}^{n})$ to $L_{\infty,\gamma}(\mathbb{R}^{n}$. However, as its substitute, we know that $R_{\gamma}^{(k)*}$ is bounded from $L_{(Q+k)/\alpha,\gamma}(\mathbb{R}^{n})$ to $\mathrm{BMO}(\mathbb{R}^{n})$. From [5], we proved that for any given $0<\alpha<Q-\frac{n}{q}$, high order Riesz Bessel transform is bounded from $L_{Q-\frac{n}{q}}(\mathbb{R}^{n})$ to the space $\mathrm{BMO}(\mathbb{R}^{n})$ when $\Omega$ satisfies some smoothness conditions on $\mathbb{S}^{n-1}$. Meng and Chen [21] obtained that, under some smoothness assumptions on $\Omega$, for

any given $\alpha \in (0,n)$ and $\lambda \in (0,n)$, the Calderon Zygmund operator is bounded from the Morrey space $L^{\lambda/\alpha,\lambda}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$. Therefore, from [21], we obtain that

$$\|R_\gamma^{(k)} f\|_{\mathrm{BMO}(\mathbb{R}^n_+)} \leq C\|f\|_{L_{Q-\frac{n}{q}}(\mathbb{R}^n_+)}, \tag{10}$$

where the positive constant $C$ depends only on $n$ and $\gamma, \alpha, k$.

Therefore, an interesting question arises for the singular integral operator $R_\gamma^{(k)*}$ defined as in (7), which is generated by generalized shift operator $T^y$ for $y \in \mathbb{R}^n_+$ whether or not there exists an alternative inequality for (9) or (10) such that the boundedness constant appearing in such inequalities does not depend on the parameter $\alpha$. We can give a positive answer to this question for (9) on under $\Omega$ different assumptions. We should also note that Chen and Guo [2] obtained a boundedness estimate for the Calderón-Zygmund singular integral operator on $\Omega$ without any regularity assumption.

Finally, we have established some rules regarding representations. Throu-ghout the study, we will denote a positive constant $C$, that is independent of the main parameters. However, according to this fixed inequality, it may vary from one row to another. The symbol $f \lesssim g$ means $f \leq Cg$. If $f \lesssim g$ and $g \lesssim f$, then $f \sim g$ is written. For any subset $B \subset \mathbb{R}^n_+$, we denote the complement of $B^{\complement}$ and its characteristic function by $\mathbf{1}_B$ . For any given $q \in [1,\infty]$, the conjugate exponent is denoted by $q'$; $1/q + 1/q' = 1$.

## 2. **Main Results**

In order to present the main results of this study, we will first recall some necessary concepts. Firstly, we will recall the definition of the BMO space.

**Definition 2.1.** *Let $T^y$ be generalized shift operator for $y \in \mathbb{R}^n_+$, and $f \in L^1_{loc,\gamma}(\mathbb{R}^n_+)$. Then $f$ is said to belong to the space $\mathrm{BMO}(\mathbb{R}^n_+)$ if*

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n_+)} := \sup_{\substack{r>0 \\ x\in\mathbb{R}^n_+}} \frac{1}{|B(0,r)|_\nu} \int_{B(0,r)} \left|T^y f(x) - f_{B(0,r)}\right| y_n^\gamma dy < \infty, \tag{11}$$

*where the supremum is taken over all balls $B \subset \mathbb{R}^n_+$, and*

$$f_{B(0,r)} := \frac{1}{|B|_\gamma} \int_{B(0,r)} T^y f(x) y_n^\gamma dy,$$

*(see [7, 8]).*

The $\mathrm{BMO}(\mathbb{R}^n)$ space plays a very important role in harmonic analysis. It is well known that the $\mathrm{BMO}(\mathbb{R}^n)$ space is the dual of the Hardy space $H^1(\mathbb{R}^n)$. Furthermore, the $\mathrm{BMO}(\mathbb{R}^n)$ space is a suitable alternative to the Lebesgue space $L^\infty(\mathbb{R}^n)$ for examining the boundedness of certain operators or solving certain partial differential equations. It is also known that high order Riesz transforms are not bounded in Lebesgue spaces for $0 < p < 1$. This study will therefore facilitate the investigation of the boundedness of these transforms in Hardy spaces for $0 < p < 1$. It should be noted that the singular integral operators studied here have been obtained using generalized shift operators $T^y$ associated with the Laplace-Bessel operator.

Then, for the space $\mathrm{BMO}(\mathbb{R}^n_+)$ with respect to norm, we have the following boundedness estimate for higher order the Riesz Bessel transform $R_\gamma^{(k)*}$.

**Theorem 2.1.** *Let $1 < q < \infty$, $0 < \alpha < Q - \frac{n}{q}$, and $1 < q' < \infty$ be such that $\frac{1}{q} + \frac{1}{q'} = 1$. We assume that $\Omega \in L^{\infty}\big(\mathbb{S}^{n-1}\big)$ satisfies* (5), (6), *and* (8). *Then there exists a positive constants $C_1(Q,q)$ and $C_2 = \big(n - q(Q-\alpha)\big)^{-1/q} \alpha^{Q-\frac{n}{q}}$ such that*

$$\|R_{\gamma}^{(k)*} f\|_{\mathrm{BMO}(\mathbb{R}_{+}^{n})} \leq C_1 \big[\|f\|_{\mathrm{BMO}(\mathbb{R}_{+}^{n})} + C_2 \|f\|_{L_{q',\gamma}(\mathbb{R}_{+}^{n})}\big] \tag{12}$$

*for any $f \in \mathrm{BMO}(\mathbb{R}_{+}^{n}) \cap L_{q',\gamma}(\mathbb{R}_{+}^{n})$, $R_{\gamma}^{(k)*} f \in \mathrm{BMO}(\mathbb{R}_{+}^{n})$.*

*Proof.* To prove the theorem, we shall consider the approach used in [26, 27, 28] and split the singular integral operator $R_{\gamma}^{(k)*}$ into two parts. Let us denote the part near the origin as $\mathcal{R}_{\gamma_1}^{(k)}$ and the part far from the origin as $\mathcal{R}_{\gamma_2}^{(k)}$. Obtaining an estimate for $\mathcal{R}_{\gamma_2}^{(k)} f\|_{\mathrm{BMO}(\mathbb{R}_{+}^{n})}$ is straightforward. However, to estimate $\|\mathcal{R}_{\gamma_1}^{(k)} f\|_{\mathrm{BMO}(\mathbb{R}_{+}^{n})}$, we apply the Dini-type condition given in equation (3). Let $\Phi \in C_c^{\infty}(\mathbb{R})$ be a nonnegative, smooth and compactly supported function. Then, for any $t \in \mathbb{R}$, let us take

$$\Phi(t) := \begin{cases} 1 & \text{when } |t| \leq 1, \\ 0 & \text{when } |t| \geq 2. \end{cases}$$

For given $\lambda \in (0,\infty)$ and $t \in \mathbb{R}$. let $\Phi_{\lambda}(t) := \Phi(\lambda t)$. Furthermore, let us define

$$\mathcal{R}_{\gamma_1}^{(k)} f(x) := \ \text{p.v.} \int_{\mathbb{R}_{+}^{n}} \frac{\Omega(y)}{|y|^{Q-\alpha}} \Phi_{\alpha}(|y|) T^{y} f(x) y_n^{\gamma} dy \tag{13}$$

and

$$\mathcal{R}_{\gamma_2}^{(k)} f(x) := \ \text{p.v.} \int_{\mathbb{R}_{+}^{n}} \frac{\Omega(y)}{|y|^{Q-\alpha}} \big[1 - \Phi_{\alpha}(|y|)\big] T^{y} f(x) y_n^{\gamma} dy \tag{14}$$

for a suitable function $f$ on $\mathbb{R}_{+}^{n}$ and any $y \in \mathbb{R}_{+}^{n}$. Then the singular integral operator $R_{\gamma}^{(k)*}$ in (7) can be written as $R_{\gamma}^{(k)*} = \mathcal{R}_{\gamma_1}^{(k)} + \mathcal{R}_{\gamma_2}^{(k)}$. Hence, we obtain the following boundedness results for the operators $\mathcal{R}_{\gamma_1}^{(k)}$ and $\mathcal{R}_{\gamma_2}^{(k)}$. ☐

**Lemma 2.1.** *Let $1 < q < \infty$, $0 < \alpha < Q - \frac{n}{q}$, and $\Omega \in L^{\infty}\big(\mathbb{S}^{n-1}\big)$ satisfies* (5), (6) *and* (3). *There exists a positive constant $C(Q,q)$ such that*

$$\big\|\mathcal{R}_{\gamma_1}^{(k)} f\big\|_{\mathrm{BMO}(\mathbb{R}_{+}^{n})} \leq C \|f\|_{\mathrm{BMO}(\mathbb{R}_{+}^{n})}$$

*for any $f \in \mathrm{BMO}(\mathbb{R}_{+}^{n}) \cap L_{q',\gamma}(\mathbb{R}_{+}^{n})$ and $\mathcal{R}_{\gamma_1}^{(k)} f \in \mathrm{BMO}(\mathbb{R}_{+}^{n})$.*

**Lemma 2.2.** *Let $1 \leq p < \infty$ and any $B := B(x,r) \subset \mathbb{R}_{+}^{n}$ be a ball. There exists a positive constant $C(Q,q)$ such that the inequality*

$$\left[\frac{1}{2^k |B|_{\gamma}} \int_{2^k B} |T^y f(x) - f_B|^p y_n^{\gamma} dy\right]^{\frac{1}{p}} \leq C \|f\|_{\mathrm{BMO}(\mathbb{R}_{+}^{n})}$$

*holds for any $f \in \mathrm{BMO}(\mathbb{R}_{+}^{n})$ and $k \in \mathbb{N}$.*

Now, we prove Lemma 2.1 using Lemma 2.2.

*Proof of Lemma 2.1.* Let $1 < q < \infty$, $0 < \alpha < Q - \frac{n}{q}$, and $f \in \mathrm{BMO}(\mathbb{R}_{+}^{n}) \cap L_{q',\gamma}(\mathbb{R}_{+}^{n})$. By [2, Theorem 2.1], we obtain the result that $\mathcal{R}_{\gamma_1}^{(k)}$ is bounded on $L_{q',\gamma}(\mathbb{R}_{+}^{n})$. This implies that $\mathcal{R}_{\gamma_1}^{(k)} f \in L_{q',\gamma}(\mathbb{R}_{+}^{n})$. Therefore, for almost every $x \in \mathbb{R}_{+}^{n}$, we obtain the result that $|\mathcal{R}_{\gamma_1}^{(k)} f(x)| < \infty$. Without loss of generality, we may assume that

$$|\mathcal{R}_{\gamma_1}^{(k)} f(x)| < \infty$$

for any $x \in \mathbb{R}^n_+$. Let $B := B\big(x_0, r\big)$ be an arbitrary ball in $\mathbb{R}^n_+$. Then we get

$$\begin{aligned} T^y f &= f_B + \big(T^y f - f_B\big)\mathbf{1}_{4B} + \big(T^y f - f_B\big)\mathbf{1}_{(4B)^{\mathrm{c}}} \\ &= f_B + g_B + h_B, \end{aligned}$$

where

$$f_B := \int_B T^y f(x) y_n^\gamma dy,$$

and $\mathbf{1}_{(4B)^{\mathrm{c}}}$ is the characteristic function of $(4B)^{\mathrm{c}}$.

Since $f_B$ is a constant, it follows that $\mathcal{R}^{(k)}_{\gamma_1} f_B = 0$. Now, we obtain the $\mathcal{R}^{(k)}_{\gamma_1} g_B$ estimate. By Hölder's inequality and the boundedness of the operator $\mathcal{R}^{(k)}_{\gamma_1}$ on $L_{2,\gamma}(\mathbb{R}^n_+)$ (see [6]), we have

$$\begin{aligned} \int_B \big|\mathcal{R}^{(k)}_{\gamma_1} g_B(y)\big| dy &\le |B|^{\frac{1}{2}} \Big[\int_B \big|\mathcal{R}^{(k)}_{\gamma_1} g_B(y)\big|^2 dy\Big]^{\frac{1}{2}} \le C|B|^{\frac{1}{2}} \big\|g_B\big\|_{L^2(\mathbb{R}^n_+)} \\ &= C|B|^{\frac{1}{2}} \Big[\int_{4B} \big|T^y f(x) - f_B\big|^2 y_n^\gamma dy\Big]^{\frac{1}{2}} \\ &\le C|B|^{\frac{1}{2}} \left\{ \Big[\int_{4B} \big|T^y f(x) - f_{4B}\big|^2 y_n^\gamma dy\Big]^{\frac{1}{2}} + \Big[\int_{4B} \big|f_{4B} - f_B\big|^2 y_n^\gamma dy\Big]^{\frac{1}{2}} \right\} \\ &= C|B|^{\frac{1}{2}} \big(\mathrm{I}_1 + \mathrm{I}_2\big). \end{aligned} \tag{15}$$

From Definition 2.1, it follows that

$$\mathrm{I}_1 = \Big[\int_{4B} \big|T^y f(x) - f_{4B}\big|^2 y_n^\gamma dy\Big]^{\frac{1}{2}} \le C|B|^{\frac{1}{2}} \|f\|_{\mathrm{BMO}(\mathbb{R}^n_+)} \tag{16}$$

and

$$\begin{aligned} \mathrm{I}_2 = \Big[\int_{4B} \big|f_{4B} - f_B\big|^2 y_n^\gamma dy\Big]^{\frac{1}{2}} &\le \Big[\frac{1}{|B|} \int_B \big|T^y f(x) - f_{4B}\big| y_n^\gamma dy\Big] |4B|^{\frac{1}{2}} \\ &\le C\Big[\frac{1}{|4B|} \int_{4B} \big|T^y f(x) - f_{4B}\big| y_n^\gamma dy\Big] |B|^{\frac{1}{2}} \le C|B|^{\frac{1}{2}} \|f\|_{\mathrm{BMO}(\mathbb{R}^n_+)}. \end{aligned} \tag{17}$$

By (15)-(17), we get

$$\int_B \big|\mathcal{R}^{(k)}_{\gamma_1} g_B(y)\big| dy \le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n_+)}. \tag{18}$$

Now let's study $\mathcal{R}_{\gamma_1}^{(k)} h_B$. Let $x \in B$. Then, we have

$$\begin{aligned}
&\mathcal{R}_{\gamma_1}^{(k)} h_B(x) - \mathcal{R}_{\gamma_1}^{(k)} h_B(x_0) = \\
&= \int_{(\mathbb{R}_+^n)} \Big[\frac{T^y\Omega(x)}{|y|^{Q-\alpha}} - \frac{T^y\Omega(x_0)}{|y|^{Q-\alpha}}\Big]\Phi_\alpha(|y|) h_B(y) y_n^\gamma dy \\
&= \int_{(\mathbb{R}_+^n)} \big[\Omega(y) T^y |x|^{-Q+\alpha} - \Omega(y) T^y |x_0|^{-Q+\alpha}\big]\Phi_\alpha(|y|) h_B(y) y_n^\gamma dy \\
&= \int_{(\mathbb{R}_+^n)} \Omega(y)\big[T^y |x|^{-Q+\alpha} - T^y |x_0|^{-Q+\alpha}\big] h_B(y)\Phi_\alpha(|y|) y_n^\gamma dy \\
&= \int_{(\mathbb{R}_+^n)} \Omega(y)\big[|x-y|^{-Q+\alpha} - |x_0-y|^{-Q+\alpha}\big] T^y h_B(x)\Phi_\alpha(|y|) y_n^\gamma dy \\
&\le \int_{(\mathbb{R}_+^n)} \Omega(y)\big[|y|^{-Q+\alpha}\big] T^y h_B(x)\Phi_\alpha(|y|) y_n^\gamma dy \\
&\le \int_{(4B)^{\complement}} \frac{\Omega(y)}{|y|^{Q-\alpha}} T^y h_B(x)\Phi_\alpha(|y|) y_n^\gamma dy \\
&\le \int_{(4B)^{\complement}} \frac{\Omega(y)}{|y|^{Q-\alpha}} \big[T^y f(x) - f_B\big]\Phi_\alpha(|y|) y_n^\gamma dy \\
&+ \int_{(4B)^{\complement}} \frac{\Omega(y)}{|y|^{Q-\alpha}} \big[f_B - f_{(4B)^{\complement}}\big]\big[\Phi_\alpha(|y|)\big] y_n^\gamma dy \\
&=: J_1 + J_2.
\end{aligned}$$

Furthermore, based on the fact that $\lim_{\alpha\to 0^+} \alpha^{-\alpha} = 1$, we obtain that there exists a constant $C > 0$ such that, for any given $0 < \alpha < Q - \frac{n}{q}$,

$$\alpha^{-\alpha} \le C \tag{19}$$

Now let's obtain the $J_1$ estimate. For any $j \in \mathbb{N}$ with $j \ge 2$, let $A_j := \big(2^{j+1}B\big)\backslash\big(2^j B\big)$. By Hölder's inequality and (2.8), we have that

$$\begin{aligned}
\big|J_1\big| &\le C\Big[\int_{(4B)^{\complement}} \Big|\frac{\Omega(y)}{|y|^{Q-\alpha}}\Big|^{\frac{Q}{Q-\alpha}} \big|T^y f(x) - f_B\big|^{\frac{Q}{Q-\alpha}} y_n^\gamma dy\Big]^{1-\frac{\alpha}{Q}} \\
&\quad\times \Big[\int_{(4B)^{\complement}} \big|\Phi_\alpha(|y|)\big|^{\frac{Q}{\alpha}} y_n^\gamma dy\Big]^{\frac{\alpha}{Q}} \\
&\le C_\gamma\Big[\sum_{j=2}^{\infty}\int \Big|\frac{\Omega(y)}{|y|^{Q-\alpha}}\Big|^{\frac{Q}{Q-\alpha}} \big|T^y f(x) - f_B\big|^{\frac{Q}{Q-\alpha}} y_n^\gamma dy\Big]^{1-\frac{\alpha}{Q}} \\
&\le C_\gamma\Big[\sum_{j=2}^{\infty}\int \Big|\frac{\Omega(y)}{|y|^{Q-\alpha}}\Big|^{\frac{Q}{Q-\alpha}} \big|T^y f(x) - f_B\big|^{\frac{Q}{Q-\alpha}} y_n^\gamma dy\Big]^{1-\frac{\alpha}{Q}}.
\end{aligned} \tag{20}$$

Moreover, for $y \in B$ and any given $x \in A_j$ with $j \in \mathbb{N}$, in [6], by Theorem 3 and 4, we obtain that there exists $y' \in B$ such that

$$\Big|\frac{\Omega(y)}{|y|^{Q-\alpha}}\Big| \le \frac{A}{|y|^{Q-\alpha}}. \tag{21}$$

Furthermore, it is easy to find that, for any $y \in B$ and $x \in A_j$ with $j \ge 1$,

$$|x|/4 \le |y|, \tag{22}$$

and

$$|x-y| \geq 4|y|. \tag{23}$$

Thus, by (21), (23) and $|y| \leq r$, for any $x \in A_j$ with $j \geq 2$, it follows that

$$\begin{aligned}\Big|\frac{\Omega(y)}{|y|^{Q-\alpha}}\Big|^{\frac{Q}{Q-\alpha}} &\leq \Big[\frac{A.C_\alpha}{|y|^{Q-\alpha}}\Big]^{\frac{Q}{Q-\alpha}} \leq \Big[\frac{A.C_\alpha}{\big(2^j r\big)^{Q-\alpha}}\Big]^{\frac{Q}{Q-\alpha}} = \frac{A.C_{\alpha,Q}}{\big(2^j r\big)^{Q}} \\ &\leq \frac{A.C_{\alpha,Q}}{(2^j)^Q r^Q} \leq \frac{A.C_{\alpha,Q} 2^{-j}}{\big|2^{j+1}B\big|}.\end{aligned}$$

Using previous estimate and Lemma 2.2, we find that

$$\begin{aligned}&\sum_{j=2}^{\infty}\int_{A_j}\Big|\frac{\Omega(y)}{|y|^{Q-\alpha}}\Big|^{\frac{Q}{Q-\alpha}}\big|T^y f(x) - f_B\big|^{\frac{Q}{Q-\alpha}} y_n^\gamma dy \\ &\leq C_{\alpha,Q}\sum_{j=2}^{\infty}\frac{2^{-j}}{\big|2^{j+1}B\big|}\int_{2^{j+1}B}\big|T^y f(x) - f_B\big|^{\frac{Q}{Q-\alpha}} y_n^\gamma dy \\ &\leq C_{\alpha,Q}\sum_{j=2}^{\infty}2^{-j}(j+1)^{\frac{Q}{Q-\alpha}}\|f\|_{\mathrm{BMO}(\mathbb{R}^n_+)}^{\frac{Q}{Q-\alpha}}.\end{aligned} \tag{24}$$

By previous estimate and (20), we conclude that

$$\big|J_1\big| \leq C\|f\|_{\mathrm{BMO}(\mathbb{R}^n_+)}\Big[\sum_{j=2}^{\infty}2^{-j}(j+1)^{\frac{Q}{Q-\alpha}}\Big]^{1-\frac{\alpha}{Q}} \leq C\|f\|_{\mathrm{BMO}(\mathbb{R}^n_+)}. \tag{25}$$

Now, we will try to obtain $J_2$. By Hölder's inequality and (19), we have

$$\begin{aligned}\big|J_2\big| &\leq C\Big[\int_{(4B)^\complement}\frac{|\Omega(y)|^{\frac{Q}{Q-\alpha}}}{|y|^Q}|f_B - f_{(4B)^\complement}|^{\frac{Q}{Q-\alpha}} y_n^\gamma dy\Big]^{1-\frac{\alpha}{Q}}\Big[\int_{(4B)^\complement}|\Phi_\alpha(|y|)|^{\frac{Q}{\alpha}} y_n^\gamma dy\Big]^{\frac{\alpha}{Q}} \\ &\leq C_{\alpha,Q}\Big[\int_{(4B)^\complement}\frac{|\Omega(y)|^{\frac{Q}{Q-\alpha}}}{|y|^Q}|f_B - f_{(4B)^\complement}|^{\frac{Q}{Q-\alpha}} y_n^\gamma dy\Big]^{1-\frac{\alpha}{Q}}.\end{aligned} \tag{26}$$

Similarly to (21), in [6], by Theorem 3 and 4, we find that, for $y \in B$, there exists a constant $C$ such that

$$\Big|\frac{\Omega(y)}{|y|}\Big| \leq \frac{A}{|y|}.$$

If our previous estimate and (23) holds, for any given $y \in B$, and $x \in A_j$ with $j \geq 2$, then we see easily that

$$\big|\Omega(y)\big| = \big|\Omega\big(\frac{y}{|y|}\big)\big| \leq \omega\big(\frac{y}{|y|}\big) \leq \omega\big(\frac{16r}{|y|}\big) \leq \omega\big(\frac{16}{2^{j-1}}\big). \tag{27}$$

By (27) and Lemma 2.2, we have

$$
\begin{aligned}
&\Big[\int_{(4B)^{\complement}} \frac{\big|\Omega(y)\big|^{\frac{Q}{Q-\alpha}}}{|y|^{Q}} \big|f_{(4B)^{\complement}} - f_B\big|^{\frac{Q}{Q-\alpha}} y_n^{\gamma} dy\Big]^{1-\frac{\alpha}{Q}} \\
&= \Big[\sum_{j=2}^{\infty} \int_{A_j} \frac{\big|\Omega(y)\big|^{\frac{Q}{Q-\alpha}}}{|y|^{Q}} \big|f_{(4B)^{\complement}} - f_B\big|^{\frac{Q}{Q-\alpha}} y_n^{\gamma} dy\Big]^{1-\frac{\alpha}{Q}} \\
&\le C_{\alpha,Q} \sum_{j=2}^{\infty} \omega\big(\frac{16}{2^{j-1}}\big)\Big[\int_{A_j} \frac{1}{|y|^{Q+k}} \big|T^y f(x) - f_B\big|^{\frac{Q}{Q-\alpha}} y_n^{\gamma} dy\Big]^{1-\frac{\alpha}{Q}} \\
&\le C_{\alpha,Q} \sum_{j=2}^{\infty} \omega\big(\frac{16}{2^{j-1}}\big)\Big[\int_{2^{j+1}B} \big|T^y f(x) - f_B\big|^{\frac{Q}{Q-\alpha}} y_n^{\gamma} dy\Big]^{1-\frac{\alpha}{Q}} \\
&\le C_{\alpha,Q} \sum_{j=2}^{\infty} \omega\big(\frac{16}{2^{j-1}}\big)(j+1)\|f\|_{\mathrm{BMO}(\mathbb{R}_{+}^{n})}. \qquad (28)
\end{aligned}
$$

If we now combine the estimates (25), (27), and (28), we have

$$\big|J_2\big| \le C\|f\|_{\mathrm{BMO}(\mathbb{R}_{+}^{n})}. \qquad (29)$$

$$
\begin{aligned}
\int_B \big|\mathcal{R}_{\gamma_1}^{(k)} f(y) - \mathcal{R}_{\gamma_1}^{(k)} h_B\big(x_0\big)\big| dy &\le \int_B \big|\mathcal{R}_{\gamma_1}^{(k)} g_B(y)\big| dy \\
&\quad + \int_B \big|\mathcal{R}_{\gamma_1}^{(k)} h_B(x) - \mathcal{R}_{\gamma_1}^{(k)} h_B\big(y_0\big)\big| dy \\
&\le C\|f\|_{\mathrm{BMO}(\mathbb{R}_{+}^{n})}. \qquad (30)
\end{aligned}
$$

Thus we conclude that

$$
\begin{aligned}
\int_B \big|\mathcal{R}_{\gamma_1}^{(k)} f(y) - \big(\mathcal{R}_{\gamma_1}^{(k)} f\big)_B\big| dy &\le 2\int_B \big|\mathcal{R}_{\gamma_1}^{(k)} f(y) - \mathcal{R}_{\gamma_1}^{(k)} h_B\big(x_0\big)\big| dy \\
&\le C\|f\|_{\mathrm{BMO}(\mathbb{R}_{+}^{n})}.
\end{aligned}
$$

Finally, we obtain $\mathcal{R}_{\gamma_1}^{(k)} f \in \mathrm{BMO}(\mathbb{R}_{+}^{n})$ and

$$\big\|\mathcal{R}_{\gamma_1}^{(k)} f\big\|_{\mathrm{BMO}(\mathbb{R}_{+}^{n})} \le C\|f\|_{\mathrm{BMO}(\mathbb{R}_{+}^{n})}.$$

Thus, Lemma 2.1 is proved.

□

**Lemma 2.3.** *Let* $1 < q < \infty$, $0 < \alpha < Q - \frac{n}{q}$, *and suppose that* $\Omega \in L^{\infty}\big(\mathbb{S}^{n-1}\big)$ *satisfies* (5) *and* (6). *Then there exists a positive constant* $C(Q,q)$ *such that,*

$$\big\|\mathcal{R}_{\gamma_2}^{(k)} f\big\|_{\mathrm{BMO}(\mathbb{R}_{+}^{n})} \le C\alpha^{Q-\frac{n}{q}}\big[n - q(Q-\alpha)\big]^{-1/q}\|f\|_{L_{q',\gamma}(\mathbb{R}_{+}^{n})} \qquad (31)$$

*for any* $f \in \mathrm{BMO}(\mathbb{R}_{+}^{n}) \cap L_{q',\gamma}(\mathbb{R}_{+}^{n})$, $\mathcal{R}_{\gamma_2}^{(k)} f \in \mathrm{BMO}(\mathbb{R}_{+}^{n})$.

*Proof.* Let $f \in \mathrm{BMO}(\mathbb{R}^n_+) \cap L_{q',\gamma}(\mathbb{R}^n_+)$. First assume that $\Omega \in L^\infty\big(\mathbb{S}^{n-1}\big)$ and (19), Then we have, for any ball $B \subset (\mathbb{R}^n_+)$,

$$\begin{aligned}
&\frac{1}{|B|}\int_B \big|\mathcal{R}^{(k)}_{\gamma_2} f(x) - \big(\mathcal{R}^{(k)}_{\gamma_2} f\big)_B\big| dx \le \frac{2}{|B|}\int_B \big|\mathcal{R}^{(k)}_{\gamma_2} f(x)\big| dx \\
&\le \frac{C}{|B|}\int_B \int_{\left\{y\in(\mathbb{R}^n_+):|y|>\frac{1}{\alpha}\right\}} \frac{|\Omega(y)|}{|y|^{(Q-\alpha)}}\big[1-\Phi_\alpha(|y|)\big] T^y |f(y)| y_n^\gamma dy dx \\
&\le \frac{C}{|B|}\int_B \Big[\int_{(\mathbb{R}^n_+)} |T^y f(x)|^{q'} y_n^\gamma dy\Big]^{\frac{1}{q'}} \Big[\int_{\left\{y\in(\mathbb{R}^n_+):|y|>\frac{1}{\alpha}\right\}} \frac{|\Omega(y)|^q}{|y|^{q(Q-\alpha)}}\big[1-\Phi_\alpha(|y|)\big]^q dy\Big]^{\frac{1}{q}} dx \\
&\le \frac{C_{Q,q}}{|B|}\int_B \Big[\int_{(\mathbb{R}^n_+)} T^y |f(x))|^{q'} y_n^\gamma dy\Big]^{\frac{1}{q'}} \Big[\int_{\mathbb{S}^{n-1}}\int_{\frac{1}{\alpha}}^\infty t^{-q(Q-\alpha)} t^{n-1} dt d\sigma\big(t'\big)\Big]^{\frac{1}{q}} dx \\
&\le \frac{C_{Q,q}}{|B|}\int_B \Big[\int_{(\mathbb{R}^n_+)} |f(y))|^{q'} y_n^\gamma dy\Big]^{\frac{1}{q'}} \Big[\int_{\mathbb{S}^{n-1}}\int_{\frac{1}{\alpha}}^\infty t^{-q(Q-\alpha)} t^{n-1} dt d\sigma\big(t'\big)\Big]^{\frac{1}{q}} dx \\
&\le \frac{C_{Q,q}}{|B|}\|f\|_{L_{q',\gamma}(\mathbb{R}^n_+)} \alpha^{Q-\frac{n}{q}} \big[n-q(Q-\alpha)\big]^{-1/q} \alpha^{-\alpha}|B| \\
&\le C_{Q,q}\alpha^{Q-\frac{n}{q}} \big[n-q(Q-\alpha)\big]^{-1/q} \|f\|_{L_{q',\gamma}(\mathbb{R}^n_+)}
\end{aligned}$$

which implies that $\mathcal{R}^{(k)}_{\gamma_2} f \in \mathrm{BMO}(\mathbb{R}^n_+)$ and

$$\big\|\mathcal{R}^{(k)}_{\gamma_2} f\big\|_{\mathrm{BMO}(\mathbb{R}^n_+)} \le C\alpha^{Q-\frac{n}{q}} \big[n-q(Q-\alpha)\big]^{-1/q} \|f\|_{L_{q',\gamma}(\mathbb{R}^n_+)},$$

where the constant $C$ is independent of $f$ and $\alpha$. This finishes the proof of Lemma 2.3. □

By Lemmas 2.1 and 2.2, it is easy to prove the Theorem 2.1

*Proof of Theorem 2.1.* Let $1 < q < \infty$ and $0 < \alpha < Q - \frac{n}{q}$. Assume that $f \in \mathrm{BMO}(\mathbb{R}^n_+) \cap L_{q',\gamma}(\mathbb{R}^n_+)$. Using Lemma 2.1, we deduce that that there exits a constant $C$ such that

$$\big\|\mathcal{R}^{(k)}_{\gamma_1} f\big\|_{\mathrm{BMO}(\mathbb{R}^n_+)} \le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n_+)}.$$

Moreover, by Lemma 2.3, we see easily that there exists a constant $C$ such that

$$\big\|\mathcal{R}^{(k)}_{\gamma_2} f\big\|_{\mathrm{BMO}(\mathbb{R}^n_+)} \le C\alpha^{Q-\frac{n}{q}} \big[n-q(Q-\alpha)\big]^{-1/q} \|f\|_{L_{q',\gamma}(\mathbb{R}^n_+)},$$

where $C$ is finite constant that are independent of $f$ and that depend only on only on $q$, $Q$, and $A$. Therefore, we have

$$\begin{aligned}
\|R^{(k)}_{\gamma,\alpha} f\|_{\mathrm{BMO}(\mathbb{R}^n_+)} &\le \big\|\mathcal{R}^{(k)}_{\gamma_1} f\big\|_{\mathrm{BMO}(\mathbb{R}^n_+)} + \big\|\mathcal{R}^{(k)}_{\gamma_2} f\big\|_{\mathrm{BMO}(\mathbb{R}^n_+)} \\
&\le C\big[\|f\|_{\mathrm{BMO}(\mathbb{R}^n_+)} + \alpha^{Q-\frac{n}{q}} \big[n-q(Q-\alpha)\big]^{-1/q} \|f\|_{L_{q',\gamma}(\mathbb{R}^n_+)}\big].
\end{aligned}$$

as desired. □

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