A NONLINEAR ELLIPTIC SYSTEM WITH THE SIGN-CHANGING LOGARITHMIC NONLINEARITIES

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In the present paper, we consider the multiple solutions for a class of nonlinear elliptic system with the sign-changing logarithmic. Under some conditions we obtain at least two nontrivial solutions by logarithmic Sobolov inequality and analysis of the relationship between the Nehari manifold and fibering maps.

Keywords: Nehari manifold, minimizer, fibering maps, critical point, logarithmic nonlinearities.
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1. **Introduction**

In this paper we shall discuss the existence of least two nontrivial solution of the nonlinear elliptic with the sign-changing logarithmic nonlinearities boundary-value problem

$$\begin{cases} -\Delta u = a(x)\, u\, \log|u| + \dfrac{\alpha}{\alpha+\beta}\, |u|^{\alpha-2}\, u\, |v|^{\beta} & in\,\Omega, \\ -\Delta v = b(x)\, v\, \log|v| + \dfrac{\alpha}{\alpha+\beta}\, |u|^{\alpha}\, v\, |v|^{\beta-2} & in\,\Omega, \\ \ \ u \equiv v \equiv 0, & on\ \partial\Omega, \end{cases} \tag{1}$$

Where Ω is a bounded region with smooth boundary in $\mathbb{R}^n$. $\alpha > 1$, $\beta > 1$ satisfying
$2 < \alpha+\beta < 2^*$; ($2^* = \frac{2n}{n\_2}$ if $n \geq 3$, $2^* \underline{=} \infty$ if $n = 2$) and $a,\, b : \overline{\Omega} \to \mathbb{R}$ are smooth functions which change sign in Ω.
In the past years, there have been increasing interests in studying logarithmic nonlinearity due to its relevance in quantum mechanics, quantum optics,

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nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity and Bose-Einstein consideration (see [13] and the reference therein).
In a recent paper [2], Tian studied the multiple solutions for a semi-linear elliptic equation on a bounded domain with the sign-changing logarithmic nonlinearity.
Namely she prove that the following problem

$$\begin{cases} -\Delta u = a(x)\, u \,\log|u| & in\, \Omega, \\ u \equiv 0, & on\ \partial\Omega, \end{cases} \tag{2}$$

has at least two nontrivial solutions provided that $a(x) \in c(\underline{\Omega})$ changes on Ω, and

$$\max_{\overline{\Omega}} |a(x)| < 2\pi \exp(2 - \frac{4|\Omega|n}{ne}) \tag{3}$$

Tian's results are quite different from these in the polynomial nonlinearities case, see [4, 5, 15]. It's proof is based on the consideration of the Nehari's manifold associated with the energy function and the using logarithmic Sobolev's inequality.
The logarithmic Schrodinger equation given by

$$- i\,\frac{\partial\psi}{\partial t} = -\triangle\psi + (w(x) + w)\psi - |\psi|^{p-1}\log|\psi|,$$
$$\psi : [0, +\infty) \times \mathbb{R}^N \to \mathbb{C},\ N \geq 1,$$

has also received a special attention. For the elliptic equation with logarithmic nonlinearity, we can refer to [1, 7, 8, 9, 10, 12, 14] and the references therein. The authors in [1], considered the following logarithmic elliptic equations of the type

$$\begin{cases} -\Delta u + u = u\,\log u^2 & in\, \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), & on\ \partial\Omega, \end{cases}$$

the author's obtained solutions for this equation by applying the non-smooth critical point theory.
Also recently, a great deal of attention has been focused on studying of problems involving fractional Sobolev spaces and corresponding nonlocal equations, both from a pure mathematical point of view and for cocreate, applications, since they naturally arise in many different contexts, such as, among the others, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves. For more details, we can see [2, 5, 6].
Our main result can be stated as follows:

Theorem 1.1. *Let $a(x), b(x) \in C(\overline{\Omega})$ and changes in $\overline{\Omega}$ satisfying*

$$\max_{\overline{\Omega}} |a(x)| < 2\pi.\exp(2-\frac{4|\Omega|n}{ne}) \quad , \quad \max_{\overline{\Omega}} |b(x)| < 2\pi.\exp(2-\frac{4|\Omega|n}{ne}), \tag{4}$$

where $|\Omega|_n$ is the volume of Ω in $\mathbb{R}^n$. Then (1) *possesses at last two nontrivial solutions.*

This paper is organized as follows:
In section 2, we obtain the inequalities and some estimates which will be used to prove. Our main theorem. In section 3, we discuss the Nehari manifold and Xamine carefully and the connection between the Nehari manifold and the fibering maps. In section 4, we analysis of the fibering maps. In section 5, we proof of the main result. Also throughout this paper we will note :

$$\begin{aligned}\|u,v\|_{H_0^1(\Omega)\times H_0^1(\Omega)} &= \big(\|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\big)^{\frac{1}{2}},\\ &= \Big(\int_\Omega |\nabla u|^{\frac{1}{2}} dx + \int_\Omega |\nabla v|^{\frac{1}{2}} dx\Big)^{\frac{1}{2}}.\end{aligned}$$

2. **Some priliminary results**

Proposition 2.1. ***(Logarithmic Sobolev equality*** [11]***)*** *: Let $u(x)$ be any function in $H_0^1(\mathbb{R}^n)$, we have*

$$2\int_{\mathbb{R}^n} |u(x)|^2 \log \frac{|u(x)|}{\|u\|_{L^2_{\mathbb{R}^n}}} dx + n(1+\log c)\|u\|^2_{L^2(\mathbb{R}^n)} \le \frac{c^2}{\pi}\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \tag{5}$$

where $c = e^{\frac{2|\Omega|_n}{ne}-1}$. For $u(x) \in H_0^1(\Omega)$, we can define $u(x) = 0$ for $x \in \mathbb{R}^n - \Omega$ then it holds;

$$2\int_{\Omega} |u(x)|^2 \log \frac{|u(x)|}{\|u\|_{L^2_{\mathbb{R}^n}}} dx + n(1+\log c)\|u\|^2_{L^2(\mathbb{R}^n)} \le \frac{c^2}{\pi}\int_{\Omega} |\nabla u(x)|^2 dx. \tag{6}$$

Next, we define the energy functional of problem (1)*, $J(u,v) : E \to \mathbb{R}$:*

$$\begin{aligned} J(u,v) = \frac{1}{2}\|(u,v)\|^2 &- \frac{1}{2}\int_\Omega a(x)u^2 \log|u| dx - \frac{1}{2}\int_\Omega b(x)v^2 \log|v| dx \\ &+ \frac{1}{4}\int_\Omega a(x)u^2\, dx + \frac{1}{4}\int_\Omega b(x)v^2 dx - \frac{1}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^\beta dx, \end{aligned} \tag{7}$$

for $(u,v) \in E$. By a direct calculation we have that, $J(u,v) \in C^1(E,R)$.

Lemma 2.1. *For $u, v \in H_0^1(\Omega)$ and $\int_\Omega a(x)u^2\, dx = 0$ and $\int_\Omega b(x)v^2\, dx = 0$, let*

$$M = \max_{\overline{\Omega}} |a(x)| \quad \textit{and} \quad N = \max_{\overline{\Omega}} |b(x)|,$$

Then it holds :

$$J(u,v) \geq \Big(\frac{1}{2} - \frac{M^{\frac{4|\Omega|n}{ne}} - 2}{4\pi} - \frac{1}{(\alpha+\beta)\, S^{\alpha+\beta}}\Big)\|(u,v)\|^2_{L^2(\Omega)\times L^2(\Omega)}. \tag{8}$$

Proof. Using the fact $\int_\Omega a(x)u^2\,dx = 0$ and $\int_\Omega b(x)v^2\,dx$, we have

$$\begin{aligned} J(u,v) &= \frac{1}{2}\|(u,v)\|^2 \frac{1}{2}\|(u,v)\|^2 - \frac{1}{2}\int_\Omega a(x)u^2 \log|u|dx \\ &- \frac{1}{2}\int_\Omega b(x)v^2 \log|v|dx - \frac{1}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^\beta dx \end{aligned} \tag{9}$$

Let $\overline{u}(x) = \dfrac{u(x)}{\|u\|_{L^2(\Omega)}}$ and $\overline{v}(x) = \dfrac{v(x)}{\|v\|_{L^2(\Omega)}}$, then

$$\begin{aligned} &\int_\Omega a(x)u^2 \log\frac{u(x)}{\|u\|_{L^2(\Omega)}}dx + \int_\Omega b(x)v^2 \log\frac{v(x)}{\|v\|_{L^2(\Omega)}}dx \\ &= \int_{\Omega_1} a(x)u^2 \log\frac{u(x)}{\|u\|_{L^2(\Omega)}}dx + \int_{\Omega_1} b(x)v^2 \log\frac{v(x)}{\|v\|_{L^2(\Omega)}}dx \\ &+ \int_{\Omega_2} a(x)u^2 \log\frac{u(x)}{\|u\|_{L^2(\Omega)}}dx + \int_{\Omega_2} b(x)v^2 \log\frac{v(x)}{\|v\|_{L^2(\Omega)}}dx, \end{aligned} \tag{10}$$

Where $\Omega_1 = \{x \in \Omega, |\overline{u}(x)| < 1\}$,and $\Omega_1 = \{x \in \Omega, |\overline{u}(x)| < 1\}$, By direct calculations, we know :

$$\begin{aligned} \int_{\Omega_1} a(x)u^2 \log|\overline{u}(x)|dx + \int_{\Omega_1} b(x)v^2 \log|\overline{v}(x)|dx &\leq \frac{|\Omega|_n M}{2e}\|u\|_{L^2(\Omega)} \\ &+ \frac{|\Omega|_n N}{2e}\|v\|_{L^2(\Omega)} \end{aligned} \tag{11}$$

Also, by logarithmic Sobolev inequality and (4), (11), we have :

$$\begin{aligned} &\int_{\Omega_2} a(x)u^2 \log|\overline{u}(x)|dx + \int_{\Omega_2} b(x)v^2 \log|\overline{v}(x)|dx \\ &\leq M\left(\frac{c^2}{2\pi}\int_\Omega |\nabla u|^2 dx - \frac{n}{2}(1+\log c)\|u\|^2_{L^2(\Omega)} - \frac{|\Omega|_n}{2e}\|u\|^2_{L^2(\Omega)}\right) \\ &+ N\left(\frac{c^2}{2\pi}\int_\Omega |\nabla v|^2 dx - \frac{n}{2}(1+\log c)\|v\|^2_{L^2(\Omega)} - \frac{|\Omega|_n}{2e}\|v\|^2_{L^2(\Omega)}\right) \end{aligned} \tag{12}$$

The combining (9), (10), (11), (12) and $c = e^{\frac{2|\Omega|_n}{ne}-1}$, we have

$$J(u,v) \geq \left(\frac{1}{2} - \frac{Mc^2}{4\pi}\right)\|u\|^2_{L^2(\Omega)} + \left(\frac{1}{2} - \frac{Nc^2}{4\pi}\right)\|v\|^2_{L^2(\Omega)} - \frac{1}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^\beta dx,$$

also, by Sobolov embedding theorem and Young's inequality that there exist positive constants such that,

$$J(u,v) \geq \left(\frac{1}{2} - \frac{Mc^2}{4\pi} - \frac{1}{(\alpha+\beta)\, S^{\alpha+\beta}}\right) \|(u,v)\|^2_{L^2(\Omega)\times L^2(\Omega)}.$$

□

3. Fibering maps and Nehari manifold

In many problems such as (1) Euler functional is not bounded below on X, but it is bounded below on an appropriate subset of and a minimizer on the set (if it exist), may give rise to solutions of the corresponding differential equation. When J is bounded below on X. J has a minimizer on X which is a weak soluion of (1). A 'good coordinate' for an appropriate subset of X is called Nehari manifold, which is defined by

$$\mathcal{N} = \Big\{ (u,v) \in E - \{0,0\},\ \langle J'(u,v),(u,v)\rangle = 0 \Big\}$$

where $\langle . \rangle$ denotes the usual duality between X and X^*. It follows from (8) J is coercive and bounded below on $\mathcal{N}$. Thus, $(u,v) \in \mathcal{N}$ if and only if

$$\begin{aligned} \Big\langle J'(u,v),(u,v) \Big\rangle &= \|(u,v)\|^2_H - \int_\Omega (a\, u^2 \log|u| + b\, v^2 \log|v|) dx \\ &- \int_\Omega |u|^\alpha\, |v|^\beta dx \\ &= 0. \end{aligned}$$

Now we define $\varphi_{(u,v)} : \mathbb{R}^+ \to \mathbb{R}$, much known as fibering maps, as $\varphi_{(u,v)}(t) = J(tu,tv)$. For $(u,v) \in E$, we have :

$$\begin{aligned} \varphi_{(u,v)}(t) &= \frac{t^2}{2}\|(u,v)\|^2 - \frac{t^2}{2}\int_\Omega (a(x)\, u^2 \log|tu| + b(x)\, v^2 \log|tv|) dx \\ &- \frac{t^{\alpha+\beta}}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^\beta dx \end{aligned}$$

$$\begin{aligned} \varphi'_{(u,v)}(t) &= t\,\|(u,v)\|^2 - t\int_\Omega (a(x)\, u^2 \log|tu| + b(x)\, v^2 \log|tv|) dx \\ &- t^{\alpha+\beta-1}\int_\Omega |u|^\alpha |v|^\beta dx, \end{aligned} \tag{13}$$

$$\begin{aligned} \varphi''_{(u,v)}(t) &= \|(u,v)\|^2 - t\int_\Omega (a(x)\, u^2 \log|tu| + b(x)\, v^2 \log|tv|) dx \\ &- \int_\Omega (a\,u^2 + b\,v^2) dx - (\alpha+\beta-1)t^{\alpha+\beta-2}\int_\Omega |u|^\alpha |v|^\beta dx. \end{aligned} \tag{14}$$

Lemma 3.1. *Let* $(u,v) \in E - \{(0,0)\}$*, then* $(tu,tv) \in \mathcal{N}$ *if and only if* $\varphi'_{(u,v)}(t) = 0$*.*

Proof. The result is a consequence of the fact that $\varphi'_{(u,v)}(t) = \langle J'(tu,tv),(u,v)\rangle$. □

However, $(u,v) \in E$, if and only if, $\varphi'_{(u,v)}(1) = 0$, and

$$\varphi''_{(u,v)}(t) = -\int_\Omega (a\,u^2 + b\,v^2)dx - (\alpha+\beta-2)\int_\Omega |u|^\alpha |v|^\beta dx. \qquad (15)$$

Thus, it is natural to split $\mathcal{N}$ into three parts corresponding to local minima, local maxima and points of inflection. For tis, we set

$$\mathcal{N}^+ = \Big\{(u,v) \in \mathcal{N},\, \varphi''_{u,v}(1) > 0\Big\} = \Big\{(u,v) \in \mathcal{N},\, \int_\Omega a\,u^2dx < 0,\, \int_\Omega b\,v^2dx < 0\Big\},$$

$$\mathcal{N}^0 = \Big\{(u,v) \in \mathcal{N},\, \varphi''_{u,v}(1) = 0\Big\} = \Big\{(u,v) \in \mathcal{N},\, \int_\Omega a\,u^2dx < 0,\, \int_\Omega b\,v^2dx < 0\Big\},$$

$$\mathcal{N}^- = \Big\{(u,v) \in \mathcal{N},\, \varphi''_{u,v}(1) < 0\Big\} = \Big\{(u,v) \in \mathcal{N},\, \int_\Omega a\,u^2dx < 0,\, \int_\Omega b\,v^2dx < 0\Big\}.$$

Lemma 3.2. *Suppose that* (u_0,v_0) *is a local minimizer of* J *on* $\mathcal{N}$ *and* $(u_0,v_0) \notin \mathcal{N}^0$*, then* (u_0,v_0) *is a critical point of* J*.*

Proof. If (u_0,v_0) is a local minimizer of J on $\mathcal{N}$. by the theory of lagrange multipliers, there exists $\sigma \in \mathbb{R}$ such that $J'(u_0,v_0) = \sigma\delta'(u_0,v_0)$. Set $\delta(u_0,v_0) = \big\langle J'(u_0,v_0),(u_0,v_0)\big\rangle$, since $(u_0,v_0) \in \mathcal{N}$, then

$$\big\langle J'(u_0,v_0),(u_0,v_0)\big\rangle = \sigma\,\big\langle \delta'(u_0,v_0),(u_0,v_0)\big\rangle = 0,$$

On the other hand, from $u_0 \notin \mathcal{N}^0$, we can see

$$\big\langle \delta'(u_0,v_0),(u_0,v_0)\big\rangle = \varphi''_{(u_0,v_0)}(1) \neq 0,$$

Then $\sigma = 0$, and $J'(u_0,v_0) = 0$. □

4. **Analysis of the Fibering maps**

In this section we give a fairly complete description of the fibering maps associated with the problem. We will find it useful to consider the function

$$\mu_{(u,v)}(t) = t^{2-(\alpha+\beta)}\,\|(u,v)\|^2 - t^{2-(\alpha+\beta)}\int_\Omega \big(a\,u^2\,\log|t\,u| + b\,v^2\,\log|t\,v|\big)dx \quad (16)$$

Clearly, for $t > 0$, $(tu,tv) \in \mathcal{N}$ if and only if t is a solution of $\mu_{(u,v)}(t) =$

$\int_\Omega |u|^\alpha\, |v|^\beta dx$.

Lemma 4.1. *Suppose $(u,v) \in E - \{(0,0)\}$, the function $\mu_{(u,v)}$ satisfies the following properties:*

- *a) $\mu_{(u,v)}(t)$ has a unique critical point at*

$$t = t_{\max}(u,v) = \exp\left(\frac{\|(u,v)\|^2 - \int_\Omega (a\,u^2 \,\log|u| + b\,v^2 \,\log|v|)dx}{\int_\Omega (a\,u^2 + b\,v^2)dx} - \frac{1}{2-(\alpha+\beta)} \right); \tag{17}$$

- *b) $\mu_{(u,v)}(t)$ is strictly increasing on $(0, t_{\max}(u,v))$ and strictly decreasing on $(t_{\max}(u,v), +\infty)$);*
- *c) $\lim_{t\to+\infty} \mu_{(u,v)}(t) = 0$ and $\lim_{t\to 0^+} \mu_{(u,v)}(t) = -\infty$.*

Proof. By a direct computation, we have

$$\begin{aligned} &\mu'_{(u,v)}(t) \\ &= (2-(\alpha+\beta))\, t^{1-(\alpha+\beta)} \|(u,v)\|^2 \\ &- (2-(\alpha+\beta)) \int_\Omega \big(a\,u^2 \,\log|t\,u| + b\,v^2 \,\log|t\,v|\big)dx \\ &- t^{1-(\alpha+\beta)} \int_\Omega \big(a\,u^2 + b\,v^2\big)dx. \end{aligned}$$

Set $\mu_{(u,v)}(t) = 0$; there exist t_0 such that $\mu'_{(u,v)}(t_0) = 0$ and $\mu''_{(u,v)}(t_0) < 0$, with $t_0 = t_{\max}$ 17, moreover, we have $\mu'_{(u,v)}(t) > 0$ for $(0, t_{\max}(u,v))$ and $\mu'_{(u,v)}(t) < 0$ for $t \in (t_{\max}(u,v), +\infty)$.
Since $\alpha + \beta > 2$, then (c) holds. □

Lemma 4.2. $(tu, tv) \in \mathcal{N}^+$ *(or $\mathcal{N}^-$) if and only if $\mu'_{(u,v)}(t_0) > 0$ (or < 0).*

Proof. is clear that for $t > 0$, $(tu, tv) \in \mathcal{N}$ if and only if $\mu_{(u,v)}(t) = \int_\Omega |u|^\alpha\, ||v|^\beta dx$, moreover,

$$\begin{aligned} \mu'_{(u,v)}(t) &= (2-(\alpha+\beta))\, t^{1-(\alpha+\beta)} \|(u,v)\|^2 \\ &- (2-(\alpha+\beta)) \int_\Omega \big(a\,u^2 \,\log|t\,u| + b\,v^2 \,\log|t\,v|\big)dx \\ &- t^{1-(\alpha+\beta)} \int_\Omega \big(a\,u^2 + b\,v^2\big)dx, \end{aligned} \tag{18}$$

and so, if $(tu, tv) \in \mathcal{N}$, then $t^{\alpha+\beta}\mu'_{(u,v)}(t) = \varphi''_{(u,v)}(t)$, hence $(tu, tv) \in \mathcal{N}^+$ (or $\mathcal{N}^-$) if and only if $\mu'_{(u,v)}(t_0) > 0$ (or < 0). □

Lemma 4.3. *Both $\mathcal{N}^+$ and $\mathcal{N}^-$ are non-empty.*

Proof. Proof. By lemma 4.1 and 4.2 $\mathcal{N}^+ \neq \varnothing$ and $\mathcal{N}^- \neq \varnothing$ □

Lemma 4.4. $\mathcal{N}^0 = \emptyset$

Proof. Suppose otherwise, then for $(u,v) \in \mathcal{N}^0$ we have

$$\varphi''_{(u,v)} = -\int_\Omega \left(a\,u^2 + b\,v^2\right)dx - (\alpha+\beta-2)\int_\Omega |u|^2\,|v|^2 dx = 0,$$

and $\int_\Omega a\,u^2 dx = 0$, $\int_\Omega b\,v^2 dx = 0$, $\alpha+\beta > 2$, however $\int_\Omega |u|^\alpha\,|v|^\beta dx = 0$, which is a contradiction. Thus we have $\mathcal{N}^0 = \emptyset$. □

By lemma 4.2, we write $\mathcal{N} \neq \mathcal{N}^+ \cup \mathcal{N}^-$.

Lemma 4.5. *Suppose $(u,v) \in E - \{(0,0)\}$, there exists t_1, t_2 such that $t_1 < t_{\max}(u,v) < t_2$ and $(t_1u, t_1v) \in \mathcal{N}^+$ and $(t_2u, t_2v) \in \mathcal{N}^-$ and*

$$J(t_1u, t_1v) = \inf_{0 \le t \le t_{\max}} J(tu, tv) \qquad , \qquad J(t_2u, t_2v) = \inf_{t \ge 0} J(tu, tv)$$

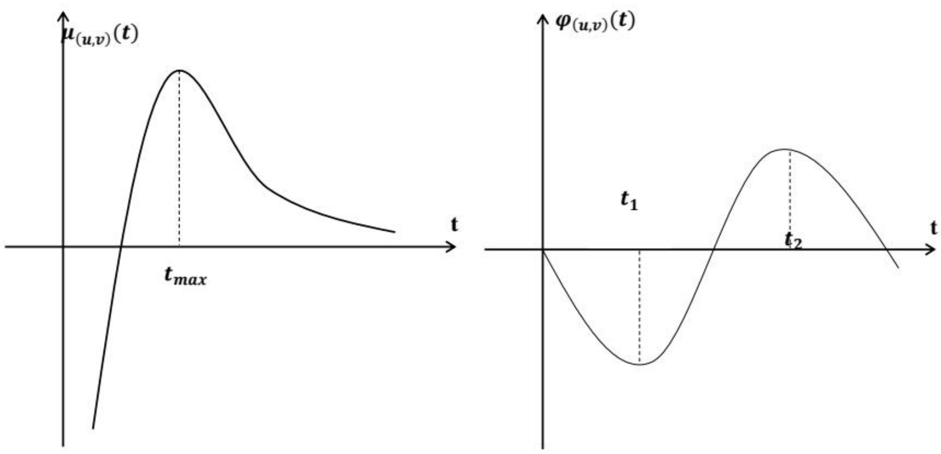

Fig 1. The graphs of $\mu_{(u,v)}(t)$ and $\varphi_{(u,v)}(t)$

Proof. By the fact that $\int_\Omega |u|^\alpha\,|v|^\beta dx > 0$, we have $0 < \int_\Omega |u|^\alpha\,|v|^\beta dx < \mu(t_{\max}(u,v))$ and there exist t_1, t_2 such that $t_1 < t_{\max} < t_2$, $\mu_{(u,v)}(t_1) = \mu_{(u,v)}(t_2) = \int_\Omega |u|^\alpha\,|v|^\beta dx$ and $\mu'_{(u,v)}(t_1) > 0$, $\mu'_{(u,v)}(t_2) < 0$ and with 16 , 13 we have :

$$t^{\alpha+\beta-1}\varphi'_{(u,v)}(t) = t^{2-(\alpha+\beta)}\,\|(u,v)\|^2 - t^{2-(\alpha+\beta)}\int_\Omega \left(a\,u^2\,\log|tu| + b\,v^2\,\log|tv|\right)dx - \int_\Omega |u|^\alpha |v|^\beta dx = 0.$$

However, $\varphi'_{(u,v)}(t_1) = \varphi'_{(u,v)}(t_2) = 0$ By proof of lemma 4.2 we have that $\varphi''_{(u,v)}(t_1) > 0$, $\varphi''_{(u,v)}(t_2) < 0$, these facts imply that the fibering map $\varphi_{(u,v)}$ has a local minimum at t_1 and a local maximum at t_2 such that $(t_1u, t_2v) \in \mathcal{N}^+$ and $(t_2u, t_2v) \in \mathcal{N}^-$ and since $\varphi_{(u,v)}(t) = J(tu, tv)$, we have $J(t_1u, t_1v) \leq J(tu, tv) \leq J(t_2u, t_2v)$ for each $t \in [t_1, t_2]$ and $J(t_1u, t_1v) \leq J(tu, tv)$ for each $t \in [0, t_1]$, thus $J(t_1u, t_1v) = \inf_{0\leq t\leq t_{\max}} J(tu, tv)$,
$J(t_2u, t_2v) = \sup_{t\geq 0} J(tu, tv)$, by a direct computation, we have $\varphi(0) = 0$, $\lim_{t\to 0^+} \varphi_{(u,v)}(t) = 0$ and $\lim_{t\to+\infty} \varphi_{(u,v)}(t) = -\infty$. So, the graphs of $\mu_{(u,v)}(t)$ and $\varphi_{(u,v)}(t)$ can be seen in Fig. 1. □

5. **Proof of the main result**

Theorem 5.1. *The functional J has a minimizer (u_1, v_1) in $\mathcal{N}^+$ and it satisfies,*

- *a) $J(u_1, v_1) = \inf_{(u,v)\in\mathcal{N}^+} J(u, v) < 0$;*
- *b) (u_1, v_1) is a solution problem 1.*

Proof. If $(u, v) \in \mathcal{N}^+$, we have

$$J(u,v) = \frac{1}{4}\int_\Omega \left(a\,u^2 + b\,v^2\right)dx - \frac{\alpha+\beta-2}{2(\alpha+\beta)}\int_\Omega |u|^\alpha\,|v|^\beta dx, \text{ of } \alpha+\beta$$
$$> 2 \text{ and } \int_\Omega |u|^\alpha\,|v|^\beta dx$$
$$> 0,$$

we have $J(u, v) < 0$, hence $\inf_{(u,v)\in\mathcal{N}^+} J(u, v) < 0$. Since J is bounded from below on $\mathcal{N}^+$, there exist a minimizing sequence $\{(u_n, v_n)\} \subset \mathcal{N}^+$ Such that,

$$\lim_{n\to\infty} J(u_n, v_n) = \inf_{(u,v)\in\mathcal{N}^+} J(u, v).$$

By lemma 2.1, J is corrective and bounded from below on $\mathcal{N}$, then $\{u_n, v_n\}$ is bounded in E. Hence, there exist $(u_1, v_1) \in E$, up to subsequence, such that $u_n \rightharpoonup u_1$, $v_n \rightharpoonup v_1$ in $H_0^1(\Omega)$ as $n \to \infty$ and $u_n \to u_1$, $v_n \to v_1$ in $L^2(\Omega)$, $L^{\alpha+\beta}(\Omega)$ a.e in Ω as $n \to \infty$.
We shall prove $(u_n, v_n) \to (u_1, v_1)$ in E as $n \to \infty$, otherwise

$$\|(u_1, v_1)\|_E < \liminf_{n\to\partial} \|(u_n, v_n)\|_E,$$

thus,

$$\begin{aligned}
&\lim_{n\to\infty} \varphi'_{(u_n,v_n)}(t_1)\\
&= \lim_{n\to\infty}\left(t_1\|(u_n,v_n)\|^2 - t_1\int_\Omega \big(a\,u_n^2\,\log|t_1\,u_n| + b\,v_n^2\,\log|t_1\,v_n|\big)dx\right)\\
&- t_1^{\alpha+\beta-1}\int_\Omega \big(|u_n|^\alpha|v_n|^\beta\big)dx,\\
&> t_1\|(u_1,v_n)\|^2 - t_1\int_\Omega \big(a\,u_1^2\,\log|t_1\,u_n| + b\,v_1^2\,\log|t_1\,v_1|\big)dx\\
&- t_1^{\alpha+\beta-1}\int_\Omega \big(|u_1|^\alpha|v_1|^\beta\big)dx = \varphi'_{(u,v)}(t_1) = 0,
\end{aligned}$$

that is $\varphi'_{(u_n,v_n)}(t_1) > 0$ for n large enough. Since $(u_n, v_n) = (1.u_n, 1.v_n) \in \mathcal{N}^+$, and we can see that $\varphi'_{(u,v)}(t_1) < 0$ for $t \in (0,1)$ and $\varphi'_{(u,v)}(1) = 0$ for all n. Then we must have $t_1 > 1$. On the other hand, $\varphi_{(u_1,v_1)}(t)$ is decreasing on $(0,t_1)$, and so $J(u_1,v_1) = \varphi'_{(u_1,v_1)}(t_1) < \varphi'_{(u_1,v_1)}(1) = J(u_1,v_1) < \lim_{n\to\infty} J(u_n,v_n) = \inf_{(u,v)\in\mathcal{N}^+} J(u,v)$, which is a contradiction. Hence $(u_n,v_n) \to (u_1,v_1)$ strongly in E. this implies that

$$J(u_1,v_1) = \inf_{(u,v)\in\mathcal{N}^+} J(u,v) \text{ as } n\to\infty$$

Namely, (u_1,v_1) is a minimize if J on $\mathcal{N}^+$. Using Lemma 3.2 (u_1,v_1) is a solution to 1. □

Theorem 5.2. *The functional J has a minimizer (u_2,v_2) in $\mathcal{N}^-$ and satisfies:*

- *a) $J(u_2,v_2) = \inf_{(u,v)\in\mathcal{N}^-} J(u,v) < 0$;*
- *b) (u_2,v_2) is a solution problem 1.*

Proof. Suppose that $(u,v)\in\mathcal{N}^-$, then $\varphi''_{(u,v)}(1) < 0$, of Young's inequality, we have

$$2-(\alpha+\beta)\,S^{\frac{-(\alpha+\beta)}{2}}\|(u,v)\|^{\alpha+\beta} < 2-(\alpha+\beta)\int_\Omega |u|^\alpha\,|v|^\beta dx < \int_\Omega \big(a\,u^2 + b\,v^2\big)dx,$$

hence,

$$J(u,v) > \frac{1}{4}\int_\Omega \left(a\,u^2 + b\,v^2\right)dx - \frac{2-(\alpha+\beta)}{(\alpha+\beta)\,S^{\frac{-(\alpha+\beta)}{2}}}\|(u,v)\|^{\alpha+\beta} = d_0 > 0,$$

since J is bounded from below on $\mathcal{N}^-$, there exist a minimizing sequence $(\widetilde{u}_n,\widetilde{v}_n) \subset \mathcal{N}^-$ such that

$$\lim_{n\to\infty} J(\widetilde{u}_n,\widetilde{v}_n) = \inf_{(u,v)\in\mathcal{N}^-} J(u,v) = d_0 > 0$$

By the same argument given in the proof of theorem 5.1 there exists $(u_2,v_2)\in E$, such that up to a subsequence $\widetilde{u}_n \to u_2$, $\widetilde{v}_n \to v_2$ strong in E. and $J(u_2,v_2) = \inf_{(u,v)\in\mathcal{N}^-} J(u,v)$ and (u_2,v_2) is solution to 1 □

6. **Proof of Theorem 1.1**

Proof. By theorem 5.1 and 5.2 and Lemma 3.2, we get that problem 1 has two solutions $(u_1, v_1) \in \mathcal{N}^+$ and $(u_2, v_2) \in \mathcal{N}^-$ in E. since $\mathcal{N}^+ \cup \mathcal{N}^- = \emptyset$, then two solutions are distinct, moreover $J(u_1, v_1) < 0$ and $J(u_2, v_2) > 0$.
We will show that solutions (u_1, v_1) and (u_2, v_2) are not semi-trivial solutions. We note that $(u, 0)$ (or $(0, v)$) is a semi-trivial solution of problem (1, that is

$$\begin{cases} -\Delta u = a(x)\, u \,\log|u| & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

see [11]. Next we prove that (u_1, v_1) is not a semi-trivial solution. Indeed, without loss of generality, we may assume that $v_1 = 0$, then u_1 is a nontrivial solution of problem 2, and $\|(u, 0)\|^2 = \|u_1\|^2 > 0$. Moreover, we may choose $w \in H_0^1(\Omega) - \{0\}$ such that,
$\|(w, 0)\|^2 = \|w\|^2 > 0$. By lemma 4.5 there exists a unique $0 < t_1 < \widetilde{t}_{\max}(u, w)$ such that $(t_1 u, t_1 w) \in \mathcal{N}^+$. Moreover, $\widetilde{t}_{\max}(u_1, w) > 1$ and $J(t_1 u, t_1 w) = \inf_{0 \le t \le \widetilde{t}_{\max}} J(tu, tw)$, this implies,

$$\inf_{(u,v)\in\mathcal{N}^+} J(u, v) \le J(t_1 u, t_1 w) \le J(u_1, w) \le J(u_1, 0) = \inf_{(u,v)\in\mathcal{N}^+} J(u, v),$$

Which is a contradiction. Hence, (u_1, v_1) is not a semi-trivial solution. By the same argument (u_2, v_2) is not a semi-trivial solution of problem 1.
This finishes the proof. □

REFERENCES

[1] *P. d'Avenia, E. Montefusco, and M. Squassina*, On the logarithmic Schrödinger equation, *Commun. Contemp. Math.*, **16**(2014), no. 2, 1350032.

[2] *S. Tian*, Multiple solutions for the semilinear elliptic equations with the sign-changing logarithmic nonlinearity, *J. Math. Anal. Appl.*, **454**(2017), 816-828.

[3] *K. J. Brown and T. F. Wu*, A semilinear elliptic system involving nonlinear boundary condition and sign-changing weight function, *J. Math. Anal. Appl.*, **337**(2008), 1326-1336.

[4] *K. J. Brown*, The Nehari manifold for a semilinear elliptic equation involving a sublinear term, *Calc. Var. Partial Differential Equations*, **22**(2004), 483-494.

[5] *K. J. Brown and Y. Zhang*, The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function, *J. Differential Equations*, **193**(2003), 481-499.

[6] *L. Caffarelli*, Non-local diffusions, drifts and games, in *Nonlinear Partial Differential Equations: The Abel Symposium 2010*, Springer, 2012, pp. 37-52.

[7] *W. Chen and S. Deng*, The Nehari manifold for a fractional p-Laplacian system involving concave–convex nonlinearities, *Nonlinear Anal. Real World Appl.*, **27**(2016), 80-92.

[8] *S. Chen and X. Tang*, Ground state sign-changing solutions for elliptic equations with logarithmic nonlinearity, *Acta Math. Hungar.*, **157**(2019), 27-38.

[9] *J. Mo and Z. Yang*, Existence of solutions to p-Laplace equations with logarithmic nonlinearity, *Electron. J. Differential Equations*, **2009**(2009), Paper No. 88.

[10] *H. Liu, Z. Liu, and Q. Xiao*, Ground state solution for a fourth-order nonlinear elliptic problem with logarithmic nonlinearity, *Appl. Math. Lett.*, **79**(2018), 176-181.

[11] *M. Squassina and A. Szulkin*, Multiple solutions to logarithmic Schrödinger equations with periodic potential, *Calc. Var. Partial Differential Equations*, **54**(2015), 585-597.

[12] *S. Tian*, Multiple solutions for the semilinear elliptic equations with the sign-changing logarithmic nonlinearity, *J. Math. Anal. Appl.*, **454**(2017), 816-828.

[13] *K. G. Zloshchastiev*, Logarithmic nonlinearity in theories of quantum gravity: Origin of time and observational consequences, *Grav. Cosmol.*, **16**(2010), 288-297.

[14] *L. Wen, X. Tang, and S. Chen*, Ground state sign-changing solutions for Kirchhoff equations with logarithmic nonlinearity, *Electron. J. Qual. Theory Differ. Equ.*, **2019**(2019), Paper No. 47, 1-13.

[15] *T. F. Wu*, Multiple positive solutions for a class of concave–convex elliptic problems in $\mathbb{R}^N$ involving sign-changing weight, *J. Funct. Anal.*, **258**(2010), 99-131.